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GENERICITY AND MINIMAX OPTIMIZATION

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Abstract. In this paper we study a class of minimax problems $\max\{f(x), g(x)\} \to \min,\ x \in R^n$ where $f, g \in C^1(R^n)$ and f is convex. We show that the subclass of all problems for which there exists a point of minimum $z \in R^1$ such that $f(z) = g(z)$ and $\nabla f(z) = \nabla g(z)$ is small.

1. Introduction

The study of minimax problems is one of central topics in optimization theory. See, for example, [3-5] and the references mentioned therein. In this paper we consider a class of minimax problems

$$\max\{f(x), g(x)\} \to \min,\ x \in R^n$$

where $f, g \in C^1(R^n)$ and f is convex. We show that the subclass of all problems for which there exists a point of minimum $z \in R^n$ such that

$$f(z) = g(z) \text{ and } \nabla f(z) = \nabla g(z)$$

is small. It means that for a typical problem, if its solution z satisfies $f(z) = g(z)$, then the cost function $\max\{f, g\}$ is not differentiable at z. Here, instead of considering a certain property for a single minimax problem, we investigate it for a class of minimax problems and show that this property holds for most of the problems in the class. This approach has already been successfully applied in many areas of Analysis. See, for example, [1, 2, 7-9].

Let X be a nonempty set. For each $f : X \to R^1$ define

$$\inf(f) = \inf\{f(x) :\ x \in X\}.$$

For each $f, g : X \to R^1$ define a function $\max\{f, g\} : X \to R^1$ by

$$\max\{f, g\}(x) = \max\{f(x), g(x)\},\ x \in X. \tag{1.1}$$

Denote by $\mathbf{Z}$ the set of all integers and by $\mathbf{Z}_+$ the set of all nonnegative integers.

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Let $n \geqslant 1$ be an integer. We consider the n-dimensional Euclidean space R^n with the Euclidean norm $|\cdot|$. Denote by (x,y) the scalar product of vectors $x, y \in R^n$. Let $k \geqslant 1$ be an integer and let $\phi : R^n \to R^1$ be a bounded from below function such that

$$\lim_{|x|\to\infty} \phi(x) = \infty. \tag{1.2}$$

Denote by $\mathscr{M}$ the set of all pairs of functions (f,g) such that $f,g : R^n \to R^1$, $f,g \in C^k(R^n)$, f is convex and

$$\max\{f(x), g(x)\} \geqslant \phi(x) \text{ for all } x \in R^n. \tag{1.3}$$

Denote by $\mathscr{M}_{co}$ the set of all $(f,g) \in \mathscr{M}$ such that g is convex.

For each $r > 0$ set $B(r) = \{z \in R^n : \ |z| \leqslant r\}$. For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}^n_+$ set

$$|\alpha| = \sum_{i=1}^n \alpha_i.$$

We equip the set $\mathscr{M}$ with the uniformity determined by the following base:

$$\begin{aligned} E_s(\epsilon, m) = \{((f_1,g_1),(f_2,g_2)) \in \mathscr{M} \times \mathscr{M} :& \\ |\partial^{|\alpha|} f_1(z)/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n} - \partial^{|\alpha|} f_2(z)/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}| \leqslant \epsilon& \\ \text{for all } z \in B(m) \text{ and all } \alpha \in \mathbf{Z}^n_+ \text{ such that } |\alpha| \leqslant k& \\ \text{and } |\partial^{|\alpha|} g_1(y)/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n} - \partial^{|\alpha|} g_2(y)/\partial x_1^{\alpha_1} \dots, \partial x_n^{\alpha_n}| \leqslant \epsilon& \\ \text{for all } y \in R^n \text{ and all } \alpha \in \mathbf{Z}^n_+ \text{ such that } |\alpha| \leqslant k\},& \end{aligned} \tag{1.4}$$

where $\epsilon, m > 0$.

It is not difficult to see that the space $\mathscr{M}$ with this uniformity is metrizable [6] (by a metric d_s) and complete. The topology induced by this uniformity is denoted by τ_s and it is called the strong topology.

