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MULTILINEAR JENSEN TYPE MAPPINGS IN BANACH MODULES OVER A C^*–ALGEBRA

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Abstract. We prove the stability of multilinear Jensen type functional equations in Banach modules over a unital C^*-algebra.

1. Introduction

Let E_1 and E_2 be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geqslant 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th.M. Rassias [5] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leqslant \frac{2\epsilon}{2 - 2^p} ||x||^p$$

for all $x \in E_1$.

In [4], the author showed the stability of multilinear functional equations in Banach modules over a unital C^*-algebra. In [3], the author proved the stability of multi-quadratic mappings in Banach spaces.

Throughout this paper, let A be a unital C^*-algebra with norm $|\cdot|$, $\mathscr{U}(A)$ the unitary group of A, $A_1 = \{a \in A \mid |a| = 1\}$, and A_1^+ the set of positive elements in A_1. Let d be a positive integer and ${}_A\mathscr{B}_s$ a left A-module with norm $||\cdot||$ for each $s = 1, \cdots, d$. Let ${}_A\mathscr{D}$ be a left Banach A-module with norm $\|\cdot\|$.

The following is useful to prove the stability of linear functional equations.

LEMMA A [2, Theorem 1]. *Let $a \in A$ and $|a| < 1 - \frac{2}{m}$ for some integer m greater than 2. Then there are m elements $u_1, \cdots, u_m \in \mathscr{U}(A)$ such that $ma = u_1 + \cdots + u_m$.*

The main purpose of this paper is to prove the stability of multilinear Jensen type functional equations in Banach modules over a unital C^*-algebra.

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2. Stability of multilinear Jensen type functional equations in Banach modules over a C^*-algebra

For a given mapping $f : \prod_{s=1}^{d} {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ and given $a_1, \cdots, a_d \in A$, we set

$$D_{a_1,\cdots,a_d} f(x_1, y_1, \cdots, x_d, y_d) := \sum_{s=1}^{d} 2f(x_1, \cdots, x_{s-1}, \frac{a_s x_s + a_s y_s}{2}, x_{s+1}, \cdots, x_d)$$

$$- \sum_{s=1}^{d} a_s f(x_1, \cdots, x_{s-1}, x_s, x_{s+1}, \cdots, x_d) - \sum_{s=1}^{d} a_i f(x_1, \cdots, x_{s-1}, y_s, x_{s+1}, \cdots, x_d)$$

for all $(x_1, \cdots, x_d)$, $(y_1, \cdots, y_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$.

THEOREM 1. *Let* $f : \prod_{s=1}^{d} {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ *be a mapping for which there exists a function* $\varphi : \prod_{s=1}^{d} {}_A\mathscr{B}_s^2 \to [0, \infty)$ *such that*

$$\widetilde{\varphi}(x_1, y_1, \cdots, x_d, y_d) := \sum_{j=0}^{\infty} \sum_{s=1}^{d} 2^{s-1+jd} \varphi(\frac{x_1}{2^{j+1}}, \frac{x_1}{2^{j+1}}, \cdots, \frac{x_{s-1}}{2^{j+1}}, \frac{x_{s-1}}{2^{j+1}}, \frac{x_s}{2^j}, \frac{y_s}{2^j}, \frac{x_{s+1}}{2^j}, \frac{x_{s+1}}{2^j}, \cdots, \frac{x_d}{2^j}, \frac{x_d}{2^j}) < \infty \tag{2.i}$$

$$\|D_{u_1,\cdots,u_d} f(x_1, y_1, \cdots, x_d, y_d)\| \leqslant \varphi(x_1, y_1, \cdots, x_d, y_d) \tag{2.ii}$$

for all $u_1, \cdots, u_d \in \mathscr{U}(A)$ *and all* $(x_1, \cdots, x_d)$, $(y_1, \cdots, y_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$. *Assume that* $f(x_1, \cdots, x_d) = 0$ *if* $x_s = 0$ *for some* $s = 1, \cdots, d$. *Then there exists a unique* A*-multilinear mapping* $M : \prod_{s=1}^{d} {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ *such that*

$$\|f(x_1, \cdots, x_d) - M(x_1, \cdots, x_d)\| \leqslant \widetilde{\varphi}(x_1, 0, \cdots, x_d, 0) \tag{2.iii}$$

for all $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$.

