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SOME OPERATIONS PRESERVING LOG–CONCAVITY OF NONNEGATIVE FUNCTIONS

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Abstract. In this paper we introduce the notions of k-*log-concave* and $k,[p,q]$-*concave functions*, as generalizations of *log-concave* and *concave sequences*, respectively. We also present sufficient conditions under which some operations with such functions yield k-log-concave functions. In particular, we infer that the convolution of a positive 1-log-concave sequence and a nonnegative sequence of $k+1$ elements is k-log-concave.

1. Introduction

A finite sequence of nonnegative real numbers $(a_0, a_1, a_2, ..., a_n)$ is said to be:

- *unimodal* if there is some $k \in \{0,1,...,n\}$, called the *mode* of the sequence, such that $a_0 \leqslant ... \leqslant a_{k-1} \leqslant a_k \geqslant a_{k+1} \geqslant ... \geqslant a_n$;
- *logarithmically concave* (*log-concave* for short), whenever $a_i^2 \geqslant a_{i-1} \cdot a_{i+1}$ for $i \in \{1,2,...,n-1\}$;
- *concave,* if $2 \cdot a_i \geqslant a_{i-1} + a_{i+1}$ for $i \in \{1,2,...,n-1\}$;
- *without internal zeroes* if there do not exist integers $0 \leqslant i < j < k \leqslant n$ satisfying $a_i \neq 0, a_j = 0, a_k \neq 0$.

Let us notice several simple facts on log-concave sequences.

$(F1)$ The sequence $\{1,0,0,1,0,0,1,0,0,1\}$ is log-concave, but it is not unimodal.

$(F2)$ By introducing $m \geqslant 2$ consecutive zeroes between the elements of a log-concave sequence one gets a log-concave sequence.

$(F3)$ Any concave sequence of nonnegative numbers is log-concave.

$(F4)$ Any log-concave sequence of positive numbers $(a_i)_{0 \leqslant i \leqslant n}$ is unimodal, because, for any $1 \leqslant i \leqslant n-1$, the inequality $a_i^2 \geqslant a_{i-1} \cdot a_{i+1} \geqslant \min\{a_{i-1}^2, a_{i+1}^2\}$ implies: (i) $a_i \geqslant \min\{a_{i-1}, a_{i+1}\}$, and (ii) $a_{i-1} = a_{i+1}$, whenever $a_i = \min\{a_{i-1}, a_{i+1}\}$. These two conditions together assure that (a_i) is unimodal. It is easy to check that the sequence $\{1,2,5\}$ is not log-concave, but satisfies (i) and (ii), while the sequence $\{1,2,2,3,1\}$ is unimodal, but does not fulfil (ii).

$(F5)$ If the sequence of nonnegative numbers $(a_0, a_1, a_2, ..., a_n)$ is log-concave and $a_i = 0$ for some $1 \leqslant i \leqslant n-1$, then at least one of a_{i-1}, a_{i+1} is also equal to zero.

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$(F6)$ A sequence of nonnegative numbers (a_i) is log-concave if and only if any subinterval with no more than one zero $[a_i, a_{i+1}, ..., a_{i+l}], l \geqslant 0$ is log-concave and unimodal. The necessity follows from $(F4)$ ensuring the local unimodality property, and $(F5)$ claiming that the number of consecutive zeroes of (a_i) is at least two. The converse is also true, because every subinterval $[a_{i-1}, a_i, a_{i+1}]$ with more than one zero is automatically log-concave, while subintervals $[a_{i-1}, a_i, a_{i+1}]$ with no more than one zero are log-concave according to our premises.

$(F7)$ A sequence of non-negative numbers (a_i) is log-concave if and only if it can be represented as a series of nonintersecting subintervals, each of them houses a positive log-concave sequence, and between every two of them could be found at least two zeroes.

Both unimodal and log-concave sequences arise often in combinatorics, geometry, algebra etc. The reader is referred to Brenti [3] and Stanley [7] for surveys on this subject.

Since we are interested in operations with sequences that preserve log-concavity, recall a number of known results.