We equip the set $\mathscr{M}$ with the uniformity determined by the following base:

$$\begin{aligned} E_w(\epsilon, m) = \{((f_1,g_1),(f_2,g_2)) \in \mathscr{M} \times \mathscr{M} :& \\ |\partial^{|\alpha|} f_1(z)/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n} - \partial^{|\alpha|} f_2(z)/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}| \leqslant \epsilon& \\ \text{and } |\partial^{|\alpha|} g_1(z)/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n} - \partial^{|\alpha|} g_2(z)/\partial x_1^{\alpha_1} \dots, \partial x_n^{\alpha_n}| \leqslant \epsilon& \\ \text{for all } z \in B(m) \text{ and all } \alpha \in \mathbf{Z}^n_+ \text{ such that } |\alpha| \leqslant k\},& \end{aligned} \tag{1.5}$$

where $\epsilon, m > 0$. It is not difficult to see that the space $\mathscr{M}$ with this uniformity is metrizable [6] (by a metric d_w) and complete.

We do not write down the explicit expressions for the metrics d_w and d_s because we are not going to use them in the sequel.

The topology induced by this uniformity is denoted by τ_w and it is called the weak topology.

Clearly $\mathscr{M}_{co}$ is a closed subset of $\mathscr{M}$ with the weak topology. We equip the subspace $\mathscr{M}_{co} \subset \mathscr{M}$ with the relative weak and strong topologies.

Denote by $\mathscr{M}_0$ the set of all $(f,g) \in \mathscr{M}$ for which there is $x \in R^n$ such that

$$f(x) = g(x) = \inf(\max\{f,g\}) \tag{1.6}$$

and denote by G the set of all $(f,g) \in \mathscr{M}$ such that there is $x \in R^n$ which satisfies (1.6) and the following equality:

$$\nabla f(x) = \nabla g(x). \tag{1.7}$$

Here

$$\nabla f(x) = ((\partial f/\partial x_1)(x), \dots, (\partial f/\partial x_n)(x)) \in R^n.$$

In this paper we will establsih the following results.

PROPOSITION 1.1. *$\mathscr{M}_0$ and G are closed subsets of $(\mathscr{M}, d_w)$.*

We consider the topological subspaces $\mathscr{M}_0$, $\mathscr{M}_{co} \cap \mathscr{M}_0 \subset \mathscr{M}$ with the relative weak and strong topologies.

THEOREM 1.1. *$\mathscr{M} \setminus G$ is an open (in the weak topology) everywhere dense (in the strong topology) subset of $\mathscr{M}$ and $\mathscr{M}_0 \setminus G$ is an open (in the weak topology) everywhere dense (in the strong topology) subset of $\mathscr{M}_0$.*

THEOREM 1.2. *$\mathscr{M}_{co} \setminus G$ is an open everywhere dense in the weak topology subset of $\mathscr{M}_{co}$ and $(\mathscr{M}_{co} \cap \mathscr{M}_0) \setminus G$ is an open everywhere dense in the weak topology subset of $\mathscr{M}_{co} \cap \mathscr{M}_0$.*

2. Proof of Proposition 1.1

Assume that $\{(f_q, g_q)\}_{q=1}^{\infty} \subset \mathscr{M}_0$ (respectively $\{(f_q, g_q)\}_{q=1}^{\infty} \subset G$) and

$$(f_q, g_q) \to (f, g) \text{ as } q \to \infty \text{ in } (\mathscr{M}, d_w). \tag{2.1}$$

We show that $(f,g) \in \mathscr{M}_0$ (respectively $(f,g) \in G$).

In view of (1.6) and (1.7) for each natural number q there is $x_q \in R^n$ such that

$$f_q(x_q) = g_q(x_q) = \inf(\max\{f_q, g_q\}), \tag{2.2}$$

$$\text{if } (f_q, g_q) \in G, \text{ then } \nabla f_q(x_q) = \nabla g_q(x_q). \tag{2.3}$$

By (1.2) and (1.3) there is $\bar{x} \in R^n$ such that

$$\max\{f(\bar{x}), g(\bar{x})\} = \inf(\max\{f, g\}). \tag{2.4}$$

(2.1) and (1.5) imply that

$$\max\{f(\bar{x}), g(\bar{x})\} = \lim_{q\to\infty} \max\{f_q(\bar{x}), g_q(\bar{x})\}. \tag{2.5}$$

Combined with (2.2) equality (2.5) implies that the sequence

$$\{\max\{f_q(x_q), g_q(x_q)\}\}_{q=1}^{\infty} = \{\inf(\max\{f_q, g_q\})\}_{q=1}^{\infty}$$

is bounded from above.

