Mathematical
Inequalities
& Applications
Volume 8, Number 3 (2005), 537–546

A NEW SYSTEM OF SET–VALUED VARIATIONAL INCLUSIONS WITH H–MONOTONE OPERATORS

WEN-YONG YAN, YA-PING FANG AND NAN-JING HUANG

(*communicated by R.U.Verma*)

Abstract. The purpose of this paper is to introduce and study a new system of set-valued variational inclusions with H-monotone operators in Hilbert spaces. By using the resolvent operator method associated with H-monotone operator due to Fang and Huang, we construct a new iterative algorithm for solving this kind of system of set-valued variational inclusions. We also prove the existence of solutions for the system of set-valued variational inclusions and the convergence of iterative sequences generated by the algorithm.

1. Introduction

In recent years, a significant number of publications have appeared that define generalizations of the variational inequality problems and complementarity problems (see, for example, [1-20] and the references therein). One of the most important generalization of these new problem classes is the variational inclusion, which has wide applications in mechanics and physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, etc.

In [17, 18], Verma introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions of system of variational inequalities in Hilbert spaces. Recently, Kim and Kim [13] introduced a new system of generalized nonlinear mixed variational inequalities and obtained some existence and uniqueness results of solution for the system of generalized nonlinear mixed variational inequalities in Hilbert spaces. Very recently, Cho et al. [3] introduced and studied a new system of nonlinear variational inequalities in Hilbert spaces. They proved some existence and uniqueness theorems of solutions for the system of nonlinear variational inequalities and constructed an iterative algorithm for approximating the solution of the system of nonlinear variational inequalities. Some related works, we refer to [5, 10-12, 19].

On the other hand, monotonicity was extended and applied in recent years because of its importance in the theory of variational inequalities, complementarity problems, and variational inclusions. In a recent paper [4], Fang and Huang introduced a new class

Mathematics subject classification (2000): 49J40, 47H10.

Key words and phrases: H-monotone operator, resolvent operator technique, system of set-valued variational inclusion, iterative algorithm.

This work was supported by the National Science Foundation of China.

Paper MIA-08-49

of monotone operators—H-monotone operators and defined the resolvent operator associated with an H-monotone operator. The authors of [4] also established the Lipschitz continuity of the resolvent operator and studied a new class of variational inclusions in Hilbert spaces by using the resolvent operator method. Very recently, Fang and Huang [6] introduced a new system of nonlinear variational inclusions involving H-monotone operators in Hilbert spaces and proved some existence and uniqueness theorems of solutions for the system of nonlinear variational inclusions.

Motivated and inspired by above works, in this paper, we introduce and study a new system of set-valued variational inclusions with H-monotone operators in Hilbert spaces. By using the resolvent operator method associated with H-monotone operator due to Fang and Huang [4], we construct a new iterative algorithm for solving this kind of system of set-valued variational inclusions. We also prove the existence of solutions for the system of set-valued variational inclusions and the convergence of iterative sequences generated by the algorithm. The present results improve and extend many known results in the literature.

2. Preliminaries

Let $\mathscr{H}$ be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, respectively, $2^{\mathscr{H}}$ denotes the family of all the nonempty subsets of $\mathscr{H}$. In the sequel, let us recall some concepts.

DEFINITION 2.1 Let $T, H : \mathscr{H} \to \mathscr{H}$ be two single-valued operators. T is said to be:

(i) monotone, if

$$\langle Tx - Ty, x - y\rangle \geqslant 0, \quad \forall x, y \in \mathscr{H};$$

(ii) strictly monotone, if T is monotone and

$$\langle Tx - Ty, x - y\rangle = 0$$

if and only $x = y$;

(iii) strongly monotone, if there exists some constant $r > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geqslant r\|x - y\|^2, \quad \forall x, y \in \mathscr{H};$$

(iv) strongly monotone. with respect to H if, there exists some constant $\gamma > 0$ such that

$$\langle Tx - Ty, Hx - Hy\rangle \geqslant \gamma\|x - y\|^2, \quad \forall x, y \in \mathscr{H};$$

(v) Lipschitz continuous, if there exists some constant $s > 0$ such that

$$\|Tx - Ty\| \leqslant s\|x - y\|, \quad \forall x, y \in \mathscr{H}.$$

DEFINITION 2.2 A multi-valued operator $M : \mathscr{H} \to 2^{\mathscr{H}}$ is said to be:

(i) monotone, if

$$\langle x - y, u - v\rangle \geqslant 0, \quad \forall u, v \in \mathscr{H}, x \in Mu, y \in Mv;$$

(*ii*) maximal monotone, if M is monotone and $(I+\lambda M)(\mathscr{H})=\mathscr{H}$ for all $\lambda>0$, where I denotes the identity mapping on $\mathscr{H}$.

