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ON A ČEBYŠEV–TYPE FUNCTIONAL AND GRÜSS–LIKE BOUNDS

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Abstract. The classic Čebyšev functional involves the difference between the integral mean of the product of two functions and the product of the integral means of the individual functions. A Čebyšev-type functional involving the arithmetic average of the upper and lower bounds of one of the functions rather than the integral mean is examined, providing sharp Grüss-like bounds.

The current investigation is undertaken within a measurable space setting. The results are capitalised under a variety of scenarios and in particular in obtaining sharp Grüss-like bounds for perturbed rules in numerical integration.

1. Introduction

For two measurable functions $f, g : [a, b] \to \mathbb{R}$, define the functional, which is known in the literature as Čebyšev's functional, by

$$T(f, g) := \mathcal{M}(fg) - \mathcal{M}(f)\mathcal{M}(g), \tag{1.1}$$

where the integral mean is given by

$$\mathcal{M}(f) := \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.2}$$

The integrals in (1.1) are assumed to exist.

Further, the weighted Čebyšev functional is defined by

$$T(f, g; w) := \mathcal{M}(f, g; w) - \mathcal{M}(f; w)\mathcal{M}(g; w), \tag{1.3}$$

where the weighted integral mean is given by

$$\mathcal{M}(f; w) := \frac{\int_a^b w(x) f(x)\,dx}{\int_a^b w(x)\,dx}, \tag{1.4}$$

with $0 < \int_a^b w(x)\,dx < \infty$.

We note that,

$$T(f, g; 1) \equiv T(f, g) \quad \text{and} \quad \mathcal{M}(f; 1) \equiv \mathcal{M}(f).$$

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It is worthwhile noting that a number of identities relating to the Čebyšev functional already exist.The reader is referred to [18] Chapters IX and X. Korkine's identity for the Čebyšev functional is well known, see [18, p. 296] and is given by

$$T(f,g) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(x)-f(y))(g(x)-g(y))\,dxdy. \tag{1.5}$$

It is identity (1.5) that is often used to prove an inequality due to Grüss for functions bounded above and below, [18].

The Grüss inequality is given by

$$|T(f,g)| \leqslant \frac{1}{4}(\Phi_f-\phi_f)(\Phi_g-\phi_g), \tag{1.6}$$

where $\phi_f \leqslant f(x) \leqslant \Phi_f$ for $x \in [a,b]$.

If we let $S(f)$ be an operator defined by

$$S(f)(x) := f(x) - \mathscr{M}(f), \tag{1.7}$$

which shifts a function by its integral mean, then the following identities hold. Namely,

$$T(f,g) = T(S(f),g) = T(f,S(g)) = T(S(f),S(g)), \tag{1.8}$$

and so

$$T(f,g) = \mathscr{M}(S(f)g) = \mathscr{M}(fS(g)) = \mathscr{M}(S(f)S(g)) \tag{1.9}$$

since $\mathscr{M}(S(f)) = \mathscr{M}(S(g)) = 0$.

For the last term in (1.9) or (1.10) only one of the functions needs to be shifted by its integral mean. If the other were to be shifted by any other quantity, the identities would still hold. A weighted version of (1.9) related to

$$T(f,g) = \mathscr{M}((f(x)-\gamma)S(g)) \tag{1.10}$$

for γ arbitrary was given by Sonin [20] (see [18, p. 246]).

The interested reader is also referred to Dragomir [13] and Fink [15] for extensive treatments of the Grüss and related inequalities.

2. The Čebyšev functional in a measurable space setting

Let $(\Omega,\mathscr{A},\mu)$ be a measurable space consisting of a set Ω, a σ – algebra $\mathscr{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathscr{A}$ with values in $\mathbb{R}\cup\{\infty\}$.

For a μ– measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geqslant 0$ for μ – a.e. $x\in\Omega$, and $\int_\Omega w(x)\,d\mu(x) > 0$, define the Lebesgue space

$$L_w(\Omega,\mathscr{A},\mu) := \left\{ f:\Omega\to\mathbb{R}, f \text{ is } \mu- \text{ measurable and } \int_\Omega w(x)\,|f(x)|\,d\mu(x) < \infty \right\}.$$

If $f,g:\Omega\to\mathbb{R}$ are $\mu-$ measurable functions and $f,g,fg\in L_w(\Omega,\mathscr{A},\mu)$, then we may consider the Čebyšev functional

$$\begin{aligned} T_w(f,g) &= T_w(f,g;\Omega)\\ &:= \frac{1}{\int_\Omega w(x)\,d\mu(x)}\int_\Omega w(x)f(x)\,g(x)\,d\mu(x)\\ &\quad -\frac{1}{\int_\Omega w(x)\,d\mu(x)}\int_\Omega w(x)f(x)\,d\mu(x)\times\\ &\quad \times\frac{1}{\int_\Omega w(x)\,d\mu(x)}\int_\Omega w(x)\,g(x)\,d\mu(x). \end{aligned} \tag{2.1}$$

