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BOUNDS FOR THE MULTIPLICITIES OF THE ROOTS FOR SOME CLASSES OF COMPLEX POLYNOMIALS

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Abstract. We use some of Ostrowski's conditions for nonvanishing of determinants in order to provide explicit upper bounds for the multiplicities of the roots for some classes of complex polynomials. In particular, some simple separability criteria are obtained. We finally provide bounds for the multiplicities of the roots for some classes of integral polynomials, in terms of the prime decomposition of their coefficients.

1. Introduction

In the present paper we are concerned with the problem of bounding the multiplicities of the roots for some classes of complex polynomials. Our idea is to consider the resultant between two different derivatives of a given polynomial, and then make use of certain nonvanishing results for determinants. Among the criteria for nonvanishing of determinants, one of the most famous is given by Hadamard's Theorem [3]:

If the elements of a $n \times n$ *complex matrix* $A = (a_{ij})$ *satisfy*

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \qquad i = 1, \dots, n, \tag{1}$$

then $\det(A) \neq 0$.

A complete bibliography of this theorem is contained in [11]. Several other criteria of this type can be found in [10].

Other conditions for the nonvanishing of a determinant were obtained by Ostrowski in [4], [5] and [7]–[9], using only the moduli of the elements of A and some simple combinations of these moduli. The results of Ostrowski use essentially the expressions:

$$R_i = \sum_{j \neq i} |a_{ij}|, \qquad C_i = \sum_{j \neq i} |a_{ji}| \qquad i = 1, \dots, n, \tag{2}$$

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$$R_{i,s} = (\sum_{j\neq i} |a_{ij}|^s)^{1/s}, \qquad C_{i,s} = (\sum_{j\neq i} |a_{ji}|^s)^{1/s} \qquad i = 1,\ldots,n, \tag{3}$$

$$m_i = \max_{j\neq i} |a_{ij}| = R_{i,\infty}, \qquad m_i' = \max_{j\neq i} |a_{ji}| = C_{i,\infty} \qquad i = 1,\ldots,n. \tag{4}$$

One criterion derived in [8] is that $\det(A) \neq 0$ if for an arbitrarily chosen fixed α, $0 \leqslant \alpha \leqslant 1$, we have

$$|a_{ii}| > R_i^{\alpha} C_i^{1-\alpha} \qquad i = 1,\ldots,n. \tag{5}$$

The most general criterion using R_i and C_i was then given by replacing the conditions (5) with

$$|a_{ii}a_{jj}| > R_i^{\alpha} C_i^{1-\alpha} R_j^{\alpha} C_j^{1-\alpha} \qquad i \neq j, \quad i,j = 1,\ldots,n. \tag{6}$$

As to $R_{i,s}$ and $C_{i,s}$, the corresponding criterion derived in [7] is that $\det(A) \neq 0$ if

$$\sum_{i=1}^{n} \frac{1}{1 + |a_{ii}|^q / R_{i,p}^q} < 1 \tag{7}$$

for a fixed but arbitrary choice of $p \geqslant 1$ and $q \geqslant 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{8}$$

and the criterion obtained by replacing $R_{i,p}$ by $C_{i,p}$. For $q = 1$ we have $p = \infty$ and (7) becomes

$$\sum_{i=1}^{n} \frac{1}{1 + |a_{ii}|/m_i} < 1. \tag{9}$$

Introducing a new parameter, Ostrowski derived in [5] the following criterion: $\det(A) \neq 0$ if we have

$$|a_{ii}| > R_{i,\alpha p}^{\alpha} C_{i,(1-\alpha)q}^{1-\alpha} \qquad i = 1,\ldots,n \tag{10}$$

for fixed but arbitrarily chosen α with $0 < \alpha < 1$ and p and q satisfying (8). More generally, we have $\det(A) \neq 0$ if we replace (10) by

$$|a_{ii}a_{jj}| > R_{i,\alpha p}^{\alpha} C_{i,(1-\alpha)q}^{1-\alpha} R_{j,\alpha p}^{\alpha} C_{j,(1-\alpha)q}^{1-\alpha} \qquad i \neq j, \quad i,j = 1,\ldots,n. \tag{11}$$

By using ideas from [5], some results bounding the spectral radius of an iteration matrix arising in various iterative methods have been derived in [2].

In [6] Ostrowski obtained parametric representations of the most general non-singular matrices for which equality in (10) holds for each i. Some of the results in [6] have been generalized in [1] to the case of monotonic norms, where estimates for the moduli of eigenvalues of matrices were also obtained.