THEOREM 1.1. (i) (Berenstein and Vainshtein [2]) *If* (a_i) *is a positive sequence,* (a_{i-1}/a_i) *is log-concave,* (b_i) *is a nonnegative concave sequence, and*

$$(b_i - b_{i-1})/(a_{i-1}/a_i - a_{i-2}/a_{i-1}),\ i \geqslant 1$$

is non-increasing, then the sequence $c_k = \left(\sum_{i=0}^{k} a_i \cdot b_i\right) / \left(\sum_{i=0}^{k} a_i\right)$ *is concave, and consequently, log-concave.*

(ii) (Godsil [4]) *Let* $\sum a_i \cdot x^i$ *be a polynomial of degree* n *with all its zeroes real. Then the coefficients* $a_i/\binom{n}{i}$ *form a log-concave sequence. If the coefficients* (a_i) *are also nonnegative, then they form a log-concave sequence.*

(iii) (Wang [8]) *Let* m, n *be two nonnegative integers,* (a_i) *is a nonnegative log-concave sequence without internal zeroes. Then the sequence* $c_k = \sum_{i=0}^{k} \binom{m+k}{n+i} \cdot a_i$ *is log-concave too.*

The *convolution* of two sequences $(a_i)_{0 \leqslant i \leqslant n}, (b_j)_{0 \leqslant j \leqslant m}$ is the sequence denoted by $(c_k) = (a_i) * (b_j)$ and defined by $c_k = a_0 \cdot b_k + a_1 \cdot b_{k-1} + ... + a_k \cdot b_0$, where $a_i = 0$ and $b_j = 0$ for $i > n$ and $j > m$. The convolution of two positive sequences, one log-concave and the other only unimodal, is not always log-concave, for instance, $(a_i)_{0 \leqslant i \leqslant 2} = (1, 4, 13)$, $(b_j)_{0 \leqslant j \leqslant 3} = (1, 2, 3, 15)$ and $(a_i) * (b_j) = (1, 6, 24, 53, 99, 195)$ is not log-concave, because $99^2 < 53 \cdot 195$.

In [6], Keilson and Gerber claim that the nonnegative sequence (a_i) is log-concave if and only if $(a_i) * (b_j)$ is unimodal for every nonnegative unimodal sequence (b_j). The following example shows that it is not true: $(a_i) = (1, 0, 0, 1), (b_j) = (1, 1), (a_i) * (b_j) = (1, 1, 0, 1, 1)$. Nevertheless, a slight reformulation of the claim saves the main intention of Keilson and Gerber. Moreover, the only change in their original proof is the remark that if $(a_i) * (b_j)$ is unimodal for any nonnegative unimodal sequence (b_j), then (a_i) should be without internal zeroes, otherwise $(a_i) * (1) = (a_i)$ is not unimodal.

THEOREM 1.2. *The nonnegative sequence (a_i) is log-concave without internal zeroes if and only if the convolution $(a_i) * (b_j)$ is unimodal for every nonnegative unimodal sequence (b_j).*

COROLLARY 1.3. *The convolution of two nonnegative log-concave sequences without internal zeroes is a nonnegative log-concave sequence without internal zeroes.*

If one of the nonnegative log-concave sequences $(a_i),(b_j)$ is with internal zeroes, then $(a_i) * (b_j)$ is not necessarily log-concave, e.g., the sequences $(a_i)_{0\leqslant i\leqslant 3} = (1,0,0,1), (b_i)_{0\leqslant i\leqslant 1} = (1,1)$ are log-concave, but $(a_i) * (b_j) = (1,1,0,1,1)$ is not log-concave. Moreover, even if the convolution is positive it may not be log-concave, e.g., the sequences $(a_i)_{0\leqslant i\leqslant 4} = (1,1,0,0,1), (b_i)_{0\leqslant i\leqslant 2} = (1,1,1)$ are log-concave, but $(a_i) * (b_j) = (1,2,2,1,1,1,1)$ is not log-concave.

Let us notice that $(a_i)_{0\leqslant i\leqslant 4}=(1,7,4,2,1)$ is log-concave, $(b_i)_{0\leqslant i\leqslant 4}=(1,2,3,20,1)$ is unimodal, but not log-concave, while the sequence $(a_i \cdot b_i) = (1,14,12,40,1)$ is neither log-concave, nor unimodal.

PROPOSITION 1.4. [4] *If $(a_i),(b_i)$ are two log-concave sequences, then the sequence $(a_i \cdot b_i)$ is log-concave.*

The sum of two nonnegative log-concave sequences is not necessarily log-concave or, at least, unimodal, e.g., $(a_i)_{1\leqslant i\leqslant 5} = (1,3,5,8,1), (b_i)_{1\leqslant i\leqslant 5}=(1,7,4,2,1)$ are both log-concave, but their sum $(a_i + b_i) = (2,10,9,10,2)$ is neither log-concave, nor unimodal.