DEFINITION 2.3 ([4]) Let $H:\mathscr{H}\to\mathscr{H}$ be a single-valued operator and $M:\mathscr{H}\to 2^{\mathscr{H}}$ be a multi-valued operator. M is said to be H-monotone if M is monotone and $(H+\lambda M)(\mathscr{H})=\mathscr{H}$ holds for very $\lambda>0$.

REMARK 2.1 If $H=I$, then the definition of I-monotone operators is that of maximal monotone operators. In fact, the class of H-monotone operators has close relation with that of maximal monotone operators.

EXAMPLE 2.1 ([4]) Let $H:\mathscr{H}\to\mathscr{H}$ be a strictly monotone single-valued operator and $M:\mathscr{H}\to 2^{\mathscr{H}}$ an H-monotone operator. Then M is maximal monotone.

DEFINITION 2.4 Let $H:\mathscr{H}\to\mathscr{H}$ be a single-valued operator. H is said to be coercive if

$$\lim_{\|x\|\to\infty}\frac{\langle Hx,x\rangle}{\|x\|}=+\infty.$$

DEFINITION 2.5 Let $A:\mathscr{H}\to\mathscr{H}$ be a single-valued operator. A is said to be bounded if $A(B)$ is bounded for every bounded subset B of $\mathscr{H}$. A is said to be hemi-continuous if for any fixed $x,y,z\in\mathscr{H}$, the function $t\mapsto\langle A(x+ty),z\rangle$ is continuous at 0^+.

EXAMPLE 2.2 ([4]) Let $M:\mathscr{H}\to 2^{\mathscr{H}}$ be a maximal monotone operator and $H:\mathscr{H}\to\mathscr{H}$ be a bounded, coercive, hemi-continuous and monotone operator. Then M is H-monotone.

The following example shows that a maximal monotone operator need not be H-monotone for some H.

EXAMPLE 2.3 ([4]) Let $\mathscr{H}=R, M=I$, and $H(x)=x^2$ for all $x\in\mathscr{H}$. Then it is easy to see that I is maximal monotone and the range of $H+I$ is $[-1/4,+\infty)$. Therefore, I is not H-monotone.

LEMMA 2.1. ([4]) *Let $H:\mathscr{H}\to\mathscr{H}$ be a strictly monotone operator and $M:\mathscr{H}\to 2^{\mathscr{H}}$ be an H-monotone operator. Then the operator $(H+\lambda M)^{-1}$ is single-valued.*

By Lemma 2.1, we can define the resolvent operator $J^H_{M,\lambda}$ as follows:

DEFINITION 2.6 ([4]) Let $H:\mathscr{H}\to\mathscr{H}$ be a strictly monotone operator and $M:\mathscr{H}\to 2^{\mathscr{H}}$ be an H-monotone operator. The resolvent operator $J^H_{M,\lambda}:\mathscr{H}\to\mathscr{H}$ is defined by:

$$J^H_{M,\lambda}(u)=(H+\lambda M)^{-1}(u),\quad \forall u\in H.$$

REMARK 2.2 When $H=I$, Definition 2.6 reduces to the definition of the resolvent operator of a maximal monotone operator ([20]).

LEMMA 2.2. ([4]) *Let $H:\mathscr{H}\to\mathscr{H}$ be a strongly monotone operator with constant r and $M:\mathscr{H}\to 2^{\mathscr{H}}$ be an H-monotone operator. Then the resolvent* *operator* $J^H_{M,\lambda} : \mathscr{H} \to \mathscr{H}$ *is Lipschitz continuous with constant* $1/r$*, i.e.,*

$$\|J^H_{M,\lambda}(u) - J^H_{M,\lambda}(v)\| \leqslant \frac{1}{r}\|u - v\|, \quad \forall u, v \in \mathscr{H}.$$

We define a Hausdorff pseudo-metric $\hat{H} : 2^{\mathscr{H}} \times 2^{\mathscr{H}} \to (-\infty, +\infty) \cup \{+\infty\}$ by

$$\hat{H}(A, B) = \max\{\sup_{u\in A} \inf_{v\in B} \|u - v\|, \sup_{u\in B} \inf_{v\in A} \|u - v\|\}$$

for any given $A, B \in 2^{\mathscr{H}}$. Note that if the domain of $\hat{H}$ is restricted to closed bounded subsets, then $\hat{H}$ is the Hausdorff metric.