In this paper we first use some of Ostrowski's results in order to provide upper bounds for the multiplicities of the roots for some classes of complex polynomials, in terms of the moduli of their coefficients. In particular, some simple separability criteria are obtained. In the final section we derive bounds for the multiplicities of the roots for some classes of integral polynomials, in terms of the prime decomposition of their coefficients. The results in Sections 2 and 3 below are quite flexible, and might prove to be useful in various applications.

2. Bounds for the multiplicities of the roots for some classes of complex polynomials

We use the following notation. For a nonconstant complex polynomial $f(X)$, which decomposes as

$$f(X) = a\prod_{i=1}^{s}(X-\theta_i)^{n_i}$$

with $a, \theta_1, \ldots, \theta_s \in \mathbb{C}$, θ_i pairwise distinct, and $n_i \in \mathbb{N}$, $n_i \geqslant 1$, we let

$$e(f) = \max\{n_1, \ldots, n_s\}.$$

PROPOSITION 1. *Let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{C}[X]$ be a polynomial of degree $n \geqslant 2$ and fix integers j and k such that $0 \leqslant j < k < n$. If*

$$|a_j| > \sum_{i=1}^{n-j}\binom{i+j}{i}|a_{i+j}| \quad \text{and} \quad \binom{n}{k}|a_n| > \sum_{i=0}^{n-k-1}\binom{i+k}{i}|a_{i+k}|, \tag{12}$$

then $e(f) \leqslant k$.

Proof. The proof relies on the fact that $e(f) \leqslant k < n$ if for a fixed but arbitrarily chosen integer j with $0 \leqslant j < k$, the resultant of the j-th and the k-th derivatives of f does not vanish. Let $\Delta_{j,k} = Res(f^{(j)}, f^{(k)})$. Denoting $\frac{(r+j)!}{r!}a_{r+j}$ by b_r for $r = 0, \ldots, n-j$ and $\frac{(r+k)!}{r!}a_{r+k}$ by c_r for $r = 0, \ldots, n-k$, we have

$$f^{(j)}(X) = \sum_{r=0}^{n-j} b_rX^r, \quad f^{(k)}(X) = \sum_{r=0}^{n-k} c_rX^r,$$

and

$$\Delta_{j,k} = \begin{vmatrix} b_0 & b_1 & b_2 & \cdot & \cdot & \cdot & b_{n-j} & & & \\ & b_0 & b_1 & b_2 & \cdot & \cdot & \cdot & b_{n-j} & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & b_0 & b_1 & b_2 & \cdot & \cdot & \cdot & b_{n-j} \\ c_0 & c_1 & \cdot & \cdot & c_{n-k} & & & & & \\ & c_0 & c_1 & \cdot & \cdot & c_{n-k} & & & & \\ & & c_0 & c_1 & \cdot & \cdot & c_{n-k} & & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & & c_0 & c_1 & \cdot & \cdot & c_{n-k} \end{vmatrix},$$

where there are $n-k$ rows of b's and $n-j$ rows of c's, the matrix being filled out with zeros. Recalling the definition of R_i we find that

$$R_i = \begin{cases} \sum\limits_{r=1}^{n-j}|b_r| = \sum\limits_{r=1}^{n-j}\frac{(r+j)!}{r!}|a_{r+j}|, & i = 1, \ldots, n-k \\ \sum\limits_{r=0}^{n-k-1}|c_r| = \sum\limits_{r=0}^{n-k-1}\frac{(r+k)!}{r!}|a_{r+k}|, & i = n-k+1, \ldots, 2n-j-k \end{cases}$$

and the conclusion follows now by applying to $\Delta_{j,k}$ the nonvanishing condition (1).

COROLLARY 1. *Let* $f(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{C}[X]$ *be a polynomial of degree* $n \geqslant 2$*, such that*

$$|a_0| > \sum_{i=1}^{n} |a_i| \quad \text{and} \quad n|a_n| > \sum_{i=1}^{n-1} i|a_i|.$$

Then $f(X)$ *is a separable polynomial.*

Some more complicated conditions, which produce the same upper bound for the multiplicities of the roots of a complex polynomial f, are provided by the following result.