Proof. Put $u_1 = \cdots = u_d = 1 \in \mathscr{U}(A)$. For each fixed l, let $y_1 = x_1, \cdots, y_{l-1} = x_{l-1}, y_{l+1} = x_{l+1}, \cdots, y_d = x_d$ and $y_l = 0$ in (2.ii). Then we get

$$\|2f(x_1, \cdots, x_{l-1}, \frac{x_l}{2}, x_{l+1}, \cdots, x_d) - f(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d)\|$$
$$\leqslant \varphi(x_1, x_1, \cdots, x_{l-1}, x_{l-1}, x_l, 0, x_{l+1}, x_{l+1}, \cdots, x_d, x_d)$$

for all $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$. So one can obtain

$$\|2^{l-1} f(\frac{x_1}{2}, \cdots, \frac{x_{l-1}}{2}, x_l, \cdots, x_d) - 2^l f(\frac{x_1}{2}, \cdots, \frac{x_l}{2}, x_{l+1}, \cdots, x_d)\|$$
$$\leqslant 2^{l-1} \varphi(\frac{x_1}{2}, \frac{x_1}{2}, \cdots, \frac{x_{l-1}}{2}, \frac{x_{l-1}}{2}, x_l, 0, x_{l+1}, x_{l+1}, \cdots, x_d, x_d)$$

for all $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$. Thus

$$\|f(x_1, \cdots, x_d) - 2^d f(\frac{x_1}{2}, \cdots, \frac{x_d}{2})\|$$
$$\leqslant \sum_{s=1}^{d} 2^{s-1} \varphi(\frac{x_1}{2}, \frac{x_1}{2}, \cdots, \frac{x_{s-1}}{2}, \frac{x_{s-1}}{2}, x_s, 0, x_{s+1}, x_{s+1}, \cdots, x_d, x_d)$$

for all $(x_1,\cdots,x_d)\in\prod_{s=1}^d {}_A\mathscr{B}_s$. Hence we get

$$\begin{aligned}&\|2^{jd}f(\frac{x_1}{2^j},\cdots,\frac{x_d}{2^j})-2^{(j+1)d}f(\frac{x_1}{2^{j+1}},\cdots,\frac{x_d}{2^{j+1}})\|\\&\quad\leqslant\sum_{s=1}^{d}2^{s-1+jd}\varphi(\frac{x_1}{2^{j+1}},\frac{x_1}{2^{j+1}},\cdots,\frac{x_{s-1}}{2^{j+1}},\frac{x_{s-1}}{2^{j+1}},\frac{x_s}{2^j},0,\frac{x_{s+1}}{2^j},\frac{x_{s+1}}{2^j},\cdots,\frac{x_d}{2^j},\frac{x_d}{2^j})\end{aligned}$$

for all $(x_1,\cdots,x_d)\in\prod_{s=1}^d {}_A\mathscr{B}_s$. So

$$\begin{aligned}\|f(x_1,\cdots,x_d)-2^{nd}f(\frac{x_1}{2^n},\cdots,\frac{x_d}{2^n})\|&\leqslant\sum_{j=0}^{n-1}\sum_{s=1}^{d}2^{s-1+jd}\\&\varphi(\frac{x_1}{2^{j+1}},\frac{x_1}{2^{j+1}},\cdots,\frac{x_{s-1}}{2^{j+1}},\frac{x_{s-1}}{2^{j+1}},\frac{x_i}{2^j},0,\frac{x_{s+1}}{2^j},\frac{x_{s+1}}{2^j},\cdots,\frac{x_d}{2^j},\frac{x_d}{2^j})\end{aligned}\tag{2.1}$$

for all $(x_1,\cdots,x_d)\in\prod_{s=1}^d {}_A\mathscr{B}_s$.

For each $s=1,\cdots,d$, let x_s be an element in ${}_A\mathscr{B}_s$. For positive integers n and m with $n>m$,

$$\begin{aligned}&\|2^{md}f(\frac{x_1}{2^m},\cdots,\frac{x_d}{2^m})-2^{nd}f(\frac{x_1}{2^n},\cdots,\frac{x_d}{2^n})\|\\&=2^{md}\|f(\frac{x_1}{2^m},\cdots,\frac{x_d}{2^m})-2^{(n-m)d}f(\frac{1}{2^{n-m}}\frac{x_1}{2^m},\cdots,\frac{1}{2^{n-m}}\frac{x_d}{2^m})\|\\&\leqslant 2^{md}\sum_{j=0}^{(n-m)-1}\sum_{s=1}^{d}2^{s-1+jd}\varphi(\frac{1}{2^{j+1}}\frac{x_1}{2^m},\frac{1}{2^{j+1}}\frac{x_1}{2^m},\cdots,\frac{1}{2^{j+1}}\frac{x_{s-1}}{2^m},\frac{1}{2^{j+1}}\frac{x_{s-1}}{2^m},\\&\qquad\frac{1}{2^j}\frac{x_s}{2^m},0,\frac{1}{2^j}\frac{x_{s+1}}{2^m},\frac{1}{2^j}\frac{x_{s+1}}{2^m},\cdots,\frac{1}{2^j}\frac{x_d}{2^m},\frac{1}{2^j}\frac{x_d}{2^m})\\&=2^{md}\sum_{j=m}^{n-1}\sum_{s=1}^{d}2^{s-1+(j-m)d}\varphi(\frac{1}{2^{(j-m)+1}}\frac{x_1}{2^m},\frac{1}{2^{(j-m)+1}}\frac{x_1}{2^m},\cdots,\\&\qquad\frac{1}{2^{(j-m)+1}}\frac{x_{s-1}}{2^m},\frac{1}{2^{(j-m)+1}}\frac{x_{s-1}}{2^m},\frac{1}{2^{(j-m)}}\frac{x_s}{2^m},0,\frac{1}{2^{(j-m)}}\frac{x_{s+1}}{2^m},\\&\qquad\frac{1}{2^{(j-m)}}\frac{x_{s+1}}{2^m},\cdots,\frac{1}{2^{(j-m)}}\frac{x_d}{2^m},\frac{1}{2^{(j-m)}}\frac{x_d}{2^m})\\&=\sum_{j=m}^{n-1}\sum_{s=1}^{d}2^{s-1+jd}\varphi(\frac{x_1}{2^{j+1}},\frac{x_1}{2^{j+1}},\cdots,\frac{x_{s-1}}{2^{j+1}},\frac{x_{s-1}}{2^{j+1}},\frac{x_s}{2^j},0,\\&\qquad\frac{x_{s+1}}{2^j},\frac{x_{s+1}}{2^j},\cdots,\frac{x_d}{2^j},\frac{x_d}{2^j}),\end{aligned}$$