Together with (1.3) this implies that the sequence $\{\phi(x_q)\}_{q=1}^{\infty}$ is bounded from above. In view of (1.2) the sequence $\{x_i\}_{i=1}^{\infty}$ is bounded. Choose a natural number n_0 such that

$$|x_q| \leqslant n_0 \text{ for all natural numbers } q \text{ and } |\bar{x}| \leqslant n_0. \tag{2.6}$$

(2.1) and (1.5) imply that

$$\sup\{|f(x) - f_q(x)|,\ |g(x) - g_q(x)| :\ x \in B(n_0)\} \to 0 \text{ as } q \to \infty, \tag{2.7}$$

$$\sup\{|\nabla f(x) - \nabla f_q(x)|,\ |\nabla g(x) - \nabla g_q(x)| :\ x \in B(n_0)\} \to 0 \text{ as } q \to \infty. \tag{2.8}$$

It follows from (2.6) and (2.7) that

$$\limsup_{q\to\infty}(\inf(\max\{f_q, g_q\})) \leqslant \max\{f(\bar{x}), g(\bar{x})\}. \tag{2.9}$$

By (2.2)

$$\begin{aligned}\liminf_{q\to\infty}(\inf(\max\{f_q, g_q\})) &= \liminf_{q\to\infty}(\max\{f_q(x_q), g_q(x_q)\}) \\ &= \liminf_{q\to\infty}(\max\{f(x_q), g(x_q)\}) \geqslant \inf(\max\{f, g\}).\end{aligned} \tag{2.10}$$

Combined with (2.9) and (2.4) this relation implies that

$$\lim_{q\to\infty} \inf(\max\{f_q, g_q\}) = \inf(\max\{f, g\}). \tag{2.11}$$

By extracting a subsequence and re-indexing, we may assume that there is

$$\tilde{x} = \lim_{q\to\infty} x_q. \tag{2.12}$$

It follows from (2.12), (2.7), (2.6), (2.2) and (2.11) that

$$\begin{aligned}(f(\tilde{x}), g(\tilde{x})) &= \lim_{q\to\infty}(f(x_q), g(x_q)) = \lim_{q\to\infty}(f_q(x_q), g_q(x_q)) \\ &= \lim_{q\to\infty}(\inf(\max\{f_q, g_q\}), \inf(\max\{f_q, g_q\})) \\ &= (\inf(\max\{f, g\}), \inf(\max\{f, g\}))\end{aligned}$$

and

$$f(\tilde{x}) = g(\tilde{x}) = \inf(\max\{f, g\}).$$

Thus $(f, g) \in \mathscr{M}_0$.

Assume that $(f_q, g_q) \in G$, $q = 1, 2, \ldots$ Then by (2.6), (2.8), (2.3) and (2.12)

$$\begin{aligned}\nabla f(\tilde{x}) = \lim_{q\to\infty} \nabla f(x_q) &= \lim_{q\to\infty} \nabla f_q(x_q) = \lim_{q\to\infty} \nabla g_q(x_q) \\ &= \lim_{q\to\infty} \nabla g(x_q) = \lim_{q\to\infty} \nabla g(\tilde{x})\end{aligned}$$

and $(f, g) \in G$. Proposition 1.1 is proved.

3. Auxiliary results

LEMMA 3.1. *Let* $(f, g) \in G$, $x \in R^n$,

$$f(x) = g(x) = \inf(\max\{f, g\}) \tag{3.1}$$

and

$$\nabla f(x) = \nabla g(x). \tag{3.2}$$

Then $\nabla f(x) = 0$.