As mentioned in the introduction, under a more restrictive setting, the following result is known in the literature as the Grüss inequality

$$|T_w(f,g)|\leqslant\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta), \tag{2.2}$$

provided

$$-\infty<\gamma\leqslant f(x)\leqslant\Gamma<\infty,\quad -\infty<\delta\leqslant g(x)\leqslant\Delta<\infty \tag{2.3}$$

for μ $-$a.e. $x\in\Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

With the above assumptions and if $f\in L_w(\Omega,\mathscr{A},\mu)$ then we may define

$$\begin{aligned} D_w(f) &:= D_{w,1}(f)\\ &:= \frac{1}{\int\limits_\Omega w(x)\,d\mu(x)}\int\limits_\Omega w(x)\left|f(x)-\frac{1}{\int\limits_\Omega w(y)\,d\mu(y)}\int\limits_\Omega w(y)f(y)\,d\mu(y)\right|d\mu(x). \end{aligned} \tag{2.4}$$

The following core result was proved in [5].

THEOREM 1. *Let $w,f,g:\Omega\to\mathbb{R}$ be $\mu-$measurable functions with $w\geqslant 0$ $\mu-$a.e. on Ω and $\int_\Omega w(y)\,d\mu(y)>0$. If $f,g,fg\in L_w(\Omega,\mathscr{A},\mu)$ and there exists the constants δ,Δ such that*

$$-\infty<\delta\leqslant g(x)\leqslant\Delta<\infty \ \textit{for}\ \ \mu-a.e.\ x\in\Omega, \tag{2.5}$$

then we have the inequality

$$|T_w(f,g)|\leqslant\frac{1}{2}(\Delta-\delta)\,D_w(f). \tag{2.6}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

For $f\in L_{w,p}(\Omega,\mathscr{A},\mu):=\left\{f:\Omega\to\mathbb{R},\int_\Omega w(x)\,|f(x)|^p\,d\mu(x)<\infty\right\}$, $1\leqslant p<\infty$ and $f\in L_\infty(\Omega,\mathscr{A},\mu):=\left\{f:\Omega\to\mathbb{R},\|f\|_{\Omega,\infty}:=ess\sup_{x\in\Omega}|f(x)|<\infty\right\}$, we may also define

$$D_{w,p}(f) := \left[\frac{1}{\int\limits_{\Omega} w(x)d\mu(x)} \int\limits_{\Omega} w(x)\left| f(x) - \frac{1}{\int\limits_{\Omega} w(y)\, d\mu(y)} \int\limits_{\Omega} w(y)f(y)\, d\mu(y)\right|^p d\mu(x)\right]^{\frac{1}{p}}$$
$$= \frac{\left\| f - \frac{1}{\int_{\Omega} wd\mu} \int_{\Omega} wf\, d\mu \right\|_{\Omega,p}}{\left[\int_{\Omega} w(x)d\mu(x)\right]^{\frac{1}{p}}} \tag{2.7}$$

where $\|\cdot\|_{\Omega,p}$ is the usual $p-$ norm on $L_{w,p}(\Omega, \mathscr{A}, \mu)$, namely,

$$\|h\|_{\Omega,p} := \left(\int_{\Omega} w\,|h|^p\, d\mu\right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

and on $L_{\infty}(\Omega, \mathscr{A}, \mu)$

$$\|h\|_{\Omega,\infty} := ess \sup_{x\in\Omega} |h(x)| < \infty.$$

Further, Cerone and Dragomir [5] also proved the following result.

COROLLARY 1. *With the assumptions of Theorem* 1, *we have*

$$\begin{aligned} |T_w(f,g)| &\leqslant \frac{1}{2}(\Delta - \delta)\, D_w(f) \\ &\leqslant \frac{1}{2}(\Delta - \delta)\, D_{w,p}(f) \quad iff \in L_{w,p}(\Omega, \mathscr{A}, \mu),\ 1 < p < \infty; \\ &\leqslant \frac{1}{2}(\Delta - \delta)\left\| f - \frac{1}{\int_{\Omega} wd\mu} \int_{\Omega} wf\, d\mu \right\|_{\Omega,\infty} \quad iff \in L_{\infty}(\Omega, \mathscr{A}, \mu). \end{aligned} \tag{2.8}$$

REMARK 1. The inequalities in (2.8) are in order of increasing coarseness. If we assume that $-\infty < \gamma \leqslant f(x) \leqslant \Gamma < \infty$ for μ – a.e. $x \in \Omega$, then by the Grüss inequality for $g = f$ we have for $p = 2$

$$\left[\frac{\int_{\Omega} wf^2 d\mu}{\int_{\Omega} wd\mu} - \left(\frac{\int_{\Omega} wf\, d\mu}{\int_{\Omega} wd\mu}\right)^2\right]^{\frac{1}{2}} \leqslant \frac{1}{2}(\Gamma - \gamma). \tag{2.9}$$

By (2.8), we deduce the following sequence of inequalities

$$\begin{aligned} |T_w(f,g)| &\leqslant \frac{1}{2}(\Delta - \delta)\frac{1}{\int_{\Omega} wd\mu} \int_{\Omega} w\left| f - \frac{1}{\int_{\Omega} wd\mu} \int_{\Omega} wf\, d\mu \right| d\mu \\ &\leqslant \frac{1}{2}(\Delta - \delta)\left[\frac{\int_{\Omega} wf^2 d\mu}{\int_{\Omega} wd\mu} - \left(\frac{\int_{\Omega} wf\, d\mu}{\int_{\Omega} wd\mu}\right)^2\right]^{\frac{1}{2}} \\ &\leqslant \frac{1}{4}(\Delta - \delta)(\Gamma - \gamma) \end{aligned} \tag{2.10}$$

for $f, g : \Omega \to \mathbb{R}$, μ – measurable functions such that $-\infty < \gamma \leqslant f(x) < \Gamma < \infty$, $-\infty < \delta \leqslant g(x) \leqslant \Delta < \infty$ for μ – a.e. $x \in \Omega$. Thus the first inequality in (2.10) or (2.6) is a refinement of the third which is the Grüss inequality (2.2).