which tends to zero as $m\to\infty$ by (2.i). So $\{2^{nd}f(\frac{x_1}{2^n},\cdots,\frac{x_d}{2^n})\}$ is a Cauchy sequence for all $(x_1,\cdots,x_d)\in\prod_{s=1}^d {}_A\mathscr{B}_s$. Since ${}_A\mathscr{D}$ is complete, the sequence $\{2^{nd}f(\frac{x_1}{2^n},\cdots,\frac{x_d}{2^n})\}$ converges for all $(x_1,\cdots,x_d)\in\prod_{s=1}^d {}_A\mathscr{B}_s$. We can define a mapping $M:\prod_{s=1}^d {}_A\mathscr{B}_s\to {}_A\mathscr{D}$ by

$$M(x_1,\cdots,x_d)=\lim_{j\to\infty}2^{jd}f(\frac{x_1}{2^j},\cdots,\frac{x_d}{2^j})\tag{2.2}$$

for all $(x_1,\cdots,x_d)\in\prod_{s=1}^d {}_A\mathscr{B}_s$.

For each fixed $l=1,\cdots,d$, by (2.i) and (2.2), we get

$$\begin{aligned}
&\|D_{1,\cdots,1}M(x_1,x_1,\cdots,x_{l-1},x_{l-1},x_l,y_l,x_{l+1},x_{l+1},\cdots,x_d,x_d)\|\\
&=\lim_{j\to\infty}2^{jd}\|D_{1,\cdots,1}f(\frac{x_1}{2^j},\frac{x_1}{2^j},\cdots,\frac{x_{l-1}}{2^j},\frac{x_{l-1}}{2^j},\frac{x_l}{2^j},\frac{y_l}{2^j},\frac{x_{l+1}}{2^j},\frac{x_{l+1}}{2^j},\cdots,\frac{x_d}{2^j},\frac{x_d}{2^j})\|\\
&\leqslant\lim_{j\to\infty}2^{jd}\varphi(\frac{x_1}{2^j},\frac{x_1}{2^j},\cdots,\frac{x_{l-1}}{2^j},\frac{x_{l-1}}{2^j},\frac{x_l}{2^j},\frac{y_l}{2^j},\frac{x_{l+1}}{2^j},\frac{x_{l+1}}{2^j},\cdots,\frac{x_d}{2^j},\frac{x_d}{2^j})\\
&=0
\end{aligned}$$

for all $(x_1,\cdots,x_d)\in\prod_{s=1}^d {}_A\mathscr{B}_s$ and all $y_l\in {}_A\mathscr{B}_l$. Hence

$$D_{1,\cdots,1}M(x_1,x_1,\cdots,x_{l-1},x_{l-1},x_l,y_l,x_{l+1},x_{l+1},\cdots,x_d,x_d)=0$$

for all $(x_1,\cdots,x_d)\in\prod_{s=1}^d {}_A\mathscr{B}_s$ and all $y_l\in {}_A\mathscr{B}_l$, which implies that

$$\begin{aligned}
2M(x_1,...,x_{l-1},\frac{x_l+y_l}{2},x_{l+1},...,x_d)&=M(x_1,...,x_{l-1},x_l,x_{l+1},...,x_d)\\
&\quad+M(x_1,...,x_{l-1},y_l,x_{l+1},...,x_d)
\end{aligned}\tag{2.3}$$

for all $(x_1,\cdots,x_d)\in\prod_{s=1}^d {}_A\mathscr{B}_s$ and all $y_l\in {}_A\mathscr{B}_l$. Putting $y_l=0$ in (2.3), we get

$$2M(x_1,\cdots,x_{l-1},\frac{x_l}{2},x_{i+1},\cdots,x_d)=M(x_1,\cdots,x_{l-1},x_l,x_{l+1},\cdots,x_d)\tag{2.4}$$

for all $(x_1,\cdots,x_d)\in\prod_{s=1}^d {}_A\mathscr{B}_s$. By (2.3) and (2.4), M is additive for each $l=1,\cdots,d$. Moreover, by passing to the limit in (2.1) as $n\to\infty$, we get the inequality (2.iii).

