

MONOTONICITY OF SEQUENCES INVOLVING GEOMETRIC MEANS OF POSITIVE SEQUENCES WITH MONOTONICITY AND LOGARITHMICAL CONVEXITY

BAI-NI GUO AND FENG QI

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Abstract. Let f be a positive function such that $x[f(x+1)/f(x) - 1]$ is increasing on $[1, \infty)$, then the sequence $\left\{ \sqrt[n]{\prod_{i=1}^n f(i)} / f(n+1) \right\}_{n=1}^{\infty}$ is decreasing. If f is a logarithmically concave and positive function defined on $[1, \infty)$, then the sequence $\left\{ \sqrt[n]{\prod_{i=1}^n f(i)} / \sqrt{f(n)} \right\}_{n=1}^{\infty}$ is increasing. As consequences of these monotonicities, the lower and upper bounds for the ratio $\sqrt[n]{\prod_{i=k+1}^{n+k} f(i)} / \sqrt[n+m]{\prod_{i=k+1}^{n+m+k} f(i)}$ are obtained, where k is a nonnegative integer and m a natural number. Some applications are given.

1. Introduction

It is known that, for $n \in \mathbb{N}$, the following double inequality was proved in [7]:

$$\frac{n}{n+1} < \frac{\sqrt[n]{n!}}{n+1\sqrt{(n+1)!}} < 1, \quad (1)$$

which can be rearranged as

$$[\Gamma(1+n)]^{\frac{1}{n}} < [\Gamma(2+n)]^{\frac{1}{n+1}} \quad (2)$$

and

$$\frac{[\Gamma(1+n)]^{\frac{1}{n}}}{n} > \frac{[\Gamma(2+n)]^{\frac{1}{n+1}}}{n+1}. \quad (3)$$

Using analytic method and Stirling's formula, in [2, 11, 18, 19], for $n, m \in \mathbb{N}$ and k being a nonnegative integer, the author and others proved the following inequalities:

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i \right)^{1/n} / \left(\prod_{i=k+1}^{n+m+k} i \right)^{1/(n+m)} \leq \sqrt{\frac{n+k}{n+m+k}}, \quad (4)$$

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the equality in (4) is valid for $n = 1$ and $m = 1$, which extend and refine those in (1).

In [1], the left inequality in (1) was refined by

$$\frac{n}{n+1} < \left(\frac{1}{n} \sum_{i=1}^n i^r / \frac{1}{n+1} \sum_{i=1}^{n+1} i^r \right)^{1/r} < \frac{\sqrt[n]{n!}}{n+1 \sqrt{(n+1)!}} \quad (5)$$

for all positive real numbers r . Both bounds are the best possible.

There is a rich literature on refinements, extensions, and generalizations of the inequalities in (5), for examples, [3, 9, 10, 14, 15, 20] and references therein. Note that the inequalities in (5) are direct consequences of a conjecture which states that the function $\left(\frac{1}{n} \sum_{i=1}^n i^r / \frac{1}{n+1} \sum_{i=1}^{n+1} i^r\right)^{1/r}$ is decreasing with r . Please refer to [21].

In [12, 13], using the ideas and method in [4, 6, 16] and the mathematical induction, the following inequalities were obtained: For all nonnegative integers k and natural numbers n and m , we have

$$\frac{a(n+k+1)+b}{a(n+m+k+1)+b} < \frac{\left[\prod_{i=k+1}^{n+k}(ai+b)\right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k}(ai+b)\right]^{\frac{1}{n+m}}} \leq \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}}, \quad (6)$$

where a is a positive constant, and b is a nonnegative constant. The equality in (6) is valid for $n = 1$ and $m = 1$.

In [5], the following monotonicity results for the gamma function were established: The function $[\Gamma(1 + \frac{1}{x})]^x$ decreases with $x > 0$ and $x[\Gamma(1 + \frac{1}{x})]^x$ increases with $x > 0$, which recover the inequalities in (1) which refer to integer values of r . These are equivalent to the function $[\Gamma(1+x)]^{1/x}$ being increasing and $\frac{[\Gamma(1+x)]^{1/x}}{x}$ being decreasing on $(0, \infty)$, respectively. In addition, it was proved that the function $x^{1-\gamma}[\Gamma(1 + \frac{1}{x})]^x$ decreases for $0 < x < 1$, where $\gamma = 0.57721566 \dots$ denotes the Euler's constant, which is equivalent to $\frac{[\Gamma(1+x)]^{1/x}}{x^{1-\gamma}}$ being increasing on $(1, \infty)$.