Proof. Let us assume the converse. Then there are $\Delta > 0$, $h \in R^n$ such that

$$|h| = 1, \ (\nabla f(x), h) < -\Delta. \tag{3.3}$$

There is $\delta > 0$ such that

$$(\nabla f(z), h) < -\Delta/2, \ (\nabla g(z), h) < -\Delta/2 \tag{3.4}$$

for each $z \in R^n$ satisfying

$$|z - x| \leqslant \delta. \tag{3.5}$$

Choose

$$t_0 \in (0, \delta). \tag{3.6}$$

By the mean value theorem there exist

$$t_1, t_2 \in [0, \delta] \tag{3.7}$$

such that

$$f(x + t_0 h) - f(x) = t_0(\nabla f(x + t_1 h), h), \tag{3.8}$$

$$g(x + t_0 h) - g(x) = t_0(\nabla g(x + t_2 h), h). \tag{3.9}$$

In view of (3.7), (3.3) and the choice of δ (see (3.4), (3.5))

$$(\nabla f(x + t_1 h), h), \ (\nabla g(x + t_2 h), h) < -\Delta/2.$$

Combined with (3.8) and (3.9) this inequality implies that

$$f(x + t_0 h) < f(x), \ g(x + t_0 h) < g(x),$$

$$\max\{f(x + t_0 h), g(x + t_0 h)\} < \max\{f(x), g(x)\},$$

a contradiction (see (3.1)). The contradiction we have reached proves Lemma 3.1.

4. Proof of Theorem 1.1

By Proposition 1.1 $\mathcal{M} \setminus G$ is an open subset of $\mathcal{M}$ with the weak topology and $\mathcal{M}_0 \setminus G$ is an open subset of $\mathcal{M}_0$ with the weak topology. In order to prove Theorem 1.1 it is sufficient to show that $\mathcal{M} \setminus G$ is an everywhere dense subset of $\mathcal{M}$ with the strong topology and $\mathcal{M}_0 \setminus G$ is an everywhere dense subset of $\mathcal{M}_0$ with the strong topology.

Assume that $\epsilon, m > 0$,

$$(f, g) \in G, \ \bar{x} \in R^n, \tag{4.1}$$

$$f(\bar{x}) = g(\bar{x}) = \inf(\max\{f, g\}) \tag{4.2}$$

and

$$\nabla f(\bar{x}) = \nabla g(\bar{x}). \tag{4.3}$$

In view of (4.2), (4.3) and Lemma 3.1,

$$\nabla f(\bar{x}) = \nabla g(\bar{x}) = 0. \tag{4.4}$$

Since f is convex it follows from (4.4) that

$$f(x) \geqslant f(\bar{x}) \ \text{ for all } x \in R^n. \tag{4.5}$$

There is a function $\phi : R^1 \to [0, 1]$ such that $\phi \in C^\infty(R^1)$,

$$\phi(t) = 1 \text{ if} |t| \leqslant 1/2, \ \phi(t) = 0 \ \text{ if } |t| \geqslant 1, \ 0 < \phi(t) < 1 \ \text{ if } 1/2 < |t| < 1. \tag{4.6}$$

Choose positive numbers

$$c_0 < \epsilon/16 \tag{4.7}$$

and

$$c_1 < 32^{-1}\epsilon(m + |\bar{x}| + 1)^{-2}. \tag{4.8}$$

For $x \in R^n$ set

$$\psi(x) = \phi(|x - \bar{x}|^2)((1, 1, \dots, 1), x - \bar{x}). \tag{4.9}$$

Clearly, $\psi \in C^\infty(R^n)$. By (4.9) and (4.6)

$$\psi(x) = 0 \ \text{ if } |x - \bar{x}| \geqslant 1, \tag{4.10}$$

$$\psi(x) = ((1, \dots, 1), x - \bar{x}) \ \text{ if } |x - \bar{x}| \leqslant 2^{-1/2}. \tag{4.11}$$

Choose a positive number c_2 such that

$$c_2(\sup\{|\partial^{|\alpha|}\psi(z)/\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}| : \ z \in R^n, \ \alpha \in \mathbf{Z}_+^n \ \text{ and } |\alpha| \leqslant k\} + 1) < c_0/8. \tag{4.12}$$