Now let $L:\prod_{s=1}^d {}_A\mathscr{B}_s\to {}_A\mathscr{D}$ be another multi-additive mapping satisfying

$$\|f(x_1,\cdots,x_d)-L(x_1,\cdots,x_d)\|\leqslant\widetilde{\varphi}(x_1,0,\cdots,x_d,0)$$

for all $(x_1,\cdots,x_d)\in\prod_{s=1}^d {}_A\mathscr{B}_s$.

$$\begin{aligned}
&\|M(x_1,\cdots,x_d)-L(x_1,\cdots,x_d)\|\\
&\quad=2^{kd}\|M(\frac{x_1}{2^k},\cdots,\frac{x_d}{2^k})-L(\frac{x_1}{2^k},\cdots,\frac{x_d}{2^k})\|\\
&\quad\leqslant 2^{kd}\|M(\frac{x_1}{2^k},\cdots,\frac{x_d}{2^k})-f(\frac{x_1}{2^k},\cdots,\frac{x_d}{2^k})\|\\
&\qquad\qquad+2^{kd}\|f(\frac{x_1}{2^k},\cdots,\frac{x_d}{2^k})-L(\frac{x_1}{2^k},\cdots,\frac{x_d}{2^k})\|\\
&\quad\leqslant 2^{kd+1}\widetilde{\varphi}(\frac{x_1}{2^k},0,\cdots,\frac{x_d}{2^k},0)\\
&\quad=\sum_{j=0}^{\infty}\sum_{s=1}^{d}2^{s+(j+k)d}\varphi(\frac{x_1}{2^{j+k+1}},\frac{x_1}{2^{j+k+1}},\cdots,\frac{x_{s-1}}{2^{j+k+1}},\frac{x_{s-1}}{2^{j+k+1}},\frac{x_s}{2^{j+k}},0,\\
&\qquad\qquad\frac{x_{s+1}}{2^{j+k}},\frac{x_{s+1}}{2^{j+k}},\cdots,\frac{x_d}{2^{j+k}},\frac{x_d}{2^{j+k}})
\end{aligned}$$

$$= \sum_{j=k}^{\infty} \sum_{s=1}^{d} 2^{s+jd} \varphi(\frac{x_1}{2^{j+1}}, \frac{x_1}{2^{j+1}}, \cdots, \frac{x_{s-1}}{2^{j+1}}, \frac{x_{s-1}}{2^{j+1}}, \frac{x_s}{2^j}, 0, \frac{x_{s+1}}{2^j}, \frac{x_{s+1}}{2^j}, \cdots, \frac{x_d}{2^j}, \frac{x_d}{2^j}),$$

which tends to zero as $k \to \infty$ by (2.i). Thus $M(x_1, \cdots, x_d) = L(x_1, \cdots, x_d)$ for all $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$. This proves the uniqueness of M.

By the assumption, for each fixed $u_l \in \mathscr{U}(A)$,

$$2^{jd} \| D_{1,\cdots,1,u_l,1,\cdots,1} f(\frac{x_1}{2^j}, \frac{x_1}{2^j}, \cdots, \frac{x_{l-1}}{2^j}, \frac{x_{l-1}}{2^j}, \frac{x_l}{2^j}, \frac{x_l}{2^j}, \frac{x_{l+1}}{2^j}, \frac{x_{l+1}}{2^j}, \cdots, \frac{x_d}{2^j}, \frac{x_d}{2^j}) \| \leqslant 2^{jd} \varphi(\frac{x_1}{2^j}, \frac{x_1}{2^j}, \cdots, \frac{x_{l-1}}{2^j}, \frac{x_{l-1}}{2^j}, \frac{x_l}{2^j}, \frac{x_l}{2^j}, \frac{x_{l+1}}{2^j}, \frac{x_{l+1}}{2^j}, \cdots, \frac{x_d}{2^j}, \frac{x_d}{2^j})$$

for all $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$, and so

$$2^{jd} \| D_{1,\cdots,1,u_l,1,\cdots,1} f(\frac{x_1}{2^j}, \frac{x_1}{2^j}, \cdots, \frac{x_l}{2^j}, \frac{x_l}{2^j}, \cdots, \frac{x_d}{2^j}, \frac{x_d}{2^j}) \| \to 0$$

as $j \to \infty$ for all $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$. So

$$D_{1,\cdots,1,u_l,1,\cdots,1} M(x_1, x_1, \cdots, x_d, x_d) = \lim_{j\to\infty} 2^{jd} D_{1,\cdots,1,u_i,1,\cdots,1} f(\frac{x_1}{2^j}, \frac{x_1}{2^j}, \cdots, \frac{x_d}{2^j}, \frac{x_d}{2^j}) = 0$$