THEOREM 1.5. *(i) Let $(a_i)_{0\leqslant i\leqslant n}$ be a nonnegative unimodal (log-concave) sequence and k be a positive integer. Then the sequence $c_i = a_i + a_{i-1} + \ldots + a_{i-k}$ is unimodal (log-concave, respectively).*

(ii) If $(a_i),(b_i)$ are two nonnegative log-concave sequences, then the sequence defined by $c_i = a_i \cdot b_i + a_{i-1} \cdot b_{i-1} + \ldots + a_{i-k} \cdot b_{i-k}$ is log-concave.

(iii) If $(a_i)_{0\leqslant i\leqslant n}$ is a non-decreasing nonnegative sequence, then the sequence defined by $c_i = a_i + a_{i+1} + \ldots + a_n$ is log-concave.

Proof. (i) Let (b_i) be the sequence defined by $b_j = 1, 0 \leqslant j \leqslant k$. Then the sequence $c_i = a_i + a_{i-1} + \ldots + a_{i-k}$ is, in fact, $(c_i) = (a_s) * (b_j)$, and the result follows according to Theorem 1.2, because (b_i) is log-concave.

(ii) The result follows by combining Proposition 1.4 and part (i).

(iii) Let us notice that

$$(c_i)^2 - c_{i-1} \cdot c_{i+1} = \left(a_i + \sum_{j=i+1}^{n} a_j\right)^2 - \left(a_{i-1} + a_i + \sum_{j=i+1}^{n} a_j\right) \cdot \sum_{j=i+1}^{n} a_j$$

$$= a_i^2 + (a_i - a_{i-1}) \cdot \sum_{j=i+1}^{n} a_j \geqslant 0,$$

for $1 \leqslant i \leqslant n-1$, i.e., the sequence (c_i) is log-concave. □

The main purpose of this paper is to generalize Theorem 1.5 (ii) to the situation, when (b_i) is not a sequence of numbers, but a sequence of sequences. To implement this goal from the notational point of view, in the sequel, instead of the nonnegative sequence $(a_i), i \in \{0, 1, ..., n\}$, we make use of the function $a : \mathbb{Z} \to [0, \infty)$ where $a(i) = a_i$ for $i \in \{0, 1, ..., n\}$ and $a(i) = 0$, otherwise.

2. Results

PROPOSITION 2.1. *Let $S \subseteq \mathbb{Z}$, and $a : S \to (-\infty, \infty)$ be a function satisfying $2 \cdot a(i) \geqslant a(i-1) + a(i+1)$ for any $i \in S$.*

(i) If $S = (-\infty, q] \cap \mathbb{Z}$ and $a(i_0 + 1) - a(i_0) > 0$ holds for some $i_0 \in S$, then there exists $i_1 \in S, i_1 \leqslant i_0$, such that $a(j) < 0$, for any $j \in S, j \leqslant i_1$.

(ii) If $S = [p, +\infty) \cap \mathbb{Z}$ and $a(i_0 + 1) - a(i_0) < 0$ is valid for some $i_0 \in S$, then there exists $i_1 \in S, i_0 \leqslant i_1$, so that $a(j) < 0$, for any $j \in S, i_1 \leqslant j$.

(iii) Suppose that $S = \mathbb{Z}$ and, in addition, $a(i) \geqslant 0$ for any $i \in S$. Then there is some $b \geqslant 0$, such that $a(i) = b$, for any $i \in S$.

Proof. (i) Let us denote $u = a(i_0 + 1) - a(i_0) > 0$. Then $2 \cdot a(i_0) \geqslant a(i_0 - 1) + a(i_0 + 1)$ is equivalent to $a(i_0 + 1) - a(i_0) \leqslant a(i_0) - a(i_0 - 1)$, which assures that $a(i_0 - 1) \leqslant a(i_0) - u$. Further, $a(i_0 - 2) \leqslant a(i_0 - 1) - u \leqslant a(i_0) - 2u$, and since $u > 0$, there must be some $i_1 = i_0 - s \leqslant i_0$, such that $a(i_1) \leqslant a(i_0) - s \cdot u < 0$. Hence, it follows that $a(j) < 0$, for any $j \in S, j \leqslant i_1$.