PROPOSITION 2. *Let* $f(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{C}[X]$ *be a polynomial of degree* $n \geqslant 2$ *and let integers* j *and* k *such that* $0 \leqslant j < k < n$*. If the coefficients of* f *satisfy:*

$$|a_j| > \frac{k!}{j!} \sum_{i=0}^{n-k-1} \binom{i+k}{i} |a_{i+k}| + \sum_{i=1}^{n-k-1} \binom{i+j}{i} |a_{i+j}|, \tag{13}$$

$$\binom{n}{k} |a_n| > \sum_{i=0}^{n-k-1} \binom{i+k}{i} |a_{i+k}| + \frac{j!}{k!} \max_{0 \leqslant r \leqslant k-j-1} \sum_{i=1}^{n-k} \binom{r+i+j}{r+i} |a_{r+i+j}|, \tag{14}$$

$$\binom{n}{k} |a_n| > \sum_{i=1}^{n-k-1} \binom{i+k}{i} |a_{i+k}| + \frac{j!}{k!} \sum_{i=k-j+1}^{n-j} \binom{i+j}{i} |a_{i+j}|, \tag{15}$$

then $e(f) \leqslant k$.

Proof. Using the same notation as in Proposition 1 and recalling the definition of C_i, we find succesively

$$C_i = \begin{cases} |c_0|, & i = 1 \\ \sum_{r=0}^{i-1} |c_r| + \sum_{r=1}^{i-1} |b_r|, & i = 2, \ldots, n-k \\ \sum_{r=0}^{n-k-1} |c_r| + \sum_{r=i-n+k}^{i-1} |b_r|, & i = n-k+1, \ldots, n-j \\ \sum_{r=i-n+j}^{n-k-1} |c_r| + \sum_{r=i-n+k}^{n-j} |b_r|, & i = n-j+1, \ldots, 2n-j-k-1 \\ |b_{n-j}|, & i = 2n-j-k \end{cases}$$

Therefore we have

$$\max_{1 \leqslant i \leqslant n-k} C_i = C_{n-k} \quad \text{and} \quad \max_{n-j+1 \leqslant i \leqslant 2n-j-k} C_i = C_{n-j+1}. \tag{16}$$

Applying now conditions (5) with $\alpha = 0$, we find in view of (16) that $\Delta_{j,k} \neq 0$ provided that the inequalities (13)–(15) hold, and this completes the proof of the proposition.

Conditions (13)–(15) take a particularly simple form for $j=0$ and $k=1$, when the maximum in the right side of (14) reduces to $|a_1|+\cdots+|a_{n-1}|$. In this case we find the following separability criterion.

COROLLARY 2. *Let $f(X)=a_0+a_1X+\cdots+a_nX^n\in\mathbb{C}[X]$ be a polynomial of degree $n\geqslant 2$, such that*

$$\begin{aligned}|a_0| &> (n-1)|a_{n-1}|+\sum_{i=1}^{n-2}(i+1)|a_i| \quad \text{and}\\ |a_n| &> \max\left\{\frac{1}{n}\sum_{i=1}^{n-1}(i+1)|a_i|,\ \frac{1}{n-1}\sum_{i=2}^{n-1}(i+1)|a_i|\right\}.\end{aligned}$$

Then $f(X)$ is a separable polynomial.

REMARK One may apply (5) with a fixed, arbitrarily chosen real α such that $0\leqslant\alpha\leqslant 1$, and combine conditions (12) and (13)–(15) in order to obtain bounds for the multiplicities of the roots under more general assumptions on the moduli of the a_i's. In particular, for $j=0$ and $k=1$ one finds that $f(X)=a_0+a_1X+\cdots+a_nX^n$ is a separable polynomial if the coefficients satisfy

$$\begin{aligned}|a_0| &> \left(\sum_{i=1}^{n}|a_i|\right)^{\alpha}\cdot\left((n-1)|a_{n-1}|+\sum_{i=1}^{n-2}(i+1)|a_i|\right)^{1-\alpha},\\ n|a_n| &> \left(\sum_{i=1}^{n-1}i|a_i|\right)^{\alpha}\cdot\left(\sum_{i=1}^{n-1}(i+1)|a_i|\right)^{1-\alpha} \quad \text{and}\\ n|a_n| &> \left(\sum_{i=1}^{n-1}i|a_i|\right)^{\alpha}\cdot\left(|a_n|+\sum_{i=2}^{n-1}(i+1)|a_i|\right)^{1-\alpha}.\end{aligned}$$