for all $u_l \in \mathscr{U}(A)$ and all $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$. Hence

$$D_{1,\cdots,1,u_l,1,\cdots,1} M(x_1, x_1, \cdots, x_d, x_d) = 2M(x_1, \cdots, x_{l-1}, u_l x_l, x_{l+1}, \cdots, x_d) - 2u_l M(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d) = 0$$

for all $u_l \in \mathscr{U}(A)$ and all $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$. So

$$M(x_1, \cdots, x_{l-1}, u_l x_l, x_{l+1}, \cdots, x_d) = u_l M(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d)$$

for all $u_l \in \mathscr{U}(A)$ and all $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$.

Now let $a \in A$ $(a \neq 0)$ and K an integer greater than $4|a|$. Then

$$|\frac{a}{K}| = \frac{1}{K}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By Lemma A, there exist three elements $u, v, w \in \mathscr{U}(A)$ such that $3\frac{a}{K} = u + v + w$. And

$$\begin{aligned} M(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d) &= M(x_1, \cdots, x_{l-1}, 3 \cdot \frac{1}{3} x_l, x_{l+1}, \cdots, x_d) \\ &= 3M(x_1, \cdots, x_{l-1}, \frac{1}{3} x_l, x_{l+1}, \cdots, x_d) \end{aligned}$$

for all $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$. So

$$M(x_1, \cdots, x_{l-1}, \frac{1}{3} x_l, x_{l+1}, \cdots, x_d) = \frac{1}{3} M(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d)$$

for all $(x_1,\cdots,x_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$. Thus

$$\begin{aligned} M(x_1,\cdots,ax_l,\cdots,x_d) &= M(x_1,\cdots,\frac{K}{3}\cdot 3\frac{a}{K}x_l,\cdots,x_d) \\ &= \frac{K}{3}M(x_1,\cdots,3\frac{a}{K}x_l,\cdots,x_d) \\ &= \frac{K}{3}M(x_1,\cdots,ux_l+vx_l+wx_l,\cdots,x_d) \\ &= \frac{K}{3}(u+v+w)M(x_1,\cdots,x_l,\cdots,x_d) \\ &= \frac{K}{3}\cdot 3\frac{a}{K}M(x_1,\cdots,x_l,\cdots,x_d) \\ &= aM(x_1,\cdots,x_l,\cdots,x_d) \end{aligned}$$

for all $(x_1,\cdots,x_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$. Obviously,

$$M(x_1,\cdots,0x_i,\cdots,x_d) = 0M(x_1,\cdots,x_i,\cdots,x_d)$$

for all $(x_1,\cdots,x_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$. Hence

$$\begin{aligned} &M(x_1,\cdots,x_{l-1},ax_l+by_l,x_{l+1},\cdots,x_d) \\ &\quad = M(x_1,\cdots,x_{l-1},ax_l,x_{l+1},\cdots,x_d) + M(x_1,\cdots,x_{l-1},by_l,x_{l+1},\cdots,x_d) \\ &\quad = aM(x_1,\cdots,x_{l-1},x_l,x_{l+1},\cdots,x_d) + bM(x_1,\cdots,x_{l-1},y_l,x_{l+1},\cdots,x_d) \end{aligned}$$

for all $a,b \in A$ and all $(x_1,\cdots,x_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$ and all $y_l \in {}_A\mathscr{B}_l$. So the unique multi-additive mapping $M : \prod_{s=1}^d {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ is an A-multilinear mapping, as desired. □

THEOREM 2. *Let* $f : \prod_{s=1}^d {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ *be a mapping for which there exists a function* $\varphi : \prod_{s=1}^d {}_A\mathscr{B}_s^2 \to [0,\infty)$ *satisfying* (2.i) *such that*

$$\|D_{a_1,\cdots,a_d}f(x_1,y_1,\cdots,x_d,y_d)\| \leqslant \varphi(x_1,y_1,\cdots,x_d,y_d)$$

for all $a_1,\cdots,a_d \in A_1^+ \cup \{i\}$ *and all* $(x_1,\cdots,x_d)$, $(y_1,\cdots,y_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$. *Assume that* $f(x_1,\cdots,x_d) = 0$ *if* $x_s = 0$ *for some* $s = 1,\cdots,d$. *Assume that for each fixed* $l = 1,\cdots,d$, $f(x_1,\cdots,x_{l-1},\lambda x_l,x_{l+1},\cdots,x_d)$ *is continuous in* $\lambda \in \mathbb{R}$ *for each fixed* $(x_1,\cdots,x_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$, *and that* $\{2^{jd}f(\frac{x_1}{2^j},\cdots,\frac{x_d}{2^j})\}$ *converges uniformly for all* $(x_1,\cdots,x_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$. *Then there exists a unique* A*-multilinear mapping* $M : \prod_{s=1}^d {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ *satisfying* (2.iii).

