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SHORT PROOF THAT THE ARITHMETIC MEAN IS GREATER THAN THE HARMONIC MEAN AND ITS REVERSE INEQUALITY

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(*communicated by J. Pečarić*)

Abstract. We shall give a short proof of the well known result made in its title.

THEOREM. *Let A_j be bounded linear operator such that $MI \geqslant A_j \geqslant mI$ $(M > m > 0)$ for $j = 1,2,\dots,n$, and let $\lambda_j > 0$ such that $\sum_{k=1}^n \lambda_k = 1$ for $j = 1,2,\dots,n$. Then the following inequalities hold:*

$$\frac{(m+M)^2}{4mM}\Big(\sum_{k=1}^n \lambda_k A_k^{-1}\Big)^{-1} \geqslant \sum_{k=1}^n \lambda_k A_k \geqslant \Big(\sum_{k=1}^n \lambda_k A_k^{-1}\Big)^{-1}. \tag{1}$$

Proof. Put $A = \sum_{k=1}^n \lambda_k A_k$ and $H = \left(\sum_{k=1}^n \lambda_k A_k^{-1}\right)^{-1}$. The second inequality follows by

$$A - H = \sum_{j=1}^n (I - HA_j^{-1})\lambda_j A_j (I - HA_j^{-1})^* \geqslant 0.$$

On the other hand, the first inequality follows from

$$\begin{aligned} mMH^{-1} + A &\leqslant \sum_{j=1}^n (MI - A_j)\lambda_j A_j^{-1}(A_j - mI) + mMH^{-1} + A \\ &= m + M \leqslant \frac{(m+M)^2}{4mM} H + mMH^{-1} \end{aligned}$$

by Arithmetic-Geometric inequality. $\square$

Further extensions of (1) are in [1] and Professor J.Pečarić kindly pointed out that the proof of the first inequality is essentially the same in his paper with Mond in [1].

REFERENCES

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