PROPOSITION 3. *Let $f(X)=a_0+a_1X+\cdots+a_nX^n\in\mathbb{C}[X]$ be a polynomial of degree $n\geqslant 2$, and let $p,q\geqslant 1$ such that $1/p+1/q=1$. Fix an arbitrarily chosen real number $\varepsilon>0$ and integers j and k such that $0\leqslant j<k<n$. If*

$$|a_j| > [(1+\varepsilon)(n-k)-1]^{1/q}\cdot\left(\sum_{i=1}^{n-j}\binom{i+j}{i}^{p}|a_{i+j}|^p\right)^{1/p} \quad \text{and} \tag{17}$$

$$\binom{n}{k}|a_n| > [(1+\varepsilon^{-1})(n-j)-1]^{1/q}\cdot\left(\sum_{i=0}^{n-k-1}\binom{i+k}{i}^{p}|a_{i+k}|^p\right)^{1/p}, \tag{18}$$

then $e(f)\leqslant k$.

Proof. Condition (7) applied to $\Delta_{j,k}$ may be written in the form

$$\frac{n-k}{1+\frac{|b_0|^q}{\left(\sum_{i=1}^{n-j}|b_i|^p\right)^{q/p}}}+\frac{n-j}{1+\frac{|c_{n-k}|^q}{\left(\sum_{i=0}^{n-k-1}|c_i|^p\right)^{q/p}}}<1,$$

or, equivalently

$$[M_1+1-(n-k)]\cdot[M_2+1-(n-j)]>(n-k)(n-j), \tag{19}$$

with

$$M_1 = \frac{|b_0|^q}{\left(\sum\limits_{i=1}^{n-j}|b_i|^p\right)^{q/p}} = \frac{(j!\cdot|a_j|)^q}{\left(\sum\limits_{i=1}^{n-j}\left(\frac{(i+j)!}{i!}|a_{i+j}|\right)^p\right)^{q/p}} \quad \text{and}$$

$$M_2 = \frac{|c_{n-k}|^q}{\left(\sum\limits_{i=0}^{n-k-1}|c_i|^p\right)^{q/p}} = \frac{(\frac{n!}{(n-k)!}\cdot|a_n|)^q}{\left(\sum\limits_{i=0}^{n-k-1}\left(\frac{(i+k)!}{i!}|a_{i+k}|\right)^p\right)^{q/p}}.$$

Therefore the inequality $e(f)\leqslant k$ will hold for large enough values of M_1 and M_2 fulfilling (19). Thus, for an arbitrarily fixed $\varepsilon>0$, we find $e(f)\leqslant k$ if $M_1+1-(n-k)>\varepsilon(n-k)$ and $M_2+1-(n-j)>\varepsilon^{-1}(n-j)$, which after division by $j!$ and $k!$ are precisely conditions (17) and (18) respectively.

REMARKS 1) For some particular values of ε one may obtain several simpler forms of (17) and (18). Let $A=(1+\varepsilon)(n-k)-1$ and $B=(1+\varepsilon^{-1})(n-j)-1$. Then $e(f)\leqslant k$ if

$$|a_j| > A^{1/q}\cdot\left(\sum_{i=1}^{n-j}\binom{i+j}{i}^p|a_{i+j}|^p\right)^{1/p} \quad \text{and}$$

$$\binom{n}{k}|a_n| > B^{1/q}\cdot\left(\sum_{i=0}^{n-k-1}\binom{i+k}{i}^p|a_{i+k}|^p\right)^{1/p},$$

in each one of the following four cases:

$$\begin{array}{lll} \varepsilon=1 & A=2(n-k)-1 & B=2(n-j)-1 \\ \varepsilon=n\text{–}j & A=(n-k)(n-j+1)-1 & B=n-j \\ \varepsilon=\frac{1}{n-k} & A=n-k & B=(n-j)(n-k+1)-1 \\ \varepsilon=\frac{n-j}{n-k} & A=2n-j-k-1 & B=2n-j-k-1 \end{array} \tag{20}$$

2) For $p=q=2$ we find $e(f)\leqslant k$ if for a fixed, arbitrarily chosen $\varepsilon>0$ we have

$$|a_j|^2 > [(1+\varepsilon)(n-k)-1]\cdot\sum_{i=1}^{n-j}\binom{i+j}{i}^2|a_{i+j}|^2 \quad \text{and}$$

$$\binom{n}{k}^2|a_n|^2 > [(1+\varepsilon^{-1})(n-j)-1]\cdot\sum_{i=0}^{n-k-1}\binom{i+k}{i}^2|a_{i+k}|^2.$$

In particular $e(f)\leqslant k$ if

$$|a_j|^2>A\cdot\sum_{i=1}^{n-j}\binom{i+j}{i}^2|a_{i+j}|^2 \quad \text{and} \quad \binom{n}{k}^2|a_n|^2>B\cdot\sum_{i=0}^{n-k-1}\binom{i+k}{i}^2|a_{i+k}|^2,$$

with A and B given by (20).