In [2], the following monotonicity result was obtained: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \quad (7)$$

is decreasing in $x \geq 1$ for fixed $y \geq 0$. Then, for positive real numbers x and y , we have

$$\frac{x+y+1}{x+y+2} \leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \quad (8)$$

Inequality (8) extends and generalizes inequality (4), since $\Gamma(n+1) = n!$.

Several results mentioned above can be found, for example, in three articles written by J. Sándor [22].

DEFINITION 1. [[8, p. 7]] A positive function $f : I \rightarrow \mathbb{R}$, I an interval in \mathbb{R} , is said to be logarithmically convex (log-convex, multiplicatively convex) if $\ln f$ is convex, or equivalently if for all $x, y \in I$ and all $\alpha \in [0, 1]$,

$$f(\alpha x + (1-\alpha)y) \leq f^\alpha(x) f^{1-\alpha}(y). \quad (9)$$

It is said to be logarithmically concave (log-concave) if the inequality in (9) is reversed.

DEFINITION 2. Let $\{a_i\}_{i \in \mathbb{N}}$ be a positive sequence. If $a_{i+1}a_{i-1} \geq a_i^2$ for $i \geq 2$, we call $\{a_i\}_{i \in \mathbb{N}}$ a logarithmically convex sequence; if $a_{i+1}a_{i-1} \leq a_i^2$ for $i \geq 2$, we call $\{a_i\}_{i \in \mathbb{N}}$ a logarithmically concave sequence.

2. Main results

In this article, we will further generalize the inequalities in (6) and obtain the following

THEOREM 1. Let f be a positive function such that $x \left[\frac{f(x+1)}{f(x)} - 1 \right]$ is increasing on $[1, \infty)$. Then the sequence

$$\left\{ \frac{\sqrt[n]{\prod_{i=1}^n f(i)}}{f(n+1)} \right\}_{n=1}^{\infty} \tag{10}$$

is decreasing. As a consequence, we have the following

$$\frac{\sqrt[n]{\prod_{i=k+1}^{n+k} f(i)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k} f(i)}} \geq \frac{f(n+k+1)}{f(n+m+k+1)}, \tag{11}$$

where m is a natural number and k a nonnegative integer.

REMARK 1. The relatively complicated condition that the function $x \left[\frac{f(x+1)}{f(x)} - 1 \right]$ is increasing on $[1, \infty)$ holds if f is logarithmically convex and increasing.

COROLLARY 1. Let $\{a_i\}_{i=1}^{\infty}$ be a positive sequence such that $\{i \left[\frac{a_{i+1}}{a_i} - 1 \right]\}_{i=1}^{\infty}$ is increasing, then the sequence

$$\left\{ \frac{\sqrt[n]{a_n!}}{a_{n+1}} \right\}_{n=1}^{\infty} \tag{12}$$

is decreasing. As a consequence, we have the following

$$\frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}} \geq \frac{a_{n+1}}{a_{n+m+1}}, \tag{13}$$

where m is a natural number and $a_n!$ is the sequence factorial defined by $\prod_{i=1}^n a_i$.

THEOREM 2. Let f be a logarithmically concave and positive function defined on $[1, \infty)$. Then the sequence

$$\left\{ \frac{\sqrt[n]{\prod_{i=1}^n f(i)}}{\sqrt{f(n)}} \right\}_{n=1}^{\infty} \tag{14}$$

is increasing. As a consequence, we have the following

$$\frac{\sqrt[n]{\prod_{i=k+1}^{n+k} f(i)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k} f(i)}} \leq \sqrt{\frac{f(n+k)}{f(n+m+k)}}, \quad (15)$$

where m is a natural number and k a nonnegative integer. The equality in (15) is valid for $n = 1$ and $m = 1$.

COROLLARY 2. Let $\{a_i\}_{i=1}^{\infty}$ be a logarithmically concave and positive sequence. Then the sequence

$$\left\{ \frac{\sqrt[n]{a_n!}}{\sqrt{a_n}} \right\}_{n=1}^{\infty} \quad (16)$$

is increasing. Therefore, we have

$$\frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}} \leq \sqrt{\frac{a_n}{a_{n+m}}}, \quad (17)$$

where m is a natural number and $a_n!$ is the sequence factorial defined by $\prod_{i=1}^n a_i$. The equality in (17) is valid for $n = 1$ and $m = 1$.

At last, in Section 4., some applications of Theorem 1 and Theorem 2 are given.