Further, (2.6) is also a refinement of the second inequality in (2.10). We note that all the inequalities in (2.8) – (2.10) are sharp.

The second inequality in (2.10) under a less general setting was termed as a pre-Grüss inequality by Matić, Pečarić and Ujević [17]. Bounds for the Čebyšev functional have been put to good use by a variety of authors in providing perturbed numerical integration rules, see for example the book [14].

3. A novel Čebyšev-like functional

If we extend the definition of the weighted integral mean (1.4) to a measurable space setting along the lines of Section 2., then we have for $f \in L_w(\Omega, \mathscr{A}, \mu)$

$$\mathscr{M}_w(f;\Omega) := \frac{1}{W(\Omega)} \int_\Omega w(x) f(x) d\mu(x), \tag{3.1}$$

with $W(\Omega) := \mathscr{M}_w(1;\Omega) = \int_\Omega w(x) d\mu(x) > 0$.

We may write the Čebyšev functional (2.1) in a more compact form for $f, g, fg \in L_w(\Omega, \mathscr{A}, \mu)$

$$T_w(f,g;\Omega) = \mathscr{M}_w(fg;\Omega) - \mathscr{M}_w(f;\Omega)\mathscr{M}_w(g;\Omega). \tag{3.2}$$

We now introduce a new Čebyšev-like functional

$$C_w(f,g;\Omega) := \mathscr{M}_w(fg;\Omega) - \frac{\Delta+\delta}{2}\mathscr{M}_w(f;\Omega), \tag{3.3}$$

where the constants δ, Δ are such that

$$-\infty < \delta \leqslant g(x) \leqslant \Delta < \infty \text{ for } \mu - \text{a.e.} \quad x \in \Omega.$$

Thus, rather than $g(\cdot)$ being represented by the integral mean, $\mathscr{M}_w(g;\Omega)$ as in (3.2), the arithmetic mean of its upper and lower bound takes its place in $C_w(f,g;\Omega)$ as defined by (3.3).

The functional $C_w(f,g;\Omega)$ as defined in (3.3) provides a variety of rich and interesting results. The following theorem holds.

THEOREM 2. *Let $w, f, g : \Omega \to \mathbb{R}$ be $\mu-$measurable functions with $w \geqslant 0$ $u-$a.e. on Ω and $\int_\Omega w(y) d(y) > 0$. If $f, g, fg \in L_w(\Omega, \mathscr{A}, \mu)$ and constants δ, Δ exist such that $-\infty < \delta \leqslant g(x) \leqslant \Delta < \infty$ for $\mu-$a.e. $x \in \Omega$, then we have*

$$\begin{aligned}
|C_w(f,g;\Omega)| &= \left| \mathscr{M}_w(fg;\Omega) - \frac{\Delta+\delta}{2}\mathscr{M}_w(f;\Omega) \right| \\
&\leqslant \frac{\Delta-\delta}{2} \cdot \frac{1}{W(\Omega)} \|f\|_{\Omega,1}, \ f \in L_{w,1}(\Omega, \mathscr{A}, \mu) \\
&\leqslant \frac{\Delta-\delta}{2} \cdot \frac{1}{W^{\frac{1}{p}}(\Omega)} \|f\|_{\Omega,p}, \ f \in L_{w,p}(\Omega, \mathscr{A}, \mu), \ 1 < p < \infty \\
&\leqslant \frac{\Delta-\delta}{2} \|f\|_{\Omega,\infty} = \frac{\Delta-\delta}{2} \max\{|\Gamma|, |\gamma|\}, \ f \in L_\infty(\Omega, \mathscr{A}, \mu),
\end{aligned} \tag{3.4}$$

provided

$$-\infty < \gamma \leqslant f(x) \leqslant \Gamma < \infty, \ \ \textit{for } \mu - \textit{a.e. } x \in \Omega,$$

where

$$L_{w,p}(\Omega, \mathscr{A}, \mu) := \left\{ h : \Omega \to \mathbb{R}, \ \int_\Omega w(x) \, |h(x)|^p \, d\mu(x) < \infty \right\}, \ 1 \leqslant p < \infty$$

and

$$L_\infty(\Omega, \mathscr{A}, \mu) := \left\{ h : \Omega \to \mathbb{R}, \ ess \sup_{x\in\Omega} |h(x)| < \infty \right\}$$

with $\|\cdot\|_{\Omega,p}$ *the* $p-$*norm on* $L_{w,p}(\Omega, \mathscr{A}, \mu)$, *namely,*