3) For $q = 1$ and $p = \infty$ we find that $e(f) \leqslant k$ if for a fixed, arbitrarily chosen $\varepsilon > 0$ we have

$$|a_j| \ > \ [(1+\varepsilon)(n-k)-1] \max_{1\leqslant i\leqslant n-j} \left| \binom{i+j}{i} a_{i+j} \right| \quad \text{and}$$
$$\binom{n}{k}|a_n| \ > \ [(1+\varepsilon^{-1})(n-j)-1] \max_{0\leqslant i\leqslant n-k-1} \left| \binom{i+k}{i} a_{i+k} \right|.$$

In particular $e(f) \leqslant k$ if

$$|a_j| > A \cdot \max_{1\leqslant i\leqslant n-j} \left| \binom{i+j}{i} a_{i+j} \right| \quad \text{and} \quad \binom{n}{k}|a_n| > B \cdot \max_{0\leqslant i\leqslant n-k-1} \left| \binom{i+k}{i} a_{i+k} \right|,$$

again with A and B given respectively by (20).

By considering $j = 0$ and $k = 1$ in Proposition 3, we obtain the following:

COROLLARY 3. *Let $f(X) = a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{C}[X]$ be a polynomial of degree $n \geqslant 2$, and let $p, q \geqslant 1$ such that $1/p + 1/q = 1$. Fix an arbitrarily chosen real number $\varepsilon > 0$. If*

$$|a_0| \ > \ [(1+\varepsilon)(n-1)-1]^{1/q} \cdot \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \quad \textit{and}$$
$$n|a_n| \ > \ [(1+\varepsilon^{-1})n-1]^{1/q} \cdot \left(\sum_{i=1}^{n-1} |ia_i|^p \right)^{1/p},$$

then $f(X)$ is separable.

REMARKS 1) For $j = 0$ and $k = 1$, (20) transforms into

$$\begin{array}{lll} \varepsilon = 1, & A = 2n-3, & B = 2n-1 \\ \varepsilon = n, & A = n^2-2, & B = n \\ \varepsilon = \frac{1}{n-1}, & A = n-1, & B = n^2-1 \\ \varepsilon = \frac{n}{n-1}, & A = 2n-2, & B = 2n-2 \end{array} \tag{21}$$

Therefore f is separable if

$$|a_0| > A^{1/q} \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \text{ and } |na_n| > B^{1/q} \left(\sum_{i=1}^{n-1} |ia_i|^p \right)^{1/p},$$

in each one of the four cases given by (21).

2) For $p = q = 2$ we find that f is separable if for a fixed, arbitrarily chosen $\varepsilon > 0$ we have

$$|a_0|^2 \ > \ [(1+\varepsilon)(n-1)-1] \cdot \sum_{i=1}^{n} |a_i|^2 \quad \text{and}$$
$$|na_n|^2 \ > \ [(1+\varepsilon^{-1})n-1] \cdot \sum_{i=1}^{n-1} |ia_i|^2.$$

In particular this conclusion holds if

$$|a_0|^2 > A \cdot \sum_{i=1}^{n} |a_i|^2 \text{ and } |na_n|^2 > B \cdot \sum_{i=1}^{n-1} |ia_i|^2,$$

with A and B given by (21).

3) For $q = 1$ and $p = \infty$ we find that f is separable if for a fixed, arbitrarily chosen $\varepsilon > 0$ we have

$$\begin{aligned} |a_0| &> [(1+\varepsilon)(n-1)-1] \cdot \max_{1 \leqslant i \leqslant n} |a_i| \quad \text{and} \\ |na_n| &> [(1+\varepsilon^{-1})n-1] \cdot \max_{1 \leqslant i \leqslant n-1} |ia_i|. \end{aligned}$$

For $\varepsilon \leqslant n$ one may replace the first inequality by $|a_0| > [(1+\varepsilon)(n-1)-1] \cdot |a_n|$, since in this case the second inequality implies $|a_n| > \max\{|a_1|, \ldots, |a_{n-1}|\}$. In particular we see that the polynomial f is separable if

$$|a_0| > A \cdot |a_n| \quad \text{and} \quad |na_n| > B \cdot \max_{1 \leqslant i \leqslant n-1} |ia_i|,$$

again with A and B given respectively by (21).