Define

$$f_1(x) = f(x) + c_0 + c_1|x - \bar{x}|^2, \ x \in R^n, \tag{4.13}$$

$$g_1(x) = g(x) + c_0 + c_2\psi(x), \ x \in R^n. \tag{4.14}$$

Clearly, $f_1, g_1 \in C^k(R^n)$. (4.13), (4.14), (4.4) and (4.11) imply that

$$\nabla f_1(\bar{x}) = \nabla f(\bar{x}) = 0,$$

$$\nabla g_1(\bar{x}) = \nabla g(\bar{x}) + c_2\nabla\psi(\bar{x}) = c_2\nabla\psi(\bar{x}) = c_2(1, \dots, 1)$$

and

$$\nabla f_1(\bar{x}) \neq \nabla g_1(\bar{x}). \tag{4.15}$$

(4.13) implies that

$$f_1(x) \geqslant f(x) \ \text{ for all } x \in R^n. \tag{4.16}$$

By (4.14) and (4.12)

$$g_1(x) \geqslant g(x) \ \text{ for all } x \in R^n. \tag{4.17}$$

In view of (4.16), (4.17) and (1.3)

$$\max\{f_1(x), g_1(x)\} \geqslant \phi(x) \ \text{ for all } x \in R^n. \tag{4.18}$$

Therefore $(f_1, g_1) \in \mathscr{M}$. By (4.14), (4.12) and (4.7)

$$\sup\{|\partial^{|\alpha|} g_1(z)/\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n} - \partial^{|\alpha|} g(z)/\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}| : z \in R^n,\ \alpha \in \mathbf{Z}_+^n \ \text{ and } |\alpha| \leqslant k\}$$
$$\leqslant c_0 + c_0/8 \leqslant \epsilon/4. \tag{4.19}$$

It follows from (4.13), (4.7) and (4.8) that for each $x \in R^n$ satisfying $|x| \leqslant m$ and each $\alpha \in \mathbf{Z}_+^n$ which satisfies $|\alpha| \leqslant k$ the following inequality holds:

$$\begin{aligned} &|\partial^{|\alpha|} f_1(x)/\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n} - \partial^{|\alpha|} f(x)/\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}| \\ &\quad \leqslant \max\{c_0 + c_1(m + |\bar{x}|)^2, 2c_1(m + |\bar{x}|), 2c_1\} \leqslant \epsilon/4. \end{aligned} \tag{4.20}$$

(4.19), (4.20) and (1.4) imply that

$$((f, g), (f_1, g_1)) \in E_s(\epsilon, m). \tag{4.21}$$

(4.13) implies that

$$f_1(\bar{x}) = f(\bar{x}) + c_0. \tag{4.22}$$

In view of (4.14), (4.9), (4.2) and (4.22)

$$g_1(\bar{x}) = g(\bar{x}) + c_0 = f(\bar{x}) + c_0 = f_1(\bar{x}). \tag{4.23}$$

We show that for each $x \in R^n \setminus \{\bar{x}\}$

$$\max\{f_1(\bar{x}), g_1(\bar{x})\} < \max\{f_1(x), g_1(x)\}. \tag{4.24}$$

Let $x \in R^n \setminus \{\bar{x}\}$. By (4.13), (4.5), (4.22) and (4.23)

$$\begin{aligned} \max\{f_1(x), g_1(x)\} &\geqslant f_1(x) = f(x) + c_0 + c_1|x - \bar{x}|^2 \geqslant f(\bar{x}) + c_0 + c_1|x - \bar{x}|^2 \\ &= \max\{f_1(\bar{x}), g_1(\bar{x})\} + c_1|x - \bar{x}|^2 > \max\{f_1(\bar{x}), g_1(\bar{x})\}. \end{aligned}$$

Thus (4.24) holds for all $x \in R^n \setminus \{\bar{x}\}$. By (4.24), (4.23) and (4.15) $(f_1, g_1) \in \mathscr{M}_0 \setminus G$. Since ϵ, m are arbitrary positive numbers we have shown that $\mathscr{M} \setminus G$ is an everywhere dense subset of $\mathscr{M}$ with the strong topology and $\mathscr{M}_0 \setminus G$ is an evrywhere dense subset of $\mathscr{M}_0$ with the strong topology. Theorem 1.1 is proved.