$$\|h\|_{\Omega,p} := \left(\int_\Omega w \, |h|^p \, d\mu(x) \right)^{\frac{1}{p}}, \qquad 1 \leqslant p < \infty$$

and

$$\|h\|_{\Omega,\infty} := ess \sup_{x\in\Omega} |h(x)| < \infty.$$

The $\frac{1}{2}$ *in all three inequalities in (3.4) is sharp.*

Proof. From (3.3) and using (3.1) we have the identity

$$C_w(f, g; \Omega) = \mathscr{M}_w(fg; \Omega) - \frac{\Delta + \delta}{2} \mathscr{M}_w(f; \Omega) = \mathscr{M}_w\left(f\left(g - \frac{\Delta + \delta}{2} \right) \right)$$

and so

$$C_w(f, g; \Omega) = \frac{1}{W(\Omega)} \int_\Omega w(x) f(x) \left(g(x) - \frac{\Delta + \delta}{2} \right) d\mu(x). \tag{3.5}$$

Taking the modulus of identity (3.5) gives

$$|C_w(f, g; \Omega)| \leqslant \frac{1}{W(\Omega)} \int_\Omega w(x) \, |f(x)| \left| g(x) - \frac{\Delta + \delta}{2} \right| d\mu(x). \tag{3.6}$$

Now, since $-\infty < \delta \leqslant g(x) \leqslant \Delta < \infty$, for $\mu-$a.e. $x \in \Omega$ then

$$-\frac{\Delta - \delta}{2} \leqslant g(x) - \frac{\Delta + \delta}{2} \leqslant \frac{\Delta - \delta}{2}$$

and so from (3.6)

$$|C_w(f, g; \Omega)| \leqslant \frac{\Delta - \delta}{2} \cdot \frac{1}{W(\Omega)} \int_\Omega w(x) \, |f(x)| \, d\mu(x) \tag{3.7}$$

producing the first inequality in (3.4).

We further have, using Hölder's inequality, from (3.7) for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\frac{1}{W(\Omega)} \int_\Omega w(x) \, |f(x)| \, d\mu(x) = \frac{1}{W(\Omega)} \left(\int_\Omega w(x) \cdot 1^q d\mu(x) \right)^{\frac{1}{q}} \left(\int_\Omega w(x) \, |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}$$

$$= \frac{1}{W^{1-\frac{1}{q}}(\Omega)} \left(\int_\Omega w(x) \, |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}},$$

giving the second inequality in (3.4) on noting that $1 - \frac{1}{q} = \frac{1}{p}$.

The final inequality in (3.4) follows directly from (3.7).

Now, for the sharpness of the constant $\frac{1}{2}$. Assume the first inequality in (3.4) holds with constant $K>0$. Namely,

$$\begin{aligned}\left|\int_{\Omega} w(x)f(x)g(x)d\mu(x) - \frac{\Delta+\delta}{2}\int_{\Omega} w(x)f(x)d\mu(x)\right| \\ \leqslant K(\Delta-\delta)\int_{\Omega} w(x)\left|f(x)\right| d\mu(x). \quad (3.8)\end{aligned}$$

Consider $g=f=f_0$ where $f_0:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is given by

$$f_0(x)=\begin{cases} -1, & x\in\left[a,\dfrac{a+b}{2}\right], \\ 1, & x\in\left(\dfrac{a+b}{2},b\right]. \end{cases} \tag{3.9}$$

Thus, for $w\equiv 1$, we have from (3.8)

$$\left|\int_a^b f_0^2(x)d\mu(x) - \frac{\Delta+\delta}{2}\int_a^b f_0(x)d\mu(x)\right| = b-a = \int_a^b \left|f_0(x)\right| dx$$

giving from (3.8), since $\delta=-1$, $\Delta=1$, $1\leqslant 2K$ and so $\frac{1}{2}\leqslant K$.

The same function, $f_0(x)$ from (3.9), will prove the sharpness of the last two inequalities in (3.4) or, more directly from the properties of the Hölder inequality. The theorem is now completely proved.

REMARK 2. It is interesting to compare the results of Section 2. with those of the above Theorem 2. Those of Corollary 1 obtain bounds for the Čebyšev functional $\left|T_w(f,g;\Omega)\right|$ in terms of norms of functions shifted by their integral means.

The bounds for the Čebyšev-like functional $\left|C_w(f,g;\Omega)\right|$ introduced here by (3.3), have the same coefficients to the norms however, the bound provided by (3.4) involve $\|f\|_{\Omega,p}$ rather than $\|f-\mathscr{M}_w(f;\Omega)\|_{\Omega,p}$ for $\left|T_w(f,g;\Omega)\right|$.

REMARK 3. It should be noted that the bounds in (3.4) are in order of increasing coarseness.

We may use Hölder's integral inequality directly from (3.6) to give for f and g in the appropriate $L_{w,\cdot}(\Omega,\mathscr{A},\mu)$

$$W(\Omega)\left|C_w(f,g;\Omega)\right| \leqslant \begin{cases} \|f\|_{\Omega,1}\left\|g-\dfrac{\Delta+\delta}{2}\right\|_{\Omega,\infty} = \dfrac{\Delta-\delta}{2}\|f\|_{\Omega,1}, \\ \|f\|_{\Omega,p}\left\|g-\dfrac{\Delta+\delta}{2}\right\|_{\Omega,q}, \\ \|f\|_{\Omega,\infty}\left\|g-\dfrac{\Delta+\delta}{2}\right\|_{\Omega,1}. \end{cases} \tag{3.10}$$

In (3.10) we have the first inequality recapturing the result of [5] and [11].