(ii) Let us denote $w = a(i_0 + 1) - a(i_0) < 0$. Then, the inequality

$$2 \cdot a(i_0 + 1) \geqslant a(i_0) + a(i_0 + 2)$$

leads to $a(i_0 + 2) \leqslant a(i_0 + 1) + w = a(i_0) + 2w$. Further,

$$a(i_0 + 3) \leqslant 2a(i_0 + 2) - a(i_0 + 1) \leqslant a(i_0 + 2) + w \leqslant a(i_0) + 3w,$$

and since $w < 0$, there must be some $i_1 = i_0 + s \geqslant i_0$, such that $a(i_1) \leqslant a(i_0) + s \cdot w < 0$. Hence, we infer that $a(j) < 0$, for any $j \in S, j \geqslant i_1$.

(iii) It follows from parts (i) and (ii), because $a(i) \geqslant 0$ for any $i \in S$. $\square$

If $p, q \in \mathbb{Z}, p < q$, then by $[p, q]$ we mean the set $\{i : p \leqslant i \leqslant q, i \in \mathbb{Z}\}$. Let k be a positive integer. We say that the function $a : \mathbb{Z} \to [0, \infty)$ is:

- *without internal zeroes* if it satisfies the following conditions: $a(i) = 0 \neq a(i+1)$ implies $a(i-s) = 0$, and also $a(j-1) \neq 0 = a(j)$ implies $a(j+s) = 0$, for any positive integer s;
- *k-log-concave* if $(a(i))^2 \geqslant a(i-k) \cdot a(i+k)$ is true for any $i \in \mathbb{Z}$;
- *$k, [p, q]$-concave* if $2 \cdot a(i) \geqslant a(i-k) + a(i+k)$ holds for any $i \in [p+k, q-k]$, and $a(j) = 0$ whenever $j < p$ or $j > q$. The numbers p, q may naturally get infinite values $-\infty, +\infty$ too. It should be taken into account that to care of infinite values correctly, one has to accept that $-\infty + k = -\infty$, and $+\infty - k = +\infty$.

Let us notice that:

- according to Proposition 2.1, any $1, (-\infty, +\infty)$-concave function $a : \mathbb{Z} \to [0, \infty)$ must be a constant; an $1, (-\infty, q]$-concave function is monotone non-increasing, while a $1, [p, +\infty)$-concave function is monotone non-decreasing;
- the function $a : \mathbb{Z} \to [0, \infty), a(j) = a(i) = 0$ for $j \leqslant 0, i \geqslant 5$, and $a(1) = a(4) = 1, a(2) = a(3) = 0$, is 1-log-concave with internal zeroes, but it is not $1, [1, 4]$-

concave; (actually, every 1-log-concave function with internal zeroes in $[p,q]$ is not $k,[p,q]$-concave for $k=1, q-p\geqslant 2$, and for $k=2, q-p\geqslant 4$);

• the function $a:\mathbb{Z}\to[0,\infty), a(j)=a(i)=0$ for $j\leqslant 0, i\geqslant 5$, and $a(1)=a(4)=2, a(2)=a(3)=1$, is $2,[1,4]$-concave, but it is not $1,[1,4]$-concave;

• the function $a:\mathbb{Z}\to[0,\infty), a(j)=a(i)=0$ for $j\leqslant 0, i\geqslant 6$, and $a(1)=a(5)=2, a(2)=a(4)=1, a(3)=3$, is $2,[1,5]$-concave, but it is not $1,[1,5]$-concave.

In [1], Bender and Canfield claim that a nonnegative log-concave sequence $(b_n)_{n\geqslant 0}$ satisfies the condition that $b_j\cdot b_k\geqslant b_{j-1}\cdot b_{k+1}$ for all $j\leqslant k$, if and only if the equality $b_n=0$ for some integer n implies $b_k=0$ for all $k>n$. The following example shows that it is not always true: $b_0=0, b_1=1, b_2=1, b_3=0,\ldots$, because while the inequality $b_j\cdot b_k\geqslant b_{j-1}\cdot b_{k+1}$ is true for all $j\leqslant k$, the sequence begins with zero and has positive elements in its sequel. A correct characterization of nonnegative log-concave functions without internal zeroes is presented in Lemma 2.2 (ii) (in [5], Karlin names a log-concave sequence without internal zeroes a *one-sided Pólya frequency sequence of order* 2).