DEFINITION 2.7 A set-valued mapping $A : \mathscr{H} \to 2^{\mathscr{H}}$ is said to be $\hat{H}$-Lipschitz continuous if there exists a constant $\eta > 0$ such that

$$\hat{H}(A(u), A(v)) \leqslant \eta\|u - v\|$$

for all $u, v \in \mathscr{H}$.

3. System of Variational Inclusions and Iterative Algorithm

In this section, we shall introduce a new system of set-valued variational inclusions with H-monotone operators and construct a new iterative algorithm for solving this kind of system of set-valued variational inclusions in Hilbert spaces. In what follows, unless other specified, we always suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are two real Hilbert spaces, $K_1 \subset \mathscr{H}_1$ and $K_2 \subset \mathscr{H}_2$ are two nonempty, closed and convex sets. Let $F : \mathscr{H}_1 \times \mathscr{H}_2 \to \mathscr{H}_1$, $G : \mathscr{H}_1 \times \mathscr{H}_2 \to \mathscr{H}_2$, $H_1 : \mathscr{H}_1 \to \mathscr{H}_1, H_2 : \mathscr{H}_2 \to \mathscr{H}_2$ be four nonlinear operators, $M : \mathscr{H}_1 \to 2^{\mathscr{H}_1}$ be an H_1-monotone operator and $N : \mathscr{H}_2 \to 2^{\mathscr{H}_2}$ be an H_2-monotone operator. Let $A : \mathscr{H}_1 \to 2^{\mathscr{H}_1}$ and $B : \mathscr{H}_2 \to 2^{\mathscr{H}_2}$. The system of set-valued variational inclusions is formulated by finding $(a, b) \in \mathscr{H}_1 \times \mathscr{H}_2$, $u \in A(a)$, and $v \in B(b)$ such that

$$\begin{cases} 0 \in F(a, v) + M(a); \\ 0 \in G(u, b) + N(b). \end{cases} \tag{3.1}$$

Some examples of problem (3.1) are as follows.

(I) If $M(x) = \partial\varphi(x)$ and $N(y) = \partial\phi(y)$ for all $x \in \mathscr{H}_1$ and $y \in \mathscr{H}_2$, where $\varphi : \mathscr{H}_1 \to R \cup \{+\infty\}$ and $\phi : \mathscr{H}_2 \to R \cup \{+\infty\}$ are two proper, convex and lower semi-continuous functionals, and $\partial\varphi$ and $\partial\phi$ denote the subdifferential operators of φ and ϕ, respectively, then problem (3.2) reduces to the following problem: find $(a, b) \in \mathscr{H}_1 \times \mathscr{H}_2$, $u \in A(a)$, and $v \in B(b)$ such that

$$\begin{cases} \langle F(a, v), x - a\rangle + \varphi(x) - \varphi(a) \geqslant 0, & \forall x \in \mathscr{H}_1, \\ \langle G(u, b), y - b\rangle + \phi(y) - \phi(b) \geqslant 0, & \forall y \in \mathscr{H}_2, \end{cases} \tag{3.2}$$

which is called a system of set-valued mixed variational inequalities. Some special cases of problem (3.2) can be found in [19].

(II) If $M(x) = \partial\varphi(x)$ and $N(y) = \partial\phi(y)$ for all $x \in \mathscr{H}_1$ and $y \in \mathscr{H}_2$, where $\varphi : \mathscr{H}_1 \to R \cup \{+\infty\}$ and $\phi : \mathscr{H}_2 \to R \cup \{+\infty\}$ are two proper, convex and lower semi-continuous functionals, and $\partial\varphi$ and $\partial\phi$ denote the subdifferential operators of φ and ϕ, respectively, then problem (3.2) reduces to the following problem: find $(a,b) \in \mathscr{H}_1 \times \mathscr{H}_2$ such that

$$\begin{cases} \langle F(a,b), x-a\rangle + \varphi(x) - \varphi(a) \geqslant 0, & \forall x \in \mathscr{H}_1, \\ \langle G(a,b), y-b\rangle + \phi(y) - \phi(b) \geqslant 0, & \forall y \in \mathscr{H}_2, \end{cases} \tag{3.3}$$

which is called system of nonlinear variational inequalities considered by Cho, Fang, Huang and Hwang [3]. Some special cases of problem (3.3) were studied by Kim and Kim [13], and Verma [17].