The following interesting corollary holds where the roles of $f(\cdot)$ and $g(\cdot)-\frac{\Delta+\delta}{2}$ have been interchanged.

COROLLARY 2. *Let the conditions of Theorem 2 persist, then we have*

$$\begin{aligned}
|C_w(f,g;\Omega)| &= \left|\mathscr{M}_w(fg;\Omega)-\frac{\Delta+\delta}{2}\mathscr{M}_w(f;\Omega)\right| \\
&\leqslant \|f\|_{\Omega,\infty}\cdot\frac{1}{W(\Omega)}\left\|g-\frac{\Delta+\delta}{2}\right\|_{\Omega,1}, \quad g\in L_{w,1}(\Omega,\mathscr{A},\mu) \\
&\leqslant \|f\|_{\Omega,\infty}\cdot\frac{1}{W^{\frac{1}{p}}(\Omega)}\left\|g-\frac{\Delta+\delta}{2}\right\|_{\Omega,p}, \quad g\in L_{w,p}(\Omega,\mathscr{A},\mu),\, 1<p<\infty, \\
&\leqslant \|f\|_{\Omega,\infty}\cdot\frac{1}{W(\Omega)}\left\|g-\frac{\Delta+\delta}{2}\right\|_{\Omega,\infty} \\
&= \|f\|_{\Omega,\infty}\frac{\Delta-\delta}{2}, \quad g\in L_\infty(\Omega,\mathscr{A},\mu).
\end{aligned} \tag{3.11}$$

The inequalities are sharp.

Proof. From (3.6) we have by interchanging the role of $f(\cdot)$ and $g(\cdot)-\frac{\Delta+\delta}{2}$ we have,

$$\begin{aligned}
|C_w(f,g;\Omega)| &\leqslant ess\sup_{x\in\Omega}|f(x)|\cdot\frac{1}{W(\Omega)}\int_\Omega w(x)\left|g(x)-\frac{\Delta+\delta}{2}\right|d\mu(x) \\
&= \|f\|_{\Omega,\infty}\cdot\frac{1}{W(\Omega)}\cdot\left\|g-\frac{\Delta+\delta}{2}\right\|_{\Omega,1}
\end{aligned} \tag{3.12}$$

giving the first inequality in (3.10).

Now, using Hölder's integral inequality we obtain from (3.11) for $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$

$$\begin{aligned}
&\frac{1}{W(\Omega)}\int_\Omega w(x)\left|g(x)-\frac{\Delta+\delta}{2}\right|d\mu(x) \\
&\quad\leqslant \frac{1}{W(\Omega)}\left(\int_\Omega w(x)\cdot 1^q d\mu(x)\right)^{\frac{1}{q}}\left(\int_\Omega w(x)\left|g(x)-\frac{\Delta+\delta}{2}\right|^p d\mu(x)\right)^{\frac{1}{p}} \\
&\quad= \frac{1}{W^{1-\frac{1}{q}}(\Omega)}\left(\int_\Omega w(x)\left|g(x)-\frac{\Delta+\delta}{2}\right|^p d\mu(x)\right)^{\frac{1}{p}},
\end{aligned}$$

giving the second inequality. The final inequality is procured on extracting the essential supremum of $g(x)-\frac{\Delta+\delta}{2}$ over Ω directly from (3.11), producing the stated result.

The question of sharpness of the results follows along similar reasoning to that in Theorem 2.

4. Some particular inequalities

Some specific cases of the results presented in Section 3. are worthy to be explicitly stated because of their wide use.

A. Let $w, f, g : [a,b] \to \mathbb{R}$ be Lebesgue measurable with $w \geqslant 0$ a.e. on $[a,b]$ and $W := W([a,b]) = \int_a^b w(y)\,dy > 0$. If $f, g, fg \in L_w[a,b]$ where

$$L_w[a,b] := \left\{ f : [a,b] \to \mathbb{R},\ \int_\Omega w(x)\,|f(x)|\,d\mu(x) < \infty \right\}$$

and $-\infty < \delta \leqslant g(x) \leqslant \Delta < \infty$ for a.e. $x \in [a,b]$. We then have, from (3.4) of Theorem 2,

$$\begin{aligned} &\left| \frac{1}{W} \int_a^b w(x) f(x) g(x)\,d\mu(x) - \frac{\Delta+\delta}{2} \cdot \frac{1}{W} \int_a^b w(x) f(x) d\mu(x) \right| \\ &\quad \leqslant \frac{\Delta-\delta}{2} \cdot \frac{1}{W} \|f\|_{[a,b],1} \\ &\quad \leqslant \frac{\Delta-\delta}{2} \cdot \frac{1}{W^{\frac{1}{p}}} \|f\|_{[a,b],p}, \quad f \in L_{w,p}[a,b], \quad 1 < p < \infty \\ &\quad \leqslant \frac{\Delta-\delta}{2} \cdot \|f\|_{[a,b],\infty}, \quad f \in L_\infty[a,b], \end{aligned} \tag{4.1}$$