LEMMA 2.2. *For any function* $a:\mathbb{Z}\to[0,\infty)$ *the following assertions hold:*
(i) *if* a *is* $k,[p,q]$*-concave, then it is* k*-log-concave;*
(ii) a *is* 1*-log-concave without internal zeroes if and only if*

$$a(j)\cdot a(i)\geqslant a(j-k)\cdot a(i+k)$$

holds for any $j\leqslant i, k\geqslant 1$. *Consequently,* a *is also* k*-log-concave, for any* $k\geqslant 1$.
(iii) *if* a *is* $1,[p,q]$*-concave and* $[j-k,i+k]\subseteq[p,q]$ *for some* $k\geqslant 1$*, then*

$$a(j)+a(i)\geqslant a(j-k)+a(i+k)$$

holds for any $j\leqslant i$*, and consequently,* a *is also* $k,[p,q]$*-concave.*

Proof. (i) The well-known inequalities

$$2\cdot a(i)\geqslant a(i-k)+a(i+k)\geqslant 2\cdot\sqrt{a(i-k)\cdot a(i+k)}$$

lead to $(a(i))^2\geqslant a(i-k)\cdot a(i+k)$, for any $i\in[p+k,q-k]$. In addition, if $i<p+k$, then $a(i-k)=0$, while for $i>q-k$, one has $a(i+k)=0$ and, therefore, in the both cases we infer that $(a(i))^2\geqslant a(i-k)\cdot a(i+k)$.

(ii) *If-part*: Since a is 1-log-concave, we get successively that

$$\begin{aligned}
a(j)\cdot a(j)&\geqslant a(j-1)\cdot a(j+1),\\
a(j+1)\cdot a(j+1)&\geqslant a(j)\cdot a(j+2),\\
\ldots&\geqslant\ldots,\\
a(i-1)\cdot a(i-1)&\geqslant a(i-2)\cdot a(i),\\
a(i)\cdot a(i)&\geqslant a(i-1)\cdot a(i+1),
\end{aligned}$$

which implies that

$$\begin{aligned}
&a(j)^2\cdot a(j+1)^2\ldots\cdot\ a(i-1)^2\cdot a(i)^2\geqslant\\
&\qquad\geqslant a(j-1)\cdot a(j)\cdot a(j+1)^2\cdot\ldots\cdot\ a(i-1)^2\cdot a(i)\cdot a(i+1),
\end{aligned}$$

and consequently, we obtain

$$a(j)\cdot a(j+1)^2\cdot ...\cdot\ a(i-1)^2\cdot a(i)\cdot(a(j)\cdot a(i)-a(j-1)\cdot a(i+1))\geqslant 0.$$

Therefore, $a(j)\cdot a(i)\geqslant a(j-1)\cdot a(i+1)$ is true, because a is without internal zeroes. Now, using k times this inequality, we conclude with

$$a(j)\cdot a(i)\geqslant a(j-1)\cdot a(i+1)\geqslant a(j-2)\cdot a(i+2)\geqslant ...\geqslant a(j-k)\cdot a(i+k).$$

Only-If-part: Since $a(j)\cdot a(i)\geqslant a(j-k)\cdot a(i+k)$ for any $j\leqslant i, 0\leqslant k$, the inequality is true for $j=i, k=1$ too, i.e., a is log-concave. Now, suppose that the function a owns internal zeroes, i.e., for some $j<i$ we have $a(j-1)\neq 0, a(j)=0, ..., a(i)=0$, and $a(i+1)\neq 0$. Consequently, $a(j)\cdot a(i)=0<a(j-1)\cdot a(i+1)$, which contradicts the hypothesis.

In particular, for $i=j$, we infer that $a(i)\cdot a(i)\geqslant a(i-k)\cdot a(i+k)$, i.e., the function a is k-log-concave.

(iii) Since a is $1,[p,q]$-concave, for any $j,i\in[p+1,q-1], j\leqslant i$, we get successively that

$$\begin{aligned} 2\cdot a(j)&\geqslant a(j-1)+a(j+1),\\ 2\cdot a(j+1)&\geqslant a(j)+a(j+2),\\ ...&\geqslant ...,\\ 2\cdot a(i-1)&\geqslant a(i-2)+a(i),\\ 2\cdot a(i)&\geqslant a(i-1)+a(i+1), \end{aligned}$$

which implies that

$$a(j)+a(i)\geqslant a(j-1)+a(i+1).$$

Consequently, for $[j-k,i+k]\subseteq[p,q]$ we obtain

$$a(j)+a(i)\geqslant a(j-1)+a(i+1)\geqslant a(j-2)+a(i+2)\geqslant ...\geqslant a(j-k)+a(i+k).$$