Proof. Put $a_1 = \cdots = a_d = 1 \in A_1^+$. By the same reasoning as the proof of Theorem 1, there exists a unique multi-additive mapping $M : \prod_{s=1}^d {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ satisfying (2.iii).

For each fixed $l = 1,\cdots,d$, since $f(x_1,\cdots,\lambda x_l,\cdots,x_d)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $(x_1,\cdots,x_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$, the mapping $M(x_1,\cdots,\lambda x_l,\cdots,x_d)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $(x_1,\cdots,x_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$ by the uniform convergence. By the same reasoning as in the proof of [4, Theorem], the multi-additive mapping $M : \prod_{s=1}^{d} {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ is $\mathbb{R}$-linear in the l-th variable. So the multi-additive mapping $M : \prod_{s=1}^{d} {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ is $\mathbb{R}$-multilinear.

By the same reasoning as the proof of Theorem 1,

$$M(x_1, \cdots, x_{l-1}, ax_l, x_{l+1}, \cdots, x_d) = aM(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d) \tag{2.5}$$

for all $a \in A_1^+ \cup \{i\}$ and $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$. So

$$\begin{aligned} M(x_1, \cdots, x_{l-1}, ax_l, x_{l+1}, \cdots, x_d) &= M(x_1, \cdots, x_{l-1}, |a|\frac{a}{|a|}x_l, x_{l+1}, \cdots, x_d) \\ &= |a|M(x_1, \cdots, x_{l-1}, \frac{a}{|a|}x_l, x_{l+1}, \cdots, x_d) \\ &= |a|\frac{a}{|a|}M(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d) \\ &= aM(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d) \end{aligned} \tag{2.6}$$

for all $a \in A^+ \setminus \{0\}$ and $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$.

For any element $a \in A$, $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$, and $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = (\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-$, where $(\frac{a+a^*}{2})^+$, $(\frac{a+a^*}{2})^-$, $(\frac{a-a^*}{2i})^+$, and $(\frac{a-a^*}{2i})^-$ are positive elements (see [1, Lemma 38.8]). Using the $\mathbb{R}$-multilinearity and (2.6), one can easily show that

$$M(x_1, \cdots, x_{l-1}, ax_l, x_{l+1}, \cdots, x_d) = aM(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d)$$

for all $a \in A$ and all $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$. Hence

$$\begin{aligned} &M(x_1, \cdots, x_{l-1}, ax_l + by_l, x_{l+1}, \cdots, x_d) \\ &\quad = M(x_1, \cdots, x_{l-1}, ax_l, x_{l+1}, \cdots, x_d) + M(x_1, \cdots, x_{l-1}, by_l, x_{l+1}, \cdots, x_d) \\ &\quad = aM(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d) + bM(x_1, \cdots, x_{l-1}, y_l, x_{l+1}, \cdots, x_d) \end{aligned}$$

for all $a, b \in A$, all $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$ and $y_l \in {}_A\mathscr{B}_l$. So the unique multi-additive mapping $M : \prod_{s=1}^{d} {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ is an A-multilinear mapping, as desired. □

COROLLARY 1. *Let $\theta \geqslant 0$ and let $\eta : [0, \infty) \to [0, \infty)$ be a function with $\eta(0) = 0$ such that*

$$\eta(\alpha\beta) \leqslant \eta(\alpha)\eta(\beta),$$
$$\eta(\frac{1}{2}) < (\frac{1}{2})^{2d}$$

for all $\alpha, \beta \in [0, \infty)$. Assume that a mapping $f : \prod_{s=1}^{d} {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ satisfies

$$\|D_{u_1, \cdots, u_d} f(x_1, y_1, \cdots, x_d, y_d)\| \leqslant \theta \sum_{s=1}^{d} (\eta(||x_s||) + \eta(||y_s||))$$

for all $u_1, \cdots, u_d \in \mathscr{U}(A)$*, and all* $(x_1, \cdots, x_d), (y_1, \cdots, y_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$*, and that* $f(x_1, \cdots, x_d) = 0$ *if* $x_s = 0$ *for some* $s = 1, \cdots, d$*. Then there exists a unique* A*-multilinear mapping* $M : \prod_{s=1}^d {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ *such that*

$$\begin{aligned}\|f(x_1, \cdots, x_d) - M(x_1, \cdots, x_d)\| &\\ \leqslant \theta \sum_{s=1}^{d} \sum_{j=0}^{\infty} 2^{s-1+jd} (2\ \eta(\|\frac{x_1}{2^{j+1}}\|) + \cdots + 2\ \eta(\|\frac{x_{s-1}}{2^{j+1}}\|) &\\ + \eta(\|\frac{x_s}{2^j}\|) + 2\ \eta(\|\frac{x_{s+1}}{2^j}\|) + \cdots + 2\ \eta(\|\frac{x_d}{2^j}\|))\end{aligned} \tag{2.iv}$$

for all $(x_1, \cdots, x_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$.