where

$$L_{w,p}[a,b] := \left\{ f : [a,b] \to \mathbb{R},\ \int_a^b w(x)\,|f(x)|^p\,dx < \infty \right\},$$

and

$$L_\infty[a,b] := \left\{ f : [a,b] \to \mathbb{R},\ ess \sup_{x \in [a,b]} |f(x)| < \infty \right\}$$

with $\|\cdot\|_{[a,b],p}$ the $p-$norm on $L_{w,p}[a,b]$ and $L_\infty[a,b]$, namely

$$\|h\|_{[a,b],p} := \left(\int_a^b w(x)\,|h(x)|^p\,dx \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty$$

and

$$\|h\|_{[a,b],\infty} := ess \sup_{x \in [a,b]} |h(x)|.$$

If $w(x) = 1$, $x \in [a,b]$ then $W = b - a$ in (4.1).

B. Let $\bar{a} = (a_1, \ldots, a_n)$, $\bar{b} = (b_1, \ldots, b_n)$ and $\bar{p} = (p_1, \ldots, p_n)$ be $n-$tuples of real numbers with $p_i \geqslant 0$ for $i \in \{1, 2, \ldots, n\}$ and $\sum_{i=1}^n p_i = 1$. If

$$b \leqslant b_i \leqslant B, \quad i \in \{1, \ldots, n\},$$

then from (3.4) of Theorem $2,$ one has the inequalities

$$\begin{aligned}\left|\sum_{i=1}^{n} p_i a_i b_i - \frac{B+b}{2}\sum_{i=1}^{n} p_i a_i\right| &\leqslant \frac{B-b}{2}\sum_{i=1}^{n} p_i\left|a_i\right| \\ &\leqslant \frac{B-b}{2}\left(\sum_{i=1}^{n} p_i\left|a_i\right|^r\right)^{\frac{1}{r}}, \quad 1<r<\infty \qquad (4.2)\\ &\leqslant \frac{B-b}{2}\max_{i\in\overline{1,n}}\left|a_i\right|.\end{aligned}$$

If $p_i=1,\ i=\{1,\ldots,n\}$ such that $\sum_{i=1}^{n} p_i = n$ then the following inequality may be stated,

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{B+b}{2}\cdot\frac{1}{n}\sum_{i=1}^{n} a_i\right| &\leqslant \frac{B-b}{2}\cdot\frac{1}{n}\sum_{i=1}^{n}\left|a_i\right| \\ &\leqslant \frac{B-b}{2}\left(\frac{1}{n}\sum_{i=1}^{n}\left|a_i\right|^r\right)^{\frac{1}{r}}, \quad 1<r<\infty \qquad (4.3)\\ &\leqslant \frac{B-b}{2}\max_{i\in\overline{1,n}}\left|a_i\right|.\end{aligned}$$

5. Applications for Ostrowski and trapezoid-type perturbed inequalities

For $\varphi:[a,b]\to\mathbb{R}$ an absolutely continuous function on $[a,b]$

$$S(\varphi)(x) := \varphi(x) - \frac{1}{b-a}\int_a^b \varphi(t)\,dt \qquad (5.1)$$

$$|S(\varphi)(x)| \leqslant \begin{cases} \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|\varphi'\right\|_\infty, \\ \qquad \varphi'\in L_\infty[a,b]; \\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1}+\left(\frac{b-x}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\left\|\varphi'\right\|_q, \\ \qquad \varphi'\in L_p[a,b], p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left\|\varphi'\right\|_1. \end{cases} \qquad (5.2)$$

Here the constants $\frac{1}{4},\ \frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense that they cannot be replaced by a smaller constant. The first inequality in (5.2) is attributed to Ostrowski which he proved in 1938. The book [14] is related to Ostrowski-type inequalities and their application to numerical integration. In (5.2), $\|\cdot\|_r$ are the usual Lebesgue norms on $L_r[a,b]$, namely,

$$\|h\|_p := \left(\int_a^b |h(t)|^p \, dt\right)^{\frac{1}{p}}, \quad p \in [1,\infty) \quad \text{and} \quad \|h\|_\infty := ess \sup_{t\in[a,b]} |h(t)|.$$

A simple proof of (5.1) may be procured from Montgomery's identity

$$\varphi(x) = \frac{1}{b-a}\int_a^b \varphi(t)\, dt + \frac{1}{b-a}\int_a^b K(x,t)\, \varphi'(t)\, dt, \tag{5.3}$$

where the kernel $K : [a,b]^2 \to \mathbb{R}$ is given by

$$K(x,t) := \begin{cases} t-a, & t\in[a,x], \\ t-b, & t\in(x,b]. \end{cases} \tag{5.4}$$

The following theorem gives a perturbed version of (5.2). Namely, for

$$S_p(\varphi)(x) := S(\varphi)(x) - \frac{\Delta+\delta}{2}\left(x - \frac{a+b}{2}\right).$$