Next, we apply to $\Delta_{j,k}$ the nonvanishing condition

$$\sum_{i=1}^{n} \frac{1}{1 + |a_{ii}|/m_i'} < 1. \tag{22}$$

PROPOSITION 4. *Let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{C}[X]$ be a polynomial of degree $n \geqslant 3$. Let us fix two arbitrarily chosen integers j and k such that $0 \leqslant j < k \leqslant n-2$ and let $\beta = 1 - \frac{(n-k)! \cdot (n-k)}{[(n-j)! + (n-k)!]}$. If*

$$|a_k| \geqslant \frac{j!}{k!} \max_{1 \leqslant i \leqslant n-j-1} \binom{j+i}{i} |a_{j+i}|, \qquad |a_k| \geqslant \max_{1 \leqslant i \leqslant n-k-1} \binom{k+i}{i} |a_{k+i}|, \tag{23}$$

$$\binom{n}{j} |a_n| \geqslant \frac{k!}{j!} \max_{1 \leqslant i \leqslant n-k-1} \binom{k+i}{i} |a_{k+i}|, \quad \binom{n}{j} |a_n| \geqslant \max_{k-j+1 \leqslant i \leqslant n-j-1} \binom{j+i}{i} |a_{j+i}| \tag{24}$$

and

$$\left(\frac{j!}{k!} \cdot \frac{|a_j|}{|a_k|} + 1 - \frac{n-k}{\beta}\right) \cdot \left(\binom{n}{k} \cdot \frac{|a_n|}{|a_k|} + 1 - \frac{k-j}{\beta}\right) > \frac{n-k}{\beta} \cdot \frac{k-j}{\beta}, \tag{25}$$

then $e(f) \leqslant k$.

Proof. Note that the maximum in the second inequality of (24) makes sense only for $k \leqslant n-2$. By using (23) for the first $n-j$ columns of $\Delta_{j,k}$, and (24) for the rest of $n-k$ columns, one finds according to the definition of b_i's and c_i's that

$$m_i' = \begin{cases} k! \cdot |a_k|, & 1 \leqslant i \leqslant n-j \\ \frac{n!}{(n-j)!} \cdot |a_n|, & n-j+1 \leqslant i \leqslant 2n-j-k \end{cases}. \tag{26}$$

Therefore condition (22) applied to $\Delta_{j,k}$ reads

$$\frac{n-k}{1+\frac{j!}{k!}\cdot\frac{|a_j|}{|a_k|}}+\frac{k-j}{1+\binom{n}{k}\cdot\frac{|a_n|}{|a_k|}}+\frac{n-k}{1+\frac{(n-j)!}{(n-k)!}}<1,$$

which, using the definition of β becomes

$$\frac{n-k}{1+\frac{j!}{k!}\cdot\frac{|a_j|}{|a_k|}}+\frac{k-j}{1+\binom{n}{k}\cdot\frac{|a_n|}{|a_k|}}<\beta.$$

The proof finishes by observing that this inequality is equivalent to (25).

Conditions (23)–(25) take a simple form for $j=0$ and $k=1$, when $\beta=2/(n+1)$. In this case we obtain the following separability criterion.

COROLLARY 4. *A complex polynomial $f(X)=a_0+a_1X+\cdots+a_nX^n$ of degree $n\geqslant 3$ is separable if $|a_1|\geqslant\max_{2\leqslant i\leqslant n-1}|ia_i|$, $|a_n|\geqslant\max_{2\leqslant i\leqslant n-1}|ia_i|$ and*

$$\left(2\frac{|a_0|}{|a_1|}-(n^2-3)\right)\cdot\left(2n\frac{|a_n|}{|a_1|}-(n-1)\right)>(n-1)(n+1)^2. \tag{27}$$

For a fixed, arbitrarily chosen $\varepsilon>0$, one may replace (27) by

$$|a_n|>\frac{(\varepsilon+1)(n-1)}{2n}\cdot|a_1| \quad\text{and}\quad |a_0|>\frac{(n+1)^2+\varepsilon(n^2-3)}{2\varepsilon}\cdot|a_1|. \tag{28}$$