Proof. It follows from Theorem 1. Indeed, for all

$$(x_1, \cdots, x_d), (y_1, \cdots, y_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s,$$

we have

$$\begin{aligned}&\widetilde{\varphi}(x_1, y_1, \cdots, x_d, y_d)\\ &= \theta \sum_{s=1}^{d} \sum_{j=0}^{\infty} 2^{s-1+jd} (2\ \eta(\|\frac{x_1}{2^{j+1}}\|) + \cdots + 2\ \eta(\|\frac{x_{s-1}}{2^{j+1}}\|) + \eta(\|\frac{x_s}{2^j}\|)\\ &\quad + \eta(\|\frac{y_s}{2^j}\|) + 2\ \eta(\|\frac{x_{s+1}}{2^j}\|) + \cdots + 2\ \eta(\|\frac{x_d}{2^j}\|))\\ &\leqslant \theta \sum_{s=1}^{d} \sum_{j=0}^{\infty} 2^{s-1+jd} (2\ \eta(\frac{1}{2})^j \eta(\|\frac{x_1}{2}\|) + \cdots + 2\ \eta(\frac{1}{2})^j \eta(\|\frac{x_{s-1}}{2}\|) + \eta(\frac{1}{2})^j \eta(\|x_s\|)\\ &\quad + \eta(\frac{1}{2})^j \eta(\|y_s\|) + 2\ \eta(\frac{1}{2})^j \eta(\|x_{s+1}\|) + \cdots + 2\ \eta(\frac{1}{2})^j \eta(\|x_d\|))\\ &= \theta \sum_{j=0}^{\infty} (2^d \eta(\frac{1}{2}))^j \sum_{s=1}^{d} 2^{s-1} (2\ \eta(\|\frac{x_1}{2}\|) + \cdots + 2\ \eta(\|\frac{x_{s-1}}{2}\|) + \eta(\|x_s\|)\\ &\quad + \eta(\|y_s\|) + 2\ \eta(\|x_{s+1}\|) + \cdots + 2\ \eta(\|x_d\|))\\ &= \frac{\theta}{1 - 2^d \eta(\frac{1}{2})} \sum_{s=1}^{d} 2^{s-1} (2\ \eta(\|\frac{x_1}{2}\|) + \cdots + 2\ \eta(\|\frac{x_{s-1}}{2}\|) + \eta(\|x_s\|)\\ &\quad + \eta(\|y_s\|) + 2\ \eta(\|x_{s+1}\|) + \cdots + 2\ \eta(\|x_d\|)) < \infty.\end{aligned}$$

Now

$$\begin{aligned}\widetilde{\varphi}(x_1, 0, \cdots, x_d, 0) = &\theta \sum_{s=1}^{d} \sum_{j=0}^{\infty} 2^{s-1+jd} (2\ \eta(\|\frac{x_1}{2^{j+1}}\|) + \cdots + 2\ \eta(\|\frac{x_{s-1}}{2^{j+1}}\|)\\ &+ \eta(\|\frac{x_s}{2^j}\|) + 2\ \eta(\|\frac{x_{s+1}}{2^j}\|) + \cdots + 2\ \eta(\|\frac{x_d}{2^j}\|))\end{aligned}$$

for all $(x_1, \cdots x_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$. So one can obtain the inequality (2.iv). □

COROLLARY 2. *Let* $\theta \geqslant 0$, $d < p$, *and let* $\mu : [0,\infty)^{2d} \to [0,\infty)$ *be a function such that*

$$\mu(\lambda\alpha_1, \lambda\beta_1, \cdots, \lambda\alpha_d, \lambda\beta_d) = \lambda^p \mu(\alpha_1, \beta_1, \cdots, \alpha_d, \beta_d)$$

for all $\lambda, \alpha_1, \beta_1, \cdots, \alpha_d, \beta_d \in [0,\infty)$. *Assume that a mapping* $f : \prod_{s=1}^d {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ *satisfies*

$$\|D_{u_1,\cdots,u_d} f(x_1, y_1, \cdots, x_d, y_d)\| \leqslant \theta\mu(||x_1||, ||y_1||, \cdots, ||x_d||, ||y_d||)$$

for all $u_1, \cdots, u_d \in \mathscr{U}(A)$, *and all* $(x_1, \cdots, x_d), (y_1, \cdots, y_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$, *and that* $f(x_1, \cdots, x_d) = 0$ *if* $x_s = 0$ *for some* $s = 1, \cdots, d$. *Then there exists a unique* A*-multilinear mapping* $M : \prod_{s=1}^d {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ *such that*

$$\begin{aligned} &\|f(x_1, \cdots, x_d) - M(x_1, \cdots, x_d)\| \\ &\leqslant \frac{2^p\theta}{2^p - 2^d} \sum_{s=1}^d 2^{s-1} \mu(||\frac{x_1}{2}||, ||\frac{x_1}{2}||, \cdots, ||\frac{x_{s-1}}{2}||, ||\frac{x_{s-1}}{2}||, \\ &\qquad ||x_s||, 0, ||x_{s+1}||, ||x_{s+1}||, \cdots, ||x_d||, ||x_d||) \end{aligned} \tag{2.v}$$

for all $(x_1, \cdots, x_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$.

