Mathematical
Inequalities
& Applications
Volume 8, Number 4 (2005), 571–574

ON SHAFER–FINK INEQUALITIES

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(*communicated by A. M. Fink*)

Abstract. In this paper, a new upper bound for inverse sine is established.We would point out that the numbers, 3 and π, 6 and $\pi(\sqrt{2}+\frac{1}{2})$, in Shafer-Fink inequalities, are optimal.

1. Introduction

Mitrinović [1,p.247] gives us a result which belongs to R. E. Shafer:

THEOREM 1. *If* $x>0$, *then*

$$\arcsin x > \frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} > \frac{3x}{2+\sqrt{1-x^2}}. \tag{1}$$

Fink [2] shows a upper bound for inverse sine,and obtains the following theorem:

THEOREM 2. *If* $0\leqslant x\leqslant 1$,*then*

$$\frac{3x}{2+\sqrt{1-x^2}} \leqslant \arcsin x \leqslant \frac{\pi x}{2+\sqrt{1-x^2}}. \tag{2}$$

Furthermore, 3 *and* π *are the best constants in* (2).

In this paper, we further improve the upper bound of inverse sine and obtain two results.

THEOREM 3. *If* $0\leqslant x\leqslant 1$,*then*

$$\frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} \leqslant \arcsin x \leqslant \frac{\pi(\sqrt{2}+\frac{1}{2})(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}}. \tag{3}$$

Furthermore, 6 *and* $\pi(\sqrt{2}+\frac{1}{2})$ *are the best constants in* (3).

Mathematics subject classification (2000): 26D15.
Key words and phrases: inverse sine; upper and lower bounds; Shafer-Fink inequalities.

Paper MIA-08-53

In view of the fact $\frac{\pi(\sqrt{2}+\frac{1}{2})(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} \leqslant \frac{\pi x}{2+\sqrt{1-x^2}}$ for $x \in [0,1]$,combining (2) and (3) gives

THEOREM 4. *If* $0 \leqslant x \leqslant 1$*,then*

$$\begin{aligned} \frac{3x}{2+\sqrt{1-x^2}} &\leqslant \frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} \leqslant \arcsin x \\ &\leqslant \frac{\pi(\sqrt{2}+\frac{1}{2})(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} \leqslant \frac{\pi x}{2+\sqrt{1-x^2}}. \end{aligned} \tag{4}$$

Furthermore, 3 *and* π*,* 6 *and* $\pi(\sqrt{2}+\frac{1}{2})$ *are the best constants in* (4).

Now,we build a function with a parameter. According to the monotony of this function with respect to θ, Theorem 2 and Theorem 3 will be proved in the same way.

2. One Lemma

LEMMA 1. *Let* $\theta \in [0,\frac{\pi}{2})$*,then* $f(\theta) = \frac{(\pi-2\theta)(2+\sin\theta)}{\cos\theta}$ *strictly decreases as* θ *increases on* $[0,\frac{\pi}{2})$.

Proof of Lemma 1. Since

$$f'(\theta) = \frac{(\pi-2\theta)(1+2\sin\theta)-2\cos\theta(2+\sin\theta)}{\cos^2\theta} =: \frac{g(\theta)}{\cos^2\theta},$$

we get that the existence of theorem can be ensured when proving the following inequality

$$g(\theta) = (\pi-2\theta)(1+2\sin\theta)-2\cos\theta(2+\sin\theta) < 0, \theta \in [0,\frac{\pi}{2}). \tag{5}$$

In fact,

$$g'(\theta) = 2\cos\theta(\pi-2\theta-2\cos\theta) =: 2\cos\theta p(\theta),$$

where, $p(\theta) = \pi-2\theta-2\cos\theta$. Since $p'(\theta) = -2+2\sin\theta < 0$ for $\theta \in [0,\frac{\pi}{2})$, we obtain that the function $p(\theta)$ strictly decreases as θ increases on $[0,\frac{\pi}{2})$. At the same time, $p(\frac{\pi}{2}) = 0$,then $p(\theta) > 0$ and $g'(\theta) > 0$. Now, $g(\frac{\pi}{2}) = 0$,we have $g(\theta) < 0$ for $\theta \in [0,\frac{\pi}{2})$. That is, (5) holds.

3. A new proof of Theorem 2

In view of the fact that $\frac{3x}{2+\sqrt{1-x^2}} = \arcsin x$ for $x = 0$, the proof of Theorem 2 can be completed when proving the following result.