REMARK 2. It is well known that the left hand side term in (11) or (15) is a ratio of two geometric means of sequence $\{f(i)\}_{i=1}^{\infty}$.

3. Proofs of Theorem 1 and Theorem 2

Proof. [Proof of Theorem 1] The monotonicity of the sequence (10) and inequality (11) are equivalent to the following

$$\begin{aligned} & \left(\prod_{i=1}^n \frac{f(i)}{f(n+1)} \right)^{1/n} \geq \left(\prod_{i=1}^{n+1} \frac{f(i)}{f(n+2)} \right)^{1/(n+1)}, \\ \Leftrightarrow & \frac{1}{n} \sum_{i=1}^n \ln \frac{f(i)}{f(n+1)} \geq \frac{1}{n+1} \sum_{i=1}^{n+1} \ln \frac{f(i)}{f(n+2)}, \\ \Leftrightarrow & \frac{n}{n+1} \sum_{i=1}^{n+1} \ln \frac{f(i)}{f(n+2)} \leq \sum_{i=1}^n \ln \frac{f(i)}{f(n+1)}. \end{aligned} \quad (18)$$

Since $\ln x$ is concave on $(0, \infty)$, by definition of concaveness, it follows that, for $1 \leq i \leq n$,

$$\begin{aligned} & \frac{i}{n+1} \ln \frac{f(i+1)}{f(n+2)} + \frac{n-i+1}{n+1} \ln \frac{f(i)}{f(n+2)} \\ & \leq \ln \left(\frac{i}{n+1} \cdot \frac{f(i+1)}{f(n+2)} + \frac{n-i+1}{n+1} \cdot \frac{f(i)}{f(n+2)} \right) \\ & = \ln \left(\frac{if(i+1) + (n-i+1)f(i)}{(n+1)f(n+2)} \right). \end{aligned} \quad (19)$$

Since the function $x \left[\frac{f(x+1)}{f(x)} - 1 \right]$ is increasing on $[1, \infty)$, for all $1 \leq i \leq n$, we have

$$\begin{aligned} & \frac{(n+1)f(n+2)}{f(n+1)} - (n+1) \geq \frac{nf(n+1)}{f(n)} - n, \\ \Leftrightarrow & \frac{(n+1)f(n+2)}{f(n+1)} - (n+1) \geq \frac{if(i+1)}{f(i)} - i, \\ \Leftrightarrow & \frac{if(i+1) + (n-i+1)f(i)}{f(i)} \leq \frac{(n+1)f(n+2)}{f(n+1)}, \\ \Leftrightarrow & \frac{if(i+1) + (n-i+1)f(i)}{(n+1)f(n+2)} \leq \frac{f(i)}{f(n+1)}. \end{aligned}$$

Combining the last line above with (19) yields

$$\frac{i}{n+1} \ln \frac{f(i+1)}{f(n+2)} + \frac{n-i+1}{n+1} \ln \frac{f(i)}{f(n+2)} \leq \ln \frac{f(i)}{f(n+1)}. \quad (20)$$

Summing up on both sides of (20) from 1 to n and simplifying reveals inequality (18). The proof is complete. \square

Proof. [Proof of Theorem 2] The monotonicity of the sequence (14) and inequality (15) are equivalent to the following

$$\begin{aligned} & \frac{\sqrt[n]{\prod_{i=1}^n f(i)}}{\sqrt{f(n)}} \leq \frac{\sqrt[n+1]{\prod_{i=1}^{n+1} f(i)}}{\sqrt{f(n+1)}}, \\ \Leftrightarrow & \frac{1}{n} \sum_{i=1}^n \ln f(i) - \frac{1}{n+1} \sum_{i=1}^{n+1} \ln f(i) \leq \frac{1}{2} [\ln f(n) - \ln f(n+1)], \\ \Leftrightarrow & \left(1 + \frac{1}{n}\right) \sum_{i=1}^n \ln f(i) - \sum_{i=1}^{n+1} \ln f(i) \leq \frac{n+1}{2} [\ln f(n) - \ln f(n+1)], \\ \Leftrightarrow & \frac{n+1}{2} \ln f(n) - \frac{n-1}{2} \ln f(n+1) \geq \frac{1}{n} \sum_{i=1}^n \ln f(i). \end{aligned} \quad (21)$$

For $n = 1$, the equality in (21) holds.