(III) If $M(x) = \partial\delta_{K_1}(x)$ and $N(y) = \partial\delta_{K_2}(y)$ for all $x \in K_1$ and $y \in K_2$, where $A \subset \mathscr{H}_1$ and $B \subset \mathscr{H}_2$ are two nonempty, closed and convex subsets, and δ_{K_1} and δ_{K_2} denote the indicator functions of K_1 and K_2, respectively, then problem (3.2) reduces to the following system of variational inequalities: find $(a,b) \in K_1 \times K_2$ such that

$$\begin{cases} \langle F(a,b), x-a\rangle \geqslant 0, & \forall x \in K_1, \\ \langle G(a,b), y-b\rangle \geqslant 0, & \forall y \in K_2, \end{cases} \tag{3.4}$$

which is just the problem in [11] with both F and G are single-valued.

(IV) If $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$, $K_1 = K_2 = K$, $F(x,y) = \rho T(y) + x - y$ and $G(x,y) = \gamma T(x) + y - x$ for all $x, y \in \mathscr{H}$, where $T : K \to \mathscr{H}$ ia a nonlinear mapping, $\rho > 0$ and $\gamma > 0$ are two constants, then problem (3.4) reduces to the following system of variational inequalities: find $(a,b) \in K \times K$ such that

$$\begin{cases} \langle \rho T(b) + a - b, x-a\rangle \geqslant 0, & \forall x \in K, \\ \langle \gamma T(a) + b - a, x-b\rangle \geqslant 0, & \forall x \in K, \end{cases} \tag{3.5}$$

which is the system of nonlinear variational inequalities considered by Verma [18].

(V) If A and B are both identity mappings, then problem (3.1) reduces to the following problem: find $(a,b) \in \mathscr{H}_1 \times \mathscr{H}_2$ such that

$$\begin{cases} 0 \in F(a,b) + M(a); \\ 0 \in G(a,b) + N(b), \end{cases}$$

which is called the system of variational inclusions considered by Fang and Huang [6].

In order to construct our algorithm, we give a characterization of solution of problem (3.1) as follows:

LEMMA 3.1. *Let $H_1 : \mathscr{H}_1 \to \mathscr{H}_1$ and $H_2 : \mathscr{H}_2 \to \mathscr{H}_2$ be two strictly monotone operators, $M : \mathscr{H}_1 \to \mathscr{H}_1$ be H_1-monotone, and $N : \mathscr{H}_2 \to \mathscr{H}_2$ be H_2-monotone. Then (a,b,u,v) is a solution of problem (3.1) if and only if (a,b,u,v) satisfies the relation*

$$\begin{cases} a = J^{H_1}_{M,\rho}[H_1(a) - \rho F(a,v)], \\ b = J^{H_2}_{N,\lambda}[H_2(b) - \lambda G(u,b)], \end{cases} \tag{3.6}$$

where $\rho > 0$ and $\lambda > 0$ are two constants.

Proof. The fact directly follows from Definition 2.6.

The relation (3.6) allows us to suggest the following iterative algorithm.

Algorithm 3.1

Step 1. Choose $(a_0, b_0) \in \mathscr{H}_1 \times \mathscr{H}_2$ and choose $u_0 \in A(a_0)$ and $v_0 \in B(b_0)$.

Step 2. Let

$$\begin{cases} a_{n+1} = J^{H_1}_{M,\rho}[H_1(a_n) - \rho F(a_n, v_n)], \\ b_{n+1} = J^{H_2}_{N,\lambda}[H_2(b_n) - \lambda G(u_n, b_n)]. \end{cases} \tag{3.7}$$

Step 3. Choose $u_{n+1} \in A(a_{n+1})$ and $v_{n+1} \in B(b_{n+1})$ such that

$$\begin{cases} \|u_{n+1} - u_n\| \leqslant (1 + (1+n)^{-1})\hat{H}_1(A(a_{n+1}), A(a_n)), \\ \|v_{n+1} - v_n\| \leqslant (1 + (1+n)^{-1})\hat{H}_2(B(b_{n+1}), B(b_n)), \end{cases} \tag{3.8}$$

where $\hat{H}_i(\cdot,\cdot)$ is the Hausdorff pseudo-metric on $2^{\mathscr{H}_i}$ for $i = 1, 2$.