Here too, some particular values of ε lead to simpler conditions for separability. Thus, for $\varepsilon=1$ we obtain that f is separable if

$$|a_0|>(n^2+n-1)|a_1|,\ |a_n|>\frac{n-1}{n}|a_1| \text{ and } \min\{|a_1|,|a_n|\}\geqslant\max_{2\leqslant i\leqslant n-1}|ia_i|,$$

while for $\varepsilon=(n+1)/(n-1)$ we find f separable if

$$|a_0|>(n^2-2)|a_1| \text{ and } |a_n|>|a_1|\geqslant\max_{2\leqslant i\leqslant n-1}|ia_i|.$$

Several other more complicated conditions giving a specified bound for the multiplicities of the roots, may be obtained using (6), (10) and (11). We end this section by noting that one may obtain similar results by using the same arguments above for the reciprocal $X^nf(1/X)$ instead of $f(X)$. For instance, instead of Corollary 1 and Corollary 2, one finds that $f(X)$ is separable if

$$|a_n|>\sum_{i=0}^{n-1}|a_i| \quad\text{and}\quad |a_0|>\frac{1}{n}\sum_{i=1}^{n-1}(n-i)|a_i|,$$

respectively if

$$\begin{aligned}|a_n| &> (n-1)|a_1|+\sum_{i=2}^{n-1}(n+1-i)|a_i| \quad\text{and}\\ |a_0| &> \max\left\{\frac{1}{n}\sum_{i=1}^{n-1}(n+1-i)|a_i|,\ \frac{1}{n-1}\sum_{i=1}^{n-2}(n+1-i)|a_i|\right\}.\end{aligned}$$

3. Further results for integral polynomials

Let p be a prime number. For a non-zero integer x we shall denote by $\omega_p(x)$ the exponent of p in the prime decomposition of x ($\omega_p(0)=\infty$). The results in this section rely on the following basic lemma.

LEMMA 1. *Let $A=(a_{ij})$ be a $n\times n$ matrix with integer entries. If all the elements on the main diagonal satisfy*

$$\omega_p(a_{ii}) < \omega_p(a_{ij}) \text{ for } j<i \text{ and } \omega_p(a_{ii}) \leqslant \omega_p(a_{ij}) \text{ for } j>i, \tag{29}$$

then $\det(A)\neq 0$. *The same conclusion holds if we replace (29) by one of the following three conditions:*

$$\omega_p(a_{ii}) \leqslant \omega_p(a_{ji}) \text{ for } j<i \text{ and } \omega_p(a_{ii}) < \omega_p(a_{ji}) \text{ for } j>i, \tag{30}$$

$$\omega_p(a_{ii}) \leqslant \omega_p(a_{ij}) \text{ for } j<i \text{ and } \omega_p(a_{ii}) < \omega_p(a_{ij}) \text{ for } j>i, \tag{31}$$

$$\omega_p(a_{ii}) < \omega_p(a_{ji}) \text{ for } j<i \text{ and } \omega_p(a_{ii}) \leqslant \omega_p(a_{ji}) \text{ for } j>i. \tag{32}$$

Proof. For every permutation σ of $\{1,\dots,n\}$ with $\sigma\neq id.$, the corresponding term $x_\sigma = a_{1\sigma(1)}\cdots a_{n\sigma(n)}$ appearing in the formula of $\det(A)$ must have at least one component $a_{i\sigma(i)}$ situated below the main diagonal. Therefore $\omega_p(x_\sigma) > \omega_p(a_{11}\cdots a_{nn})$ for every $\sigma\neq id.$, which prevents $\det(A)$ from vanishing. The proof is similar for the remaining cases (30)–(32).

The following results give bounds for the multiplicities of the roots, in particular separability criteria, for some classes of integral polynomials in terms of the prime decomposition of their coefficients.

