SOME COEFFICIENT INEQUALITIES AND DISTORTION BOUNDS ASSOCIATED WITH CERTAIN NEW SUBCLASSES OF ANALYTIC FUNCTIONS

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(*communicated by J. Pečarić*)

Abstract. The authors introduce and investigate two new subclasses $\mathscr{M}^*(\alpha)$ and $\mathscr{N}^*(\alpha)$ of normalized analytic functions satisfying certain coefficient inequalities in the open unit disk $\mathbb{U}$. The main results of the present paper provide various interesting properties of functions belonging to the classes $\mathscr{M}^*(\alpha)$ and $\mathscr{N}^*(\alpha)$. Some of these properties include (for example) several coefficient inequalities, distortion bounds, and inclusion relationships for the function classes which are considered here.

1. Introduction

Let $\mathscr{A}$ denote the class of functions $f(z)$ *normalized* in the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}. \tag{1.2}$$

We denote by $\mathscr{S}$ the subclass of $\mathscr{A}$ consisting of all functions $f(z)$ which are also *univalent* in $\mathbb{U}$.

Let $\mathscr{S}^*(\alpha)$ be the subclass of $\mathscr{A}$ consisting of all functions $f(z)$ which satisfy the following inequality:

$$\mathscr{R}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}; \quad 0 \leqslant \alpha < 1). \tag{1.3}$$

A function $f \in \mathscr{S}^*(\alpha)$ is said to be *starlike of order* α in $\mathbb{U}$. Furthermore, let $\mathscr{K}(\alpha)$ denote the subclass of $\mathscr{A}$ consisting of all functions $f(z)$ which satisfy the following inequality:

$$\mathscr{R}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}; \quad 0 \leqslant \alpha < 1). \tag{1.4}$$

A function $f \in \mathcal{K}(\alpha)$ is said to be *convex of order* α in $\mathbb{U}$. We note that

$$f(z) \in \mathcal{K}(\alpha) \iff zf'(z) \in \mathcal{S}^*(\alpha).$$

(See, for details, [1] and [2]; see also [3] and [6], and the references cited therein.)

About three decades ago, Silverman [5] gave the following coefficient inequalities for the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$.

THEOREM A. (Silverman [5]). *If* $f(z) \in \mathcal{A}$ *satisfies the following coefficient inequality*:

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqslant 1-\alpha \quad (0 \leqslant \alpha < 1), \tag{1.5}$$

then

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\alpha \quad (z \in \mathbb{U}; \quad 0 \leqslant \alpha < 1), \tag{1.6}$$

that is, that $f(z) \in \mathcal{S}^*(\alpha)$.

THEOREM B. (Silverman [5]). *If* $f(z) \in \mathcal{A}$ *satisfies the following coefficient inequality*:

$$\sum_{n=2}^{\infty}n(n-\alpha)|a_n| \leqslant 1-\alpha \quad (0 \leqslant \alpha < 1), \tag{1.7}$$

then

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1-\alpha \quad (z \in \mathbb{U}; \quad 0 \leqslant \alpha < 1), \tag{1.8}$$

that is, that $f(z) \in \mathcal{K}(\alpha)$.

More recently, Sekine and Owa [4] considered the subclass of functions $f \in \mathcal{A}$ which satisfy the following inequality:

$$\left|\frac{zf'(z)}{f(z)}-a\right| < a-\alpha \quad (z \in \mathbb{U}; \quad 0 \leqslant \alpha < 1; a > \alpha). \tag{1.9}$$

In this paper, we consider a new subclass $\mathcal{M}(\alpha)$ of the class $\mathcal{A}$ consisting of functions $f(z)$ such that

$$\left|\frac{f(z)}{zf'(z)}-\frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \quad (z \in \mathbb{U}; \quad 0 < \alpha < 1). \tag{1.10}$$

We also introduce and investigate here the subclass $\mathcal{N}(\alpha)$ of the class $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following inclusion relationship:

$$zf'(z) \in \mathcal{M}(\alpha).$$

Let us now define the function $F(z)$ by

$$F(z) = \frac{zf'(z)}{f(z)} \quad \left(f \in \mathcal{M}(\alpha)\right).$$

Then $f(z)$ satisfies the inequality:

$$F(z) + \bar{F}(z) > 2\alpha \quad (z \in \mathbb{U}; \quad 0 < \alpha < 1), \tag{1.11}$$

so that

$$\mathscr{R}(F(z)) = \mathscr{R}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}; \quad 0 < \alpha < 1). \tag{1.12}$$

It follows from (1.12) that

$$\mathscr{M}(\alpha) \subset \mathscr{S}^*(\alpha) \quad \text{and} \quad \mathscr{N}(\alpha) \subset \mathscr{K}(\alpha).$$

EXAMPLE. Let us consider the function given by

$$f(z) = z + \frac{1}{k}z^2 \quad (k \geqslant 2). \tag{1.13}$$

Then we have

$$\frac{zf'(z)}{f(z)} - 1 = \frac{k+2z}{k+z} - 1 = \frac{z}{k+z}. \tag{1.14}$$

Since

$$\left|\frac{z}{k+z} + \frac{1}{k^2-1}\right| < \frac{k}{k^2-1} \quad (z \in \mathbb{U}), \tag{1.15}$$

we see that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{k-1} = 1 - \frac{k-2}{k-1} \tag{1.16}$$

which readily implies that

$$f(z) \in \mathscr{S}^*\left(\frac{k-2}{k-1}\right). \tag{1.17}$$

On the other hand, we observe that

$$\frac{f(z)}{zf'(z)} - \frac{1}{2\alpha} = \frac{k+z}{k+2z} - \frac{1}{2\alpha} = \frac{1}{2}\left(1 - \frac{1}{\alpha} + \frac{k}{k+2z}\right). \tag{1.18}$$

Noting also that

$$\left|\frac{k}{k+2z} - \frac{k^2}{k^2-4}\right| < \frac{2k}{k^2-4} \quad (z \in \mathbb{U}), \tag{1.19}$$

we have

$$\left|\frac{f(z)}{zf'(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2}A(k,\alpha), \tag{1.20}$$

where

$$A(k,\alpha) := \max\left\{\left|1 - \frac{1}{\alpha} + \frac{k}{k+2}\right|, \left|1 - \frac{1}{\alpha} + \frac{k}{k-2}\right|\right\}. \tag{1.21}$$

Thus we obtain

$$\frac{1}{\alpha} = \frac{1}{2}A(k,\alpha) \tag{1.22}$$

for $f(z) \in \mathscr{M}(\alpha)$. Let us put $\alpha = \alpha_0$. If α is given by

$$\alpha = \frac{1}{A(k,\alpha)}, \tag{1.23}$$

then $f(z) \in \mathcal{M}(\alpha_0)$. By the fact that $\mathcal{M}(\alpha) \subset \mathcal{S}^*(\alpha)$, we have

$$\alpha_0 \geqslant \frac{k-2}{k-1}. \tag{1.24}$$

If we set

$$\alpha = \frac{k-2}{k-1}, \tag{1.25}$$

then we have

$$1 - \frac{1}{\alpha} + \frac{k}{k+2} = 1 - \frac{k-1}{k-2} + \frac{k}{k+2} = \frac{(k+1