Step 4. If a_{n+1}, b_{n+1}, u_{n+1}, and v_{n+1} satisfy (3.7) to sufficient accuracy, stop; otherwise, set $n := n+1$ and return to Step 2.

4. Existence and Convergence

In this section, we will prove the existence of solutions for problem (3.1) and the convergence of iterative sequences generated by Algorithm 3.1. For the following theorem, define $C(\mathscr{H})$ to be the collection of all closed subsets of $\mathscr{H}$.

THEOREM 4.1. *Let $H_1 : \mathscr{H}_1 \to \mathscr{H}_1$ be a strongly monotone and Lipschitz continuous operator with constants γ_1 and τ_1, respectively, $H_2 : \mathscr{H}_2 \to \mathscr{H}_2$ be a strongly monotone and Lipschitz continuous operator with constants γ_2 and τ_2, respectively, $M : \mathscr{H}_1 \to 2^{\mathscr{H}_1}$ be H_1-monotone, and $N : \mathscr{H}_2 \to 2^{\mathscr{H}_2}$ be H_2-monotone. Suppose that $A : \mathscr{H}_1 \to C(\mathscr{H}_1)$ is $\hat{H}_1$-Lipschitz continuous and $B : \mathscr{H}_2 \to C(\mathscr{H}_2)$ is $\hat{H}_2$-Lipschitz continuous with constants η_1 and η_2, respectively. Let $F : \mathscr{H}_1 \times \mathscr{H}_2 \to \mathscr{H}_1$ be a nonlinear operator such that for any given $(a,b) \in \mathscr{H}_1 \times \mathscr{H}_2$, $F(\cdot, b)$ is strongly monotone with respect to H_1 and Lipschitz continuous with constants r_1 and s_1, respectively, and $F(a,\cdot)$ is Lipschitz continuous with constant θ. Let $G : \mathscr{H}_1 \times \mathscr{H}_2 \to \mathscr{H}_2$ be a nonlinear operator such that for any given $(x,y) \in \mathscr{H}_1 \times \mathscr{H}_2$, $G(x,\cdot)$ is strongly monotone with respect to H_2 and Lipschitz continuous with constant r_2 and s_2, and $G(\cdot, y)$ is Lipschitz continuous with constant ξ. If there exist constants $\rho > 0$ and $\lambda > 0$ such that*

$$\begin{cases} \gamma_2\sqrt{\tau_1^2 - 2\rho r_1 + \rho^2 s_1^2} + \lambda\xi\eta_1\gamma_1 < \gamma_1\gamma_2, \\ \gamma_1\sqrt{\tau_2^2 - 2\lambda r_2 + \lambda^2 s_2^2} + \rho\theta\eta_2\gamma_2 < \gamma_1\gamma_2, \end{cases} \tag{4.1}$$

then problem (3.1) admits a solution (a,b,u,v) and sequences $\{a_n\}$, $\{b_n\}$, $\{u_n\}$, and $\{v_n\}$ converge to a, b, u, and v, respectively, where $\{a_n\}$, $\{b_n\}$, $\{u_n\}$, and $\{v_n\}$ is the sequences generated by Algorithm 3.1.

Proof. It follows from (3.7) and Lemma 2.2 that

$$\begin{aligned}\|a_{n+1}-a_n\| &= \|J^{H_1}_{M,\rho}[H_1(a_n)-\rho F(a_n,v_n)]-J^{H_1}_{M,\rho}[H_1(a_{n-1})-\rho F(a_{n-1},v_{n-1})]\|\\ &\leqslant \frac{1}{\gamma_1}\|H_1(a_n)-H_1(a_{n-1})-\rho(F(a_n,v_n)-F(a_{n-1},v_{n-1}))\|\\ &\leqslant \frac{1}{\gamma_1}\|H_1(a_n)-H_1(a_{n-1})-\rho(F(a_n,v_n)-F(a_{n-1},v_n))\|\\ &\quad+\frac{\rho}{\gamma_1}\|F(a_{n-1},v_{n-1})-F(a_{n-1},v_n)\|\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\|b_{n+1}-b_n\| &= \|J^{H_2}_{N,\lambda}[H_2(b_n)-\lambda G(u_n,b_n)]-J^{H_2}_{N,\lambda}[H_2(b_{n-1})-\lambda G(u_{n-1},b_{n-1})]\|\\ &\leqslant \frac{1}{\gamma_2}\|H_2(b_n)-H_2(b_{n-1})-\lambda(G(u_n,b_n)-G(u_{n-1},b_{n-1}))\|\\ &\leqslant \frac{1}{\gamma_2}\|H_2(b_n)-H_2(b_{n-1})-\lambda(G(u_n,b_n)-G(u_n,b_{n-1}))\|\\ &\quad+\frac{\lambda}{\gamma_2}\|G(u_n,b_{n-1})-G(u_{n-1},b_{n-1})\|.\end{aligned} \tag{4.3}$$