PROPOSITION 5. *Let $f(X)=a_0+a_1X+\cdots+a_nX^n\in\mathbb{Z}[X]$ be a polynomial of degree $n\geqslant 2$, and let us fix two arbitrarily chosen integers j and k such that $0\leqslant j<k<n$. If there exists a prime number p such that*

$$\begin{aligned} \omega_p(a_j) &\leqslant \min_{1\leqslant i\leqslant n-j} \omega_p\left(\tbinom{j+i}{i}a_{j+i}\right), \\ \omega_p\left(\tbinom{n}{k}a_n\right) &< \min_{0\leqslant i\leqslant n-k-1} \omega_p\left(\tbinom{k+i}{i}a_{k+i}\right), \end{aligned} \tag{33}$$

or

$$\begin{aligned} \omega_p(a_j) &< \min_{1\leqslant i\leqslant n-j} \omega_p\left(\tbinom{j+i}{i}a_{j+i}\right) \\ \omega_p\left(\tbinom{n}{k}a_n\right) &\leqslant \min_{0\leqslant i\leqslant n-k-1} \omega_p\left(\tbinom{k+i}{i}a_{k+i}\right), \end{aligned} \tag{34}$$

then $e(f)\leqslant k$.

Proof. The conclusion follows by observing that conditions (33) and (34) coincide with conditions (29) and (31) applied to $\Delta_{j,k}$, respectively.

In particular, for $j=0$ and $k=1$ we obtain:

COROLLARY 5. *Let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{Z}[X]$ be a polynomial of degree $n \geqslant 2$. If there exists a prime number p such that*

$$\omega_p(a_0) \leqslant \min_{1\leqslant i\leqslant n} \omega_p(a_i) \text{ and } \omega_p(na_n) < \min_{1\leqslant i\leqslant n-1} \omega_p(ia_i),$$

or

$$\omega_p(a_0) < \min_{1\leqslant i\leqslant n} \omega_p(a_i) \text{ and } \omega_p(na_n) \leqslant \min_{1\leqslant i\leqslant n-1} \omega_p(ia_i),$$

then $f(X)$ is a separable polynomial.

REMARK In the case when f is a primitive polynomial one may replace the above condition $\omega_p(a_0) < \min_{1\leqslant i\leqslant n} \omega_p(a_i)$ by the conditions $p \mid a_1, \ldots, p \mid a_n$ and $p \nmid a_0$.

PROPOSITION 6. *Let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{Z}[X]$ be a polynomial of degree $n \geqslant 3$, and let us fix two arbitrarily chosen integers j and k such that $0 \leqslant j < k < n$. If there exists a prime number p such that*

$$\begin{aligned}
\omega_p(a_j) &\leqslant \min_{1\leqslant i\leqslant n-k-1} \omega_p\left(\binom{j+i}{i}a_{j+i}\right), \\
\omega_p\left(\frac{k!}{j!}\binom{n}{k}a_n\right) &\leqslant \min_{1\leqslant i\leqslant n-j} \omega_p\left(\binom{j+i}{i}a_{j+i}\right), \\
\omega_p(a_j) &< \min_{0\leqslant i\leqslant n-k-1} \omega_p\left(\frac{k!}{j!}\binom{k+i}{i}a_{k+i}\right), \\
\omega_p\left(\binom{n}{k}a_n\right) &< \min_{0\leqslant i\leqslant n-k-1} \omega_p\left(\binom{k+i}{i}a_{k+i}\right),
\end{aligned}$$

then $e(f) \leqslant k$.

Proof. The conclusion follows by applying conditions (30) to $\Delta_{j,k}$.

We remark that conditions (32) are not suitable to obtain an analog of Proposition 6, since they would require $\omega_p\left(\frac{k!}{j!}\binom{n}{k}a_n\right) < \min_{1\leqslant i\leqslant n-j} \omega_p\left(\binom{j+i}{i}a_{j+i}\right)$, which can not hold, since $\omega_p(\frac{n!}{(n-k)!}a_n) \geqslant \omega_p(\frac{n!}{(n-j)!}a_n)$.

COROLLARY 6. *Let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{Z}[X]$ be a polynomial of degree $n \geqslant 3$. If there exists a prime number p such that*

$$\begin{aligned}
\omega_p(a_0) &\leqslant \min_{1\leqslant i\leqslant n-2} \omega_p(a_i), \quad \omega_p(na_n) \leqslant \min_{1\leqslant i\leqslant n} \omega_p(a_i), \\
\omega_p(a_0) &< \min_{1\leqslant i\leqslant n-1} \omega_p(ia_i), \quad \omega_p(na_n) < \min_{1\leqslant i\leqslant n-1} \omega_p(ia_i),
\end{aligned}$$

then $f(X)$ is a separable polynomial. In particular, the polynomial f is separable if $p \mid a_1, \ldots, p \mid a_{n-1}$ and $p \nmid na_0a_n$.

We end by noting that the results in this section may be adapted to polynomials with coefficients in more general unique factorization domains.

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