THEOREM 3. *Assume that* $\varphi : [a,b] \to \mathbb{R}$ *is an absolutely continuous function on* $[a,b]$ *such that*

$$-\infty < \delta \leqslant \varphi'(t) \leqslant \Delta < \infty \ \ \textit{for a.e. } x \in [a,b].$$

We then have the results

$$\begin{aligned} |S_p(\varphi)(x)| &= \left|\varphi(x) - \frac{1}{b-a}\int_a^b \varphi(t)\, dt - \frac{\Delta+\delta}{2}\left(x-\frac{a+b}{2}\right)\right| \\ &\leqslant \frac{\Delta-\delta}{2}\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a) \\ &\leqslant \frac{\Delta-\delta}{2}\cdot\frac{1}{(p+1)^{\frac{1}{p}}}\left[\frac{(x-a)^{p+1}+(b-x)^{p+1}}{b-a}\right]^{\frac{1}{p}} \\ &\leqslant \frac{\Delta-\delta}{2}\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right](b-a). \end{aligned} \tag{5.5}$$

The above constants are sharp.

Proof. From the first inequality in (4.1) if we take $w(t) = 1$ and associate $K(x,t)$ as defined by (5.4) with $f(t)$ and $\varphi'(t)$ with $g(t)$ for $t\in[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b K(x,t)\varphi'(t)dt - \frac{\Delta+\delta}{2}\cdot\frac{1}{b-a}\int_a^b K(x,t)\, dt\right| \leqslant \frac{\Delta-\delta}{2}\cdot\frac{1}{b-a}\int_a^b |K(x,t)|\, dt. \tag{5.6}$$

Now, from (5.4), we have

$$\frac{1}{b-a}\int_a^b K(x,t)\,dt = x - \frac{a+b}{2}$$

and

$$\frac{1}{b-a}\int_a^b |K(x,t)|\,dt = \frac{(x-a)^2+(b-x)^2}{2(b-a)} = \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)$$

so that from (5.6) the first inequality in (5.5) results.

With the same associations described above, we have from the second inequality in (4.1)

$$\begin{aligned}\left(\frac{1}{b-a}\right)^{\frac{1}{p}}\left(\int_a^b |K(x,t)|^p\,dt\right)^{\frac{1}{p}} &= \frac{1}{(b-a)^{\frac{1}{p}}}\left(\int_a^x (t-a)^p\,dt+\int_x^b (b-t)^p\,dt\right)^{\frac{1}{p}}\\ &= \left[\frac{(x-a)^{p+1}+(b-x)^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}\end{aligned}$$

and hence the second inequality (5.5) is obtained.

The last result in (5.5) follows from the corresponding inequality in (4.1) on noting that

$$\|K(x,\cdot)\|_\infty := ess\sup_{t\in[a,b]}|K(x,t)| = \max\{x-a,b-x\} = \frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|.$$

To prove the sharpness of the constants assume

$$\left|\varphi(x)-\frac{1}{b-a}\int_a^b\varphi(t)dt-\frac{\Delta+\delta}{2}\left(x-\frac{a+b}{2}\right)\right| \leqslant K(\Delta-\delta)\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a). \tag{5.7}$$

Let $\varphi(t)=\left|t-\frac{a+b}{2}\right|$ and $x=\frac{a+b}{2}$, then from (5.7) we have, since $\frac{1}{b-a}\int_a^b\varphi(t)\,dt = \frac{b-a}{4}$ and $\delta=-1,\ \Delta=1$

$$\frac{b-a}{4}\leqslant 2K\cdot\left(\frac{b-a}{4}\right).$$

Hence $K\geqslant\frac{1}{2}$. The same function will prove the sharpness of the constants for the other two inequalities. The theorem is now completely proved.

COROLLARY 3. *Let the conditions of Theorem 3 persist, then for* $-\infty < \delta \leqslant \varphi'(t) \leqslant \Delta < \infty$,

$$\begin{aligned} \left|S(\varphi)\left(\frac{a+b}{2}\right)\right| &= \left|\varphi\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \varphi(t)\,dt\right| \\ &\leqslant \frac{\Delta-\delta}{2}\cdot\left(\frac{b-a}{4}\right) \\ &\leqslant \frac{\Delta-\delta}{2}\cdot\frac{1}{(p+1)^{\frac{1}{p}}}\cdot\frac{(b-a)}{2} \\ &\leqslant \frac{\Delta-\delta}{2}\cdot\left(\frac{b-a}{2}\right). \end{aligned} \tag{5.8}$$

Proof. Taking $x = \frac{a+b}{2}$ in (5.5) gives the stated result with no perturbation.