By assumptions, we have

$$\begin{aligned}&\|H_1(a_n)-H_1(a_{n-1})-\rho(F(a_n,v_n)-F(a_{n-1},v_n))\|^2\\ &\quad=\|H_1(a_n)-H_1(a_{n-1})\|^2-2\rho\langle F(a_n,v_n)-F(a_{n-1},v_n),H_1(a_n)-H_1(a_{n-1})\rangle\\ &\qquad\quad+\rho^2\|F(a_n,v_n)-F(a_{n-1},v_n)\|^2\\ &\leqslant(\tau_1^2-2\rho r_1+\rho^2 s_1^2)\|a_n-a_{n-1}\|^2\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}&\|H_2(b_n)-H_2(b_{n-1})-\lambda(G(u_n,b_n)-G(u_n,b_{n-1}))\|^2\\ &\quad=\|H_2(b_n)-H_2(b_{n-1})\|^2-2\lambda\langle G(u_n,b_n)-G(u_n,b_{n-1}),H_2(b_n)-H_2(b_{n-1})\rangle\\ &\qquad\quad+\lambda^2\|G(u_n,b_n)-G(u_n,b_{n-1})\|^2\\ &\leqslant(\tau_2^2-2\lambda r_2+\lambda^2 s_2^2)\|b_n-b_{n-1}\|^2.\end{aligned} \tag{4.5}$$

Furthermore, it follows from (3.8) and the Lipschitz continuity of F, G, A, and B that

$$\|F(a_{n-1},v_{n-1})-F(a_{n-1},v_n)\|\leqslant\theta\|v_n-v_{n-1}\|\leqslant\theta(1+n^{-1})\eta_2\|b_n-b_{n-1}\|, \tag{4.6}$$

and

$$\|G(u_n,b_{n-1})-G(u_{n-1},b_{n-1})\|\leqslant\xi\|u_n-u_{n-1}\|\leqslant\xi(1+n^{-1})\eta_1\|a_n-a_{n-1}\|. \tag{4.7}$$

It follows from $(4.2)-(4.7)$ that

$$\begin{cases} \|a_{n+1}-a_n\| \leqslant \gamma_1^{-1}\sqrt{\tau_1^2-2\rho r_1+\rho^2 s_1^2}\|a_n-a_{n-1}\|+\rho\theta\eta_2\gamma_1^{-1}(1+n^{-1})\|b_n-b_{n-1}\|, \\ \|b_{n+1}-b_n\| \leqslant \gamma_2^{-1}\sqrt{\tau_2^2-2\lambda r_2+\lambda^2 s_2^2}\|b_n-b_{n-1}\|+\lambda\xi\eta_1\gamma_2^{-1}(1+n^{-1})\|a_n-a_{n-1}\|. \end{cases} \tag{4.8}$$

Now (4.8) implies that

$$\begin{aligned} &\|a_{n+1}-a_n\|+\|b_{n+1}-b_n\| \\ &\quad \leqslant \left(\gamma_1^{-1}\sqrt{\tau_1^2-2\rho r_1+\rho^2 s_1^2}+\lambda\xi\eta_1\gamma_2^{-1}(1+n^{-1})\right)\|a_n-a_{n-1}\| \\ &\qquad + \left(\gamma_2^{-1}\sqrt{\tau_2^2-2\lambda r_2+\lambda^2 s_2^2}+\rho\theta\eta_2\gamma_1^{-1}(1+n^{-1})\right)\|b_n-b_{n-1}\| \\ &\quad \leqslant k_n(\|a_n-a_{n-1}\|+\|b_n-b_{n-1}\|) \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} k_n = \max\{&\gamma_1^{-1}\sqrt{\tau_1^2-2\rho r_1+\rho^2 s_1^2}+\lambda\xi\eta_1\gamma_2^{-1}(1+n^{-1}), \\ &\gamma_2^{-1}\sqrt{\tau_2^2-2\lambda r_2+\lambda^2 s_2^2}+\rho\theta\eta_2\gamma_1^{-1}(1+n^{-1})\}. \end{aligned}$$