Suppose inequality (21) is valid for some $n > 1$. Since, by the inductive hypothesis

$$\begin{aligned} \frac{1}{n+1} \sum_{i=1}^{n+1} \ln f(i) &= \frac{n}{n+1} \left[\frac{1}{n} \sum_{i=1}^n \ln f(i) \right] + \frac{\ln f(n+1)}{n+1} \\ &\leq \frac{n}{n+1} \left[\frac{n+1}{2} \ln f(n) - \frac{n-1}{2} \ln f(n+1) \right] + \frac{\ln f(n+1)}{n+1} \\ &= \frac{n}{2} \ln f(n) - \frac{n-2}{2} \ln f(n+1), \end{aligned}$$

by induction, it is sufficient to prove

$$\begin{aligned}
& \frac{n}{2} \ln f(n) - \frac{n-2}{2} \ln f(n+1) \leq \frac{n+2}{2} \ln f(n+1) - \frac{n}{2} \ln f(n+2), \\
\iff & n \ln f(n) \leq 2n \ln f(n+1) - n \ln f(n+2), \\
\iff & \ln[f(n)f(n+2)] \leq \ln f^2(n+1), \\
\iff & f(n)f(n+2) \leq f^2(n+1),
\end{aligned}$$

this follows from the logarithmic concaveness of the function f . The proof is complete. \square

REMARK 3. If the function f in Theorem 1 is differentiable, then we can give the following proof of Theorem 1 by Cauchy's mean value theorem and mathematical induction.

Proof. [Proof of Theorem 1 under condition such that f being differentiable] The monotonicity of the sequence (10) and inequality (11) are equivalent to

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \ln f(i) - \frac{1}{n+1} \sum_{i=1}^{n+1} \ln f(i) \geq \ln f(n+1) - \ln f(n+2), \\
\iff & \frac{1}{n} \sum_{i=1}^n \ln f(i) - \ln f(n+1) \geq (n+1)[\ln f(n+1) - \ln f(n+2)], \quad (22) \\
\iff & (n+2) \ln f(n+1) - (n+1) \ln f(n+2) \leq \frac{1}{n} \sum_{i=1}^n \ln f(i).
\end{aligned}$$

For $n = 1$, inequality (22) can be rewritten as $f(1)[f(3)]^2 \geq [f(2)]^3$. Since f is logarithmically convex and increasing, we have $f(1)f(3) \geq [f(2)]^2$ and $f(3) \geq f(2)$, respectively. Therefore, inequality (22) holds for $n = 1$.

Suppose inequality (22) is valid for some $n > 1$. Then, by inductive hypothesis, we have

$$\begin{aligned}
\frac{1}{n+1} \sum_{i=1}^{n+1} \ln f(i) &= \frac{n}{n+1} \left[\frac{1}{n} \sum_{i=1}^n \ln f(i) \right] + \frac{f(n+1)}{n+1} \\
&\geq \frac{n}{n+1} [(n+2) \ln f(n+1) - (n+1) \ln f(n+2)] + \frac{f(n+1)}{n+1} \\
&= (n+1) \ln f(n+1) - n \ln f(n+2).
\end{aligned}$$

hence, by induction, it is sufficient to prove the following

$$(n+1) \ln f(n+1) - n \ln f(n+2) \geq (n+3) \ln f(n+2) - (n+2) \ln f(n+3),$$

which can be rearranged as

$$(n+1)[\ln f(n+1) - \ln f(n+2)] \geq (n+2)[\ln f(n+2) - \ln f(n+3)],$$

further, since f is increasing,

$$\frac{\ln f(n+2) - \ln f(n+1)}{\ln f(n+3) - \ln f(n+2)} \leq \frac{n+2}{n+1}. \tag{23}$$

Using Cauchy's mean values applied to $g(x) = \ln f(n+1+x)$ and $h(x) = \ln f(n+2+x)$ for $x \in [0, 1]$ in inequality (23), it follows that there exists a point $\xi \in (0, 1)$ such that

$$\frac{f'(n+1+\xi)}{f(n+1+\xi)} \cdot \frac{f(n+2+\xi)}{f'(n+2+\xi)} \leq \frac{n+2}{n+1}.$$

Since the positive function f is logarithmically convex and differentiable, then $[\ln f(x)]' = \frac{f'(x)}{f(x)}$ is increasing. Thus

$$\frac{f'(n+1+\xi)}{f(n+1+\xi)} \leq \frac{f'(n+2+\xi)}{f(n+2+\xi)},$$

and then

$$\frac{f'(n+1+\xi)}{f(n+1+\xi)} \cdot \frac{f(n+2+\xi)}{f'(n+2+\xi)} \leq 1 < \frac{n+2}{n+1}.$$

Inequality (23) follows. The proof is complete. \square

4. Applications

4.1. The affine function $f(x) = ax + b$ for $x > -\frac{b}{a}$, where $a > 0$ and $b \in \mathbb{R}$ are constants, is positive and logarithmically concave. From Theorem 2 applied to this affine function, the right hand side inequality in (6) follows.