5. Proof of Theorem 1.2

By Proposition 1.1 $\mathscr{M}_{co} \setminus G$ is an open subset of $\mathscr{M}_{co}$ with the weak topology and $(\mathscr{M}_{co} \cap \mathscr{M}_0) \setminus G$ is an open subset of $\mathscr{M}_{co} \cap \mathscr{M}_0$ with the weak topology.

In order to prove Theorem 1.2 it is sufficient to show that $\mathscr{M}_{co} \setminus G$ is an everywhere dense subset of $\mathscr{M}_{co}$ with the weak topology and $(\mathscr{M}_{co} \cap \mathscr{M}_0) \setminus G$ is an everywhere dense subset of $\mathscr{M}_{co} \cap \mathscr{M}_0$ with the weak topology.

Let $\epsilon, m > 0$,

$$(f, g) \in G \cap \mathscr{M}_{co},\ \bar{x} \in R^n, \tag{5.1}$$

$$f(\bar{x}) = g(\bar{x}) = \inf(\max\{f, g\}) \tag{5.2}$$

and

$$\nabla f(\bar{x}) = \nabla g(\bar{x}). \tag{5.3}$$

By Lemma 3.1, (5.2) and (5.3)

$$\nabla f(\bar{x}) = \nabla g(\bar{x}) = 0. \tag{5.4}$$

Since f, g are convex functions the inequality (5.4) implies that

$$f(x) \geqslant f(\bar{x}),\ g(x) \geqslant g(\bar{x}) \ \text{ for all } x \in R^n. \tag{5.5}$$

Choose positive numbers c_0, c_1, c_2 such that

$$c_2 < \epsilon/8, \tag{5.6}$$

$$nc_0(m + |\bar{x}| + 4)^2 < \epsilon/8, \tag{5.7}$$

$$c_1 n(|\bar{x}| + m + 1) < \epsilon/8,\ 2c_1^2 n^2 < c_0 c_2. \tag{5.8}$$

Define

$$f_1(x) = f(x) + c_2 + c_0|x - \bar{x}|^2,\ x \in R^n, \tag{5.9}$$

$$g_1(x) = g(x) + c_2 + c_1((1, \dots, 1), x - \bar{x}) + c_0|x - \bar{x}|^2,\ x \in R^n. \tag{5.10}$$

Clearly, f_1, g_1 are convex functions, $f_1, g_1 \in C^k(R^n)$ and

$$f_1(x) \geqslant f(x) \ \text{ for all } x \in R^n. \tag{5.11}$$

Let $x \in R^n$. We show that

$$g_1(x) \geqslant g(x).$$

By (5.10)

$$g_1(x) - g(x) = c_2 + c_0|x - \bar{x}|^2 + c_1((1, 1, \dots, 1), x - \bar{x}) \geqslant c_2 + c_0|x - \bar{x}|^2 - c_1|x - \bar{x}|n. \tag{5.12}$$

There are two cases:

$$|x - \bar{x}| \leqslant c_2(2nc_1)^{-1}; \tag{5.13}$$

$$|x - \bar{x}| > c_2(2nc_1)^{-1}. \tag{5.14}$$

If (5.13) is valid, then it follows from (5.12) that

$$g_1(x) - g(x) \geqslant c_2 - c_1|\bar{x} - x|n \geqslant c_2/2. \tag{5.15}$$

Now assume that (5.14) is true. It follows from (5.12), (5.14) and (5.8) that

$$g_1(x) - g(x) \geqslant c_0|x-\bar{x}|^2 - c_1|x-\bar{x}|n \geqslant |x-\bar{x}|(c_0|x-\bar{x}| - c_1 n)$$
$$\geqslant |x-\bar{x}|(c_0c_2(2nc_1)^{-1} - c_1 n) \geqslant 0.$$

Thus in both cases

$$g_1(x) \geqslant g(x). \tag{5.16}$$

(5.11), (5.16) and (1.3) imply that for all $x \in R^n$

$$\max\{f_1(x), g_1(x)\} \geqslant \max\{f(x), g(x)\} \geqslant \phi(x).$$

Therefore $(f_1, g_1) \in \mathscr{M}_{co}$.