In particular, for $i=j$, we infer that $2a(i)\geqslant a(i-k)+a(i+k)$, i.e., the function a is $k,[p,q]$-concave. □

Let us notice that the functions $a,b:\mathbb{Z}\rightarrow[0,\infty), a(j)=a(i)=b(j)=b(i)=0$ for $j\leqslant 0, i\geqslant 4, a(1)=2, a(2)=4, a(3)=6, b(1)=1, b(2)=3, b(3)=5$, are $1,[1,3]$-concave, but $a\cdot b$ it is not $1,[1,3]$-concave, because

$$2\cdot a(2)\cdot b(2)=24<32=a(1)\cdot b(1)+a(3)\cdot b(3).$$

However, under certain conditions, the product of two $k,[p,q]$-concave functions is still $k,[p,q]$-concave.

LEMMA 2.3. *If x,y,z,u,v,w are nonnegative real numbers such that*

$$2\cdot x\geqslant y+z,\quad 2\cdot u\geqslant w+v\quad \textit{and}\quad (z-y)\cdot(w-v)\geqslant 0,$$

then

$$2\cdot x\cdot u\geqslant y\cdot w+z\cdot v.$$

Proof. The inequality $(z-y)\cdot(w-v) \geqslant 0$ is equivalent to $z\cdot w+y\cdot v \geqslant y\cdot w+z\cdot v$. Consequently, we obtain

$$4\cdot x\cdot u \geqslant (y+z)\cdot(w+v) = z\cdot w+y\cdot v+y\cdot w+z\cdot v \geqslant 2(y\cdot w+z\cdot v)$$

i.e., $2\cdot x\cdot u \geqslant y\cdot w+z\cdot v$. □

PROPOSITION 2.4. *If the functions $a,b:\mathbb{Z}\to[0,\infty)$ are $k,[p,q]$-concave and*

$$(a(i+k)-a(i-k))\cdot(b(i-k)-b(i+k)) \geqslant 0,$$

for any $i\in[p+k,q-k]$, then $a\cdot b$ is also $k,[p,q]$-concave.

Proof. For $i\in[p+k,q-k]$, let us denote

$$x=a(i),\ y=a(i-k),\ z=a(i+k),\ u=b(i),\ w=b(i-k),\ v=b(i+k).$$

Since

$$2\cdot a(i) \geqslant a(i-k)+a(i+k), \quad 2\cdot b(i) \geqslant b(i-k)+b(i+k),$$

Lemma 2.3 implies

$$2\cdot a(i)\cdot b(i) \geqslant a(i-k)\cdot b(i-k)+a(i+k)\cdot b(i+k).$$

In other words, the function $a\cdot b$ is $k,[p,q]$-concave. □

THEOREM 2.5. *If $a:\mathbb{Z}\to[0,\infty)$ is a 1-log-concave function without internal zeroes, $f_j:\mathbb{Z}\to[0,\infty)$ is $k,[p,q]$-concave for any $0\leqslant j\leqslant k$, and for each $i\in[p+k,q-k]$, the differences $f_l(i+k-l)-f_l(i-k-l), 0\leqslant l\leqslant k$, are of the same sign (i.e., either they are non-positive or they are non-negative), then the function*

$$c:\mathbb{Z}\to[0,\infty),\ c(i)=\sum_{j=0}^{k} a(i-j)\cdot f_j(i-j),$$

is k-log-concave.

Proof. Since $c(i-k)=\sum_{j=0}^{k} a(i-k-j)\cdot f_j(i-k-j)$, $c(i+k)=\sum_{j=0}^{k} a(i+k-j)\cdot f_j(i+k-j)$, we get

$$\begin{aligned}
&(c(i))^2-c(i-k)\cdot c(i+k)= \\
&\quad=\sum_{j=0}^{k}(a(i-j)\cdot f_j(i-j))^2+2\cdot\sum_{0\leqslant t<u\leqslant k} a(i-u)\cdot a(i-t)\cdot f_u(i-u)\cdot f_t(i-t) \\
&\quad\quad-\left(\sum_{u=0}^{k} a(i-k-u)\cdot f_u(i-k-u)\right)\cdot\left(\sum_{t=0}^{k} a(i+k-t)\cdot f_t(i+k-t)\right) \\
&\quad=\sum_{j=0}^{k}(a(i-j))^2\cdot(f_j(i-j))^2-\sum_{j=0}^{k} a(i-k-j)\cdot a(i+k-j)\cdot f_j(i-k-j)\cdot f_j(i+k-j) \\
&\quad\quad+2\cdot\sum_{0\leqslant t<u\leqslant k} a(i-u)\cdot a(i-t)\cdot f_u(i-u)\cdot f_t(i-t) \\
&\quad\quad-\sum_{0\leqslant t\neq u\leqslant k} a(i-k-u)\cdot a(i+k-t)\cdot f_u(i-k-u)\cdot f_t(i+k-t).
\end{aligned}$$