4.2. The left hand side inequality in (6) follows from Corollary 1.

4.3. Applying Theorem 1 or Corollary 1 to $f(x) = \Gamma(x+1)$ or $a_i = i!$ respectively yields

$$\begin{aligned} \frac{\prod_{i=2}^n (i+k)^{\frac{n+1-i}{n}}}{\prod_{i=2}^{n+m} (i+k)^{\frac{n+m+1-i}{n+m}}} &= \frac{\sqrt[n]{\prod_{i=k+1}^{n+k} (i!)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k} (i!)}} \\ &\geq \frac{(n+k+1)!}{(n+m+k+1)!} = \frac{1}{\prod_{i=1}^m (n+k+1+i)}. \end{aligned} \tag{24}$$

Similarly, we have

$$\begin{aligned} \frac{\sqrt[n]{\prod_{i=k+1}^{n+k} (i!!)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k} (i!!)}} &\geq \frac{(n+k+1)!!}{(n+m+k+1)!!}, \\ \frac{\sqrt[n]{\prod_{i=k+1}^{n+k} ((2i)!!)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k} ((2i)!!)}} &\geq \frac{[2(n+k+1)]!!}{[2(n+m+k+1)]!!}, \\ \frac{\sqrt[n]{\prod_{i=k+1}^{n+k} ((2i-1)!!)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k} ((2i-1)!!)}} &\geq \frac{[2(n+k)+1]!!}{[2(n+m+k)+1]!!}. \end{aligned} \tag{25}$$

Where n and m are natural numbers and k a nonnegative integer.

4.4. If $\{a_i\}_{i=1}^{\infty}$ is increasing and logarithmically convex, the sequence $\{i[\frac{a_{i+1}}{a_i} - 1]\}_{i=1}^{\infty}$ is increasing. In Corollary 1, considering the sequence $\{\ln a_i\}_{i=1}^{\infty}$ is increasing, convex, and positive, we obtain the following

PROPOSITION 1. *Let $\{a_i\}_{i=1}^{\infty}$ be an increasing convex positive sequence and $A_n = \frac{1}{n} \sum_{i=1}^n a_i$ an arithmetic mean. Then the sequence $A_n - a_{n+1}$ decreases. This gives a lower bound for difference of two arithmetic means:*

$$A_n - A_{n+m} \geq a_{n+1} - a_{n+m+1}, \quad (26)$$

where m is a natural number.

4.5. In Corollary 2, considering the sequence $\{\ln a_i\}_{i=1}^{\infty}$ being concave and positive, we have

PROPOSITION 2. *Let $\{a_i\}_{i=1}^{\infty}$ be a concave and positive sequence and $A_n = \frac{1}{n} \sum_{i=1}^n a_i$ an arithmetic mean. Then the sequence $A_n - \frac{a_n}{2}$ increases. This implies an upper bound for difference of two arithmetic means:*

$$A_n - A_{n+m} \leq \frac{a_n - a_{n+m}}{2}, \quad (27)$$

where m is a natural number.

4.6. For real numbers $b \geq 1$ and $c \geq 0$ such that $b^2 > 2c$, the function $x^2 + bx + c$ is logarithmically concave and satisfies conditions of Theorem 2, then we have

$$\begin{aligned} \frac{(n+k+1)^2 + b(n+k+1) + c}{(n+m+k+1)^2 + b(n+m+k+1) + c} &\leq \frac{\sqrt[n]{\prod_{i=k+1}^{n+k} (i^2 + bi + c)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k} (i^2 + bi + c)}} \\ &\leq \sqrt{\frac{(n+k)^2 + b(n+k) + c}{(n+m+k)^2 + b(n+m+k) + c}}, \end{aligned}$$

where m is a natural number and k a nonnegative integer.

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B.-N. Guo

School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City, Henan Province, 454010
China
e-mail: guobaini@hpu.edu.cn

F. Qi

Research Institute of Mathematical Inequality Theory
Henan Polytechnic University
Jiaozuo City, Henan Province, 454010
China
e-mail: qifeng@hpu.edu.cn
e-mail: fengqi618@member.ams.org
e-mail: qifeng618@hotmail.com
e-mail: qifeng618@msn.com
URL: <http://rgmia.vu.edu.au/qi.html>