Let

$$k=\max\{\gamma_1^{-1}\sqrt{\tau_1^2-2\rho r_1+\rho^2 s_1^2}+\lambda\xi\eta_1\gamma_2^{-1}, \gamma_2^{-1}\sqrt{\tau_2^2-2\lambda r_2+\lambda^2 s_2^2}+\rho\theta\eta_2\gamma_1^{-1}\}.$$

Then $k_n \to k$ as $n \to \infty$. By (4.1), we know that $0<k<1$ and so (4.9) implies that $\{a_n\}$ and $\{b_n\}$ are both Cauchy sequences. Thus, there exist $a \in \mathscr{H}_1$ and $b \in \mathscr{H}_2$ such that $a_n \to a$ and $b_n \to b$ as $n \to \infty$.

Now we prove that $u_n \to u \in A(a)$ and $v_n \to v \in B(b)$. In fact, from (4.6) and (4.7), we know that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences. Therefore, there exist $u \in \mathscr{H}_1$ and $v \in \mathscr{H}_2$ such that $u_n \to u$ and $v_n \to v$ as $n \to \infty$. Further,

$$\begin{aligned} d(u,A(a)) &= \inf\{\|u-t\| : t \in A(a)\} \\ &\leqslant \|u-u_n\|+d(u_n,A(a)) \\ &\leqslant \|u-u_n\|+\hat{H}_1(A(a_n),A(a)) \\ &\leqslant \|u-u_n\|+\eta_1\|a_n-a\| \to 0. \end{aligned}$$

Hence, since $A(a)$ is closed, we have $u \in A(a)$. Similarly, $v \in B(b)$.

By continuity, a,b,u,v satisfy the following relation

$$\begin{cases} a = J_{M,\rho}^{H_1}[H_1(a)-\rho F(a,v)], \\ b = J_{N,\lambda}^{H_2}[H_2(b)-\lambda G(u,b)]. \end{cases}$$

By Lemma 4.1, (a,b,u,v) is a solution of problem (3.1). This completes the proof.

REMARK 4.1 From Theorem 4.1, we can get some existence results of solutions for problems $(3.2)-(3.5)$.

REMARK 4.2 From Definition 2.1, we know that

$$r_1 \leqslant s_1\tau_1, \quad r_2 \leqslant s_2\tau_2.$$

REMARK 4.3 If $\rho = \lambda > 0$ such that

$$\left|\rho - \frac{\gamma_2[r_1\gamma_2 - \gamma_1^2\xi\eta_1]}{s_1^2\gamma_2^2 - \xi^2\eta_1^2\gamma_1^2}\right| < \frac{\sqrt{\gamma_2^2[r_1\gamma_2 - \gamma_1^2\xi\eta_1]^2 - (s_1^2\gamma_2^2 - \xi^2\eta_1^2\gamma_1^2)\gamma_2^2(\tau_1^2 - \gamma_1^2)}}{s_1^2\gamma_2^2 - \xi^2\eta_1^2\gamma_1^2},$$

$$\sqrt{(s_1^2\gamma_2^2 - \xi^2\eta_1^2\gamma_1^2)(\tau_1^2 - \gamma_1^2)} < r_1\gamma_2 - \gamma_1\xi\eta_1, \quad \rho\xi\eta_1 < \gamma_2, \quad \xi\eta_1\gamma_1 < s_1\gamma_2$$

and

$$\left|\rho - \frac{\gamma_1[r_2\gamma_1 - \gamma_2^2\theta\eta_2]}{s_2^2\gamma_1^2 - \theta^2\eta_2^2\gamma_2^2}\right| < \frac{\sqrt{\gamma_1^2[r_2\gamma_1 - \gamma_2^2\theta\eta_2]^2 - (s_2^2\gamma_1^2 - \theta^2\eta_2^2\gamma_2^2)\gamma_1^2(\tau_2^2 - \gamma_2^2)}}{s_2^2\gamma_1^2 - \theta^2\eta_2^2\gamma_2^2},$$

$$\sqrt{(s_2^2\gamma_1^2 - \theta^2\eta_2^2\gamma_2^2)(\tau_2^2 - \gamma_2^2)} < r_2\gamma_1 - \gamma_2\theta\eta_2, \quad \rho\theta\eta_2 < \gamma_1, \ \theta\eta_2\gamma_2 < s_2\gamma_1,$$

then condition (4.1) holds.