Lemma 2.2 (ii) ensures that a is k-log-concave, because a is 1-log-concave without internal zeroes, and therefore,

$$(a(i-j))^2 \geqslant a(i-k-j) \cdot a(i+k-j),$$

while according to Lemma 2.2 (i), the $k,[p,q]$-concavity of f_j says that

$$(f_j(i-j))^2 \geqslant f_j(i-k-j) \cdot f_j(i+k-j).$$

Consequently, we obtain

$$(a(i-j))^2 \cdot (f_j(i-j))^2 \geqslant a(i-k-j) \cdot a(i+k-j) \cdot f_j(i-k-j) \cdot f_j(i+k-j).$$

Here, to get the result, it is sufficient to show that

$$\begin{aligned} 2 \cdot a(i-u) \cdot a(i-t) \cdot f_u(i-u) \cdot f_t(i-t) & \\ \geqslant a(i-k-u) \cdot a(i+k-t) \cdot f_u(i-k-u) \cdot f_t(i+k-t) & \\ + a(i-k-t) \cdot a(i+k-u) \cdot f_t(i-k-t) \cdot f_u(i+k-u) & \end{aligned}$$

holds for any $u > t$. According to Lemma 2.2 (ii), it follows that

$$a(i-u) \cdot a(i-t) \geqslant a(i-k-u) \cdot a(i+k-t),$$

since $i-t > i-u$. On the other hand, the inequality $0 \leqslant t < u \leqslant k$ implies

$$(i-u)-(i-k-t) = (i+k-u)-(i-t) = k-(u-t) \geqslant 0.$$

Now, the log-concavity without internal zeroes of the function a together with Lemma 2.2 (ii) assure that

$$a(i-u) \cdot a(i-t) \geqslant a(i-k-t) \cdot a(i+k-u).$$

Further, the fact that all the differences

$$f_l(i+k-l) - f_l(i-k-l), 0 \leqslant l \leqslant k,$$

are of the same sign assures that

$$(f_u(i+k-u) - f_u(i-k-u)) \cdot (f_t(i+k-t) - f_t(i-k-t)) \geqslant 0$$

for any $0 \leqslant t, u \leqslant k$. Now, choosing

$$\begin{aligned} & x = f_u(i-u),\ y = f_u(i-k-u),\ z = f_u(i+k-u), \\ & u = f_t(i-t),\ w = f_t(i+k-t),\ v = f_t(i-k-t), \end{aligned}$$

we see that Lemma 2.3 implies the inequality

$$2 \cdot f_u(i-u) \cdot f_t(i-t) \geqslant f_u(i-k-u) \cdot f_t(i+k-t) + f_t(i-k-t) \cdot f_u(i+k-u) \qquad (*)$$

for $p+k \leqslant i \leqslant q-k$.

Actually, the inequality (*) is true for all integers i, since in accordance with the definition of $k,[p,q]$-concavity, if $i \leqslant p+k-1$, then $f_u(i-k-u) = 0 = f_t(i-k-t)$, and if $i \geqslant q-k+1$, then $f_t(i+k-t) = 0 = f_u(i+k-u)$. □

EXAMPLE 1. The function $a:\mathbb{Z}\to[0,\infty)$, defined by

$$a(1)=a(2)=a(3)=a(4)=a(5)=1, \text{and } a(k)=0 \text{ otherwise},$$

is 1-log-concave; the functions $f_j:\mathbb{Z}\to[0,\infty)$, defined by

$$f_j(1)=f_j(3)=f_j(5)=1,\ f_j(2)=f_j(4)=0,$$

and

$$f_j(i)=0,\ i\in\mathbb{Z}-\{1,2,3,4,5\},\ 0\leqslant j\leqslant 2,$$

are $2,[1,5]$-concave, for any $0\leqslant j\leqslant 2$. According to Theorem 2.5, the function

$$c:\mathbb{Z}\to[0,\infty),\ c(i)=\sum_{j=0}^{2}a(i-j)\cdot f_j(i-j)$$

is 2-log-concave, while it is not 1-log-concave: $c(\mathbb{Z})=\{...,0,0,1,1,\mathbf{2},\mathbf{1},\mathbf{2},1,1,0,0,...\}$.