Now we show that

$$((f,g),(f_1,g_1)) \in E_w(\epsilon, m). \tag{5.17}$$

Assume that $x \in R^n$ satisfies

$$|x| \leqslant m. \tag{5.18}$$

By (5.9), (5.18) and (5.6)

$$|f_1(x) - f(x)| = c_2 + c_0|x-\bar{x}|^2 \leqslant c_2 + c_0(|x| + |\bar{x}|)^2$$
$$\leqslant c_2 + c_0(m + |\bar{x}|)^2 < \epsilon/4. \tag{5.19}$$

In view of (5.9), (5.19), (5.8) and (5.7)

$$|\nabla f_1(x) - \nabla f(x)| \leqslant 2c_0|x-\bar{x}| \leqslant 2c_0(m+|\bar{x}|) < \epsilon/8. \tag{5.20}$$

It follows from (5.10), (5.18) and (5.6)-(5.8) that

$$|g_1(x) - g(x)| \leqslant c_2 + c_1|x-\bar{x}|n + c_0|x-\bar{x}|^2 \tag{5.21}$$
$$\leqslant c_2 + c_1 n(|\bar{x}| + m) + c_0(m + |\bar{x}|)^2 \leqslant \epsilon.$$

(5.10), (5.18) and (5.8) imply that

$$|\nabla g_1(x) - \nabla g(x)| \leqslant c_1 n + 2c_0|x-\bar{x}| \leqslant c_1 n + 2c_0(m+|\bar{x}|) \leqslant \epsilon/4. \tag{5.22}$$

In view of (5.19)-(5.22), (5.9), (5.10) and (5.7) the inclusion (5.17) is true. By (5.9), (5.10) and (5.2)

$$f_1(\bar{x}) = f(\bar{x}) + c_2 = g(\bar{x}) + c_2 = g_1(\bar{x}). \tag{5.23}$$

It follows from (5.9) and (5.4) that

$$\nabla f_1(\bar{x}) = \nabla f(\bar{x}) = 0.$$

(5.10) and (5.4) imply that

$$\nabla g_1(\bar{x}) = \nabla g(\bar{x}) + c_1(1,\ldots,1) = c_1(1,\ldots,1).$$

Thus

$$\nabla f_1(\bar{x}) \neq \nabla g_1(\bar{x}). \tag{5.24}$$

Let $x \in R^n \setminus \{\bar{x}\}$. By (5.9), (5.5) and (5.23)

$$\begin{aligned}\max\{f_1(x), g_1(x)\} &\geqslant f_1(x) = f(x) + c_2 + c_0|x-\bar{x}|^2 \\ &\geqslant f(\bar{x}) + c_2 + c_0|x-\bar{x}|^2 = f_1(\bar{x}) + c_0|x-\bar{x}|^2 \\ &> f_1(\bar{x}) = \max\{f_1(\bar{x}), g_1(\bar{x})\}.\end{aligned}$$

Thus

$$\max\{f_1(x), g_1(x)\} > \max\{f_1(\bar{x}), g_1(\bar{x})\} \tag{5.25}$$

for all $x \in R^n \setminus \{\bar{x}\}$. In view of (5.23), (5.24) and (5.25)

$$(f_1, g_1) \in (\mathscr{M}_{co} \cap \mathscr{M}_0) \setminus G. \tag{5.26}$$

Since ϵ, m are arbitrary positive numbers it follows from (5.26) and (5.17) that $\mathscr{M}_{co} \setminus G$ is an everywhere dense subset of $\mathscr{M}_{co}$ with the weak topology and $(\mathscr{M}_{co} \cap \mathscr{M}_0) \setminus G$ is an evrywhere dense subset of $\mathscr{M}_{co} \cap \mathscr{M}_0$ with the weak topology. Theorem 1.2 is proved.

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