REMARK 4. The first inequalities in (5.5) and (5.8) were obtained by Dragomir [12] by taking $\varphi(x) = \Phi(x) - \left(x - \frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)$ in the first inequality in (5.1). Here $-\infty < m < \Phi'(x) < M < \infty$. It may be shown that the generalised Trapezoidal functional

$$T(\varphi)(x) := \left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b) - \frac{1}{b-a}\int_a^b \varphi(t)\,dt \tag{5.9}$$

satisfies the identity

$$T(\varphi) = \frac{1}{b-a}\int_a^b (t-x)\,\varphi'(t)\,dt. \tag{5.10}$$

Cerone [3] has shown that $|T(\varphi)(x)|$ has the same bounds as for $|S(\varphi)(x)|$ by the symmetry of the kernels (5.4) in identity (5.3) and that of (5.10). The same bounds also hold for the perturbed generalised trapezoidal function

$$T_p(\varphi)(x) := T(\varphi)(x) - \frac{\Delta+\delta}{2}\left(x - \frac{a+b}{2}\right) \tag{5.11}$$

as for the perturbed Ostrowski functional $S_p(\varphi)(x)$ presented in Theorem 2 as (5.5) and the corresponding result to Corollary 3. Thus, from (5.9) and (5.11),

$$\begin{aligned} |T_p(\varphi)(x)| &= \left|\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b) - \frac{1}{b-a}\int_a^b \varphi(t)dt - \frac{\Delta+\delta}{2}\left(x - \frac{a+b}{2}\right)\right| \\ &\leqslant \frac{\Delta-\delta}{2}\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a) \end{aligned}$$

and $\left|T_p(\varphi)\left(\frac{a+b}{2}\right)\right| \leqslant \frac{\Delta-\delta}{8}$. Dragomir [12] obtained the first of these results in the manner specified above for the perturbed Ostrowski functional $S_p(\varphi)(x)$.

COROLLARY 4. *Let the conditions of Theorem 3 persist, then*

$$\left|\varphi(x) - \frac{1}{b-a}\int_a^b \varphi(t)\,dt - [\varphi;a,b]\left(x - \frac{a+b}{2}\right)\right| \tag{5.12}$$
$$\begin{aligned} &\leqslant \|\varphi'\|_\infty \cdot \frac{b-a}{4} \\ &\leqslant \|\varphi'\|_\infty \cdot \frac{b-a}{[2(p+1)]^{\frac{1}{p}}}, \quad 1<p<\infty \\ &\leqslant \|\varphi'\|_\infty \cdot \frac{b-a}{2}. \end{aligned}$$

Proof. Assume that (3.10) is cast in a Lebesgue measurable setting as (3.4) has been, in the form of (4.1). In this new formulation take $w(t) = 1$ and associate $K(x,t)$ from (5.4) with $g(t)$ and $\varphi'(t)$ with $f(t)$ for $t \in [a,b]$. Now, $\delta = x - b \leqslant K(x,t) \leqslant x - a = \Delta$ such that

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b K(x,t)\,\varphi'(t)\,dt - \frac{\Delta+\delta}{2}\cdot\frac{1}{b-a}\int_a^b \varphi'(t)\,dt\right| \\ &\quad = \left|\frac{1}{b-a}\int_a^b K(x,t)\,\varphi'(t)\,dt - \left(x - \frac{a+b}{2}\right)[\varphi;a,b]\right| \\ &\quad \leqslant \frac{1}{b-a}\int_a^b |\varphi'(t)|\left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,u)\,du\right| dt \\ &\quad \leqslant \|\varphi'\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,u)\,du\right| dt. \end{aligned} \tag{5.13}$$

Now, by direct calculation, we have, using Hölder's inequality

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,u)\,du\right| dt \\ &\quad = \frac{1}{b-a}\left(\int_a^b 1^q dt\right)^{\frac{1}{q}}\left(\int_a^b \left|K(x,t) - \left(x - \frac{a+b}{2}\right)\right|^p dt\right)^{\frac{1}{p}} \\ &\quad = \frac{1}{(b-a)^{1-\frac{1}{q}}}\left[\int_a^x \left|t - x + \frac{b-a}{2}\right|^p dt + \int_x^b \left|t - x - \frac{b-a}{2}\right|^p dt\right]^{\frac{1}{p}} \\ &\quad = \frac{1}{(b-a)^{1-\frac{1}{q}}}\left(\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} |u|^p\,du\right)^{\frac{1}{p}} = \frac{2}{(b-a)^{\frac{1}{p}}}\left(\int_0^{\frac{b-a}{2}} u^p du\right)^{\frac{1}{p}} \\ &\quad = \frac{b-a}{(2(p+1))^{\frac{1}{p}}}. \end{aligned}$$

Using the above result in (5.13) and the Lebesgue form of (3.10) produces the first two inequalities in (5.12).

Now,

$$ess\sup_{t\in[a,b]}\left|K(x,t)-\left(x-\frac{a+b}{2}\right)\right|=\frac{b-a}{2},$$

producing the final inequality in (3.10).

REMARK 5. From (5.13) and (5.3) we have

$$\begin{aligned}\left|\varphi(x)-\frac{1}{b-a}\int_a^b\varphi(t)\,dt-\left(x-\frac{a+b}{2}\right)[\varphi;a,b]\right|&\\ \leqslant\frac{1}{b-a}\int_a^b\varphi'(t)\left|K(x,t)-\frac{1}{b-a}\int_a^b K(x,u)\,du\right|dt&\\ \leqslant\frac{1}{2}\int_a^b\left|\varphi'(t)\right|dt.&\qquad(5.14)\end{aligned}$$

We notice that taking $x=\frac{a+b}{2}$ above produces agreement with the last result in (5.1). The bound in (5.14) is independent of x and is the best bound for the unperturbed result in (5.1).

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