Proof. It follows from Theorem 1. Indeed, for all

$$(x_1, \cdots, x_d), (y_1, \cdots, y_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s,$$

we have

$$\begin{aligned} &\widetilde{\varphi}(x_1, y_1, \cdots, x_d, y_d) \\ &= \theta \sum_{s=1}^d \sum_{j=0}^\infty 2^{s-1+jd} \mu((\frac{1}{2})^j||\frac{x_1}{2}||, (\frac{1}{2})^j||\frac{x_1}{2}||, \cdots, (\frac{1}{2})^j||\frac{x_{s-1}}{2}||, (\frac{1}{2})^j||\frac{x_{s-1}}{2}||, \\ &\qquad (\frac{1}{2})^j||x_s||, (\frac{1}{2})^j||x_s||, (\frac{1}{2})^j||x_{s+1}||, (\frac{1}{2})^j||x_{s+1}||, \cdots, (\frac{1}{2})^j||x_d||, (\frac{1}{2})^j||x_d||) \\ &= \theta \sum_{s=1}^d 2^{s-1} \sum_{j=0}^\infty 2^{jd} (\frac{1}{2})^{jp} \mu(||\frac{x_1}{2}||, ||\frac{x_1}{2}||, \cdots, ||\frac{x_{s-1}}{2}||, ||\frac{x_{s-1}}{2}||, ||x_s||, ||y_s||, ||x_{s+1}||, \\ &\qquad ||x_{s+1}||, \cdots, ||x_d||, ||x_d||) \\ &= \frac{2^p\theta}{2^p - 2^d} \sum_{s=1}^d 2^{s-1} \mu(||\frac{x_1}{2}||, ||\frac{x_1}{2}||, \cdots, ||\frac{x_{s-1}}{2}||, ||\frac{x_{s-1}}{2}||, ||x_s||, ||y_s||, ||x_{s+1}||, \\ &\qquad ||x_{s+1}||, \cdots, ||x_d||, ||x_d||) < \infty. \end{aligned}$$

Now

$$\begin{aligned} \widetilde{\varphi}(x_1, 0, \cdots, x_d, 0) = &\frac{2^p\theta}{2^p - 2^d} \sum_{s=1}^d 2^{s-1} \mu(||\frac{x_1}{2}||, ||\frac{x_1}{2}||, \cdots, ||\frac{x_{s-1}}{2}||, ||\frac{x_{s-1}}{2}||, \\ &\qquad ||x_s||, 0, ||x_{s+1}||, ||x_{s+1}||, \cdots, ||x_d||, ||x_d||) \end{aligned}$$

for all $(x_1, \cdots x_d) \in \prod_{s=1}^d {}_A\mathscr{B}_s$. So one can obtain the inequality (2.v). $\square$

COROLLARY 3. *Let* $\theta \geqslant 0$ *and* $d < p$. *Assume that a mapping* $f : \prod_{s=1}^{d} {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ *satisfies*

$$\|D_{u_1,\cdots,u_d} f(x_1, y_1, \cdots, x_d, y_d)\| \leqslant \theta \sum_{s=1}^{d} (||x_s||^p + ||y_s||^p)$$

for all $u_1, \cdots, u_d \in \mathscr{U}(A)$, *and all* $(x_1, \cdots, x_d), (y_1, \cdots, y_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$, *and that* $f(x_1, \cdots, x_d) = 0$ *if* $x_s = 0$ *for some* $s = 1, \cdots, d$. *Then there exists a unique* A*-multilinear mapping* $M : \prod_{s=1}^{d} {}_A\mathscr{B}_s \to {}_A\mathscr{D}$ *such that*

$$\begin{aligned}\|f(x_1, \cdots, x_d) - M(x_1, \cdots, x_d)\| \leqslant &\frac{2^p \theta}{2^p - 2^d} \sum_{s=1}^{d} 2^{s-1} (2||\frac{x_1}{2}||^p + \cdots + 2||\frac{x_{s-1}}{2}||^p \\ &+ ||x_s||^p + 2||x_{s+1}||^p + \cdots + 2||x_d||^p)\end{aligned}$$

for all $(x_1, \cdots, x_d) \in \prod_{s=1}^{d} {}_A\mathscr{B}_s$.

Proof. It follows either from Corollary 1 for $\eta(t) = t^p$, or from Corollary 2 for $\mu(\alpha_1, \beta_1, \cdots, \alpha_d, \beta_d) = {\alpha_1}^p + {\beta_1}^p + \cdots + {\alpha_d}^p + {\beta_d}^p$. □

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