EXAMPLE 2. Let us point out that the function $a:\mathbb{Z}\to[0,\infty)$, defined by

$$a(1)=a(2)=a(3)=a(5)=1, a(4)=10, \text{and } a(k)=0 \text{ otherwise},$$

is 2-log-concave, but not 1-log-concave; the functions

$$f_j:\mathbb{Z}\to[0,\infty),\ f_j(i)=1,\ 1\leqslant i\leqslant 5 \text{ and } f_j(i)=0,\ i\in\mathbb{Z}-\{1,2,3,4,5\},$$

are $2,[1,5]$-concave (actually, they are also $1,[1,5]$-concave), for any $0\leqslant j\leqslant 2$, but the function

$$c:\mathbb{Z}\to[0,\infty),\ c(i)=\sum_{j=0}^{2}a(i-j)\cdot f_j(i-j)=a(i)+a(i-1)+a(i-2),$$

is not 2-log-concave, because $c(\mathbb{Z})=\{...,0,0,\mathbf{1},2,\mathbf{3},12,\mathbf{12},0,0,...\}$.

COROLLARY 2.6. *If $a:\mathbb{Z}\to[0,\infty)$ is a 1-log-concave function without internal zeroes, $f_j:\mathbb{Z}\to[0,\infty)$ is $k,[p,q]$-concave and monotone non-decreasing (non-increasing) in $[p,q]$, for any $0\leqslant j\leqslant k$, then the function*

$$c:\mathbb{Z}\to[0,\infty),\ c(i)=\sum_{j=0}^{k}a(i-j)\cdot f_j(i-j),$$

is k-log-concave.

COROLLARY 2.7. *If $a:\mathbb{Z}\to[0,\infty)$ is a 1-log-concave function without internal zeroes, $b:\mathbb{Z}\to[0,\infty)$ is an arbitrary function, and k is a positive integer, then the function*

$$c:\mathbb{Z}\to[0,\infty),\ c(i)=\sum_{j=0}^{k}a(i-j)\cdot b(j),$$

is k-log-concave.

The following result extends Theorem 1.2 in a way that, while the convolution of a 1-log-concave sequence and an arbitrary sequence is not always 1-log-concave, it is still possible to reveal an interesting structure of log-concavity type in this convolution.

COROLLARY 2.8. *If the nonnegative sequence (a_i) is 1-log-concave without internal zeroes, then the convolution $(a_i) * (b_j)$ is k-log-concave for any nonnegative sequence (b_j) containing $k+1$ elements.*

The following example makes clear some obstacles on the way to strengthen the conclusion of Corollary 2.8.

EXAMPLE 3. The convolution

$$(a_i) * (b_j) = (1,2,4,8,16,32,64) * (2,1,2) = (2,5,12,24,48,96,192,128,128)$$

is 2-log-concave, but is not 1-log-concave: $128^2 - 128 * 192 < 0$. Moreover, the equality $48^2 - 12 * 192 = 0$ means that the 2-log-concave inequality can not be improved.

3. Conclusions

In this paper we show how to build k-log-concave functions starting from a 1-log-concave function and a number of $k, [p,q]$-concave functions. We deduce that the convolution of a 1-log-concave sequence and a nonnegative sequence on $k+1$ elements is k-log-concave. In particular, for $k \in \{0,1\}$, this follows also from the result of Keilson and Gerber, [6].

It is worth mentioning that any 2-log-concave function $a : \mathbb{Z} \to [0,\infty)$ gives rise to two 1-log-concave functions, namely,

$$b, c : \mathbb{Z} \to [0,\infty), \text{ where } b(i) = a(2 \cdot i), c(i) = a(2 \cdot i + 1),$$

and vice-versa, by interlacing the values of two 1-log-concave functions one gets a 2-log-concave function. In general, any k-log-concave function a can be obtained by interlacing values of k 1-log-concave functions, and, on the other hand, f gives birth to k 1-log-concave functions. From this perspective, our main result claims that the "convolution" of a 1-log-concave function and k k-concave functions can be represented as a composition of k interlacing 1-log-concave functions.

In further research we intend to get a better understanding of our findings for the situation when two sequences of functions cooperate in order to produce a log-concave function.

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