Acknowledgment. The authors would like to express their deep gratitude to Professor R. U. Verma for his helpful comments and suggestions.

REFERENCES

[1] S. ADLY, *Perturbed algorithm and sensitivity analysis for a general class of variational inclusions*, J. Math. Anal. Appl. **201** (1996), 609–630.
[2] R. AHMAD AND Q. H. ANSARI, *An iterative algorithm for generalized nonlinear variational inclusions*, Appl. Math. Lett. **13** 5 (2000), 23–26.
[3] Y. J. CHO, Y. P. FANG, N. J. HUANG AND H. J. HWANG, *Algorithms for systems of nonlinear variational inequalities*, J. Korean Math. Soc. **41** (2004), 489–499.
[4] Y. P. FANG AND N. J. HUANG, *H-Monotone operator and resolvent operator technique for variatonal inclusions*, Appl. Math. Comput. **145** (2003), 795–803.
[5] Y. P. FANG AND N. J. HUANG, *Existence results for systems of strongly implicit vector variational inequalities*, Acta Math. Hungar. **103**(2004), 265–277.
[6] Y. P. FANG AND N. J. HUANG, *H-monotone operators and system of variational inclusions*, Commun. Appl. Nonlinear Anal. **11** 1 (2004), 93–101.
[7] N. J. HUANG, *Generalized nonlinear variatonal inclusions with noncompact valued mappings*, Appl. Math. Lett. **9(3)** (1996), 25–29.
[8] N. J. HUANG, *Mann and Ishikawa type perturbed iterative algorithms for generalized nonlinear implicit quasi-variational inclusions*, Comput. Math. Appl. **35** 10 (1998), 1–7.
[9] N. J. HUANG, *A new completely general class of variational inclusions with noncompact valued mappings*, Comput. Math. Appl. **35** 10 (1998), 9–14.
[10] N. J. HUANG AND Y. P. FANG, *Fixed point theorems and a new system of multivalued generalized order complementarity problems* , Positivity **7** (2003), 257–265.
[11] G. KASSAY AND J. KOLUMBÀN, *System of multi-valued variational inequalities*, Publ. Math. Debrecen **56** (2000), 185–195.
[12] G. KASSAY, J. KOLUMBÁN AND Z. PÁLES, *Factorization of Minty and Stampacchia variational inequality system*, European J. Operational Research **143(2)** (2002), 377–389.
[13] J. K. KIM AND D. S. KIM, *A new system of generalized nonlinear mixed variational inequalities in Hilbert spaces*, J. Convex Anal. **11** (2004), 117–124.

[14] C. H. LEE, Q. H. ANSARI AND J. C. YAO, *A perturbed algorithm for strongly nonlinear variational-like inclusions*, Bull. Austral. Math. Soc. **62** (2000), 417–426.
[15] L. W. LIU AND Y. Q. LI, *On generalized set-valued variational inclusions*, J. Math. Anal. Appl. **261** 1 (2001), 231–240.
[16] S. B. NADLER, *Muliti-valued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.
[17] R. U. VERMA, *Iterative algorithms and a new system of nonlinear quasivariational inequalities*, Adv. Nonlinear Var. Inequl. **4** 1 (2001), 117–124.
[18] R. U. VERMA, *Projection methods, algorithms, and a new system of nonlinear variational inequalities*, Comput. Math. Appl. **41** (2001), 1025–1031.
[19] R. U. VERMA, *Generalized system for relaxed coercive variational inequalities and projection methods*, J. Optim. Theory Appl. **121** (2004), 203–210.
[20] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications II: Monotone Operators*, Springer-Verlag, Berlin, 1985.

(Received August 7, 2004)

Nan-jing Huang
Department of Mathematics
Sichuan University
Chengdu
Sichuan 610064
P. R. China
e-mail: nanjinghuang@hotmail.com

Wen-yong Yan
Department of Mathematics
Sichuan University
Chengdu
Sichuan 610064
P. R. China
e-mail: wenyongy@yahoo.com

Ya-ping Fang
Department of Mathematics
Sichuan University
Chengdu
Sichuan 610064
P. R. China
e-mail: fabhcn@yahoo.com.cn