COROLLARY 1. *If* $0 < x \leqslant 1$*,then*

$$\frac{3x}{2+\sqrt{1-x^2}} < \arcsin x \leqslant \frac{\pi x}{2+\sqrt{1-x^2}}. \tag{6}$$

Furthermore, 3 *and* π *are the best constants in* (6).

Proof of Corollary 1. Now, $0 < x \leqslant 1$. Let $x = \cos\theta$, then $\theta \in [0, \frac{\pi}{2})$ and (6) is equivalent to

$$\frac{3\cos\theta}{2+\sin\theta} < \frac{\pi}{2} - \theta \leqslant \frac{\pi\cos\theta}{2+\sin\theta}$$

or

$$6 < \frac{(\pi-2\theta)(2+\sin\theta)}{\cos\theta} \leqslant 2\pi. \tag{7}$$

In fact, $f(\theta) = \frac{(\pi-2\theta)(2+\sin\theta)}{\cos\theta}$ strictly decreases as θ increases on $[0, \frac{\pi}{2})$ by Lemma 1, then

$$f(\frac{\pi}{2}-0) < f(\theta) \leqslant f(0).$$

Since $f(\frac{\pi}{2}^-) = 6$ and $f(0) = 2\pi$, we have that (7) holds. At the same time, 3 and π are the best constants in (6).

4. The proof of Theorem 4

In view of the fact that $\frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} = \arcsin x$ for $x = 0$, the existence of Theorem 3 is ensured when proving the result as follows.

COROLLARY 2. *If* $0 < x \leqslant 1$*, then*

$$\frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} < \arcsin x \leqslant \frac{\pi(\sqrt{2}+\frac{1}{2})(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}}. \tag{8}$$

Furthermore, 6 *and* $\pi(\sqrt{2}+\frac{1}{2})$ *are the best constants in* (8).

Proof of Corollary 2. Now, $0 < x \leqslant 1$. Let $\sqrt{1+x} = \sqrt{2}\cos\alpha, \sqrt{1-x} = \sqrt{2}\sin\alpha$, then $\alpha \in [0, \frac{\pi}{4}), x = \cos 2\alpha$ and (8) is equivalent to

$$\frac{6\sqrt{2}(\cos\alpha-\sin\alpha)}{4+\sqrt{2}(\cos\alpha+\sin\alpha)} < \frac{\pi}{2} - 2\alpha \leqslant \frac{\pi(\sqrt{2}+\frac{1}{2})\sqrt{2}(\cos\alpha-\sin\alpha)}{4+\sqrt{2}(\cos\alpha+\sin\alpha)}$$

or

$$\frac{6\cdot 2\cos(\alpha+\frac{\pi}{4})}{4+2\sin(\alpha+\frac{\pi}{4})} < \frac{\pi}{2} - 2\alpha \leqslant \frac{\pi(\sqrt{2}+\frac{1}{2})2\cos(\alpha+\frac{\pi}{4})}{4+2\sin(\alpha+\frac{\pi}{4})}. \tag{9}$$

Now, let $\alpha + \frac{\pi}{4} = \theta$, then $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$ and (9) is equivalent to

$$\frac{6\cos\theta}{2+\sin\theta} < \pi - 2\theta \leqslant \frac{\pi(\sqrt{2}+\frac{1}{2})\cos\theta}{2+\sin\theta}, \theta \in [\frac{\pi}{4}, \frac{\pi}{2})$$

or

$$6 < \frac{(\pi-2\theta)(2+\sin\theta)}{\cos\theta} \leqslant \pi(\sqrt{2}+\frac{1}{2}), \theta \in [\frac{\pi}{4}, \frac{\pi}{2}). \tag{10}$$

In fact, $f(\theta) = \frac{(\pi-2\theta)(2+\sin\theta)}{\cos\theta}$ strictly decreases as θ increases on $[\frac{\pi}{4}, \frac{\pi}{2})$ by Lemma 1, then

$$f(\frac{\pi}{2}-0) < f(\theta) \leqslant f(\frac{\pi}{4}).$$

Since $f\left(\frac{\pi}{2}^{-}\right) = 6$ and $f\left(\frac{\pi}{4}\right) = \pi(\sqrt{2}+\frac{1}{2})$,we obtain (10). At the same time, 6 and $\pi(\sqrt{2}+\frac{1}{2})$ are the best constants in (8).

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(Received September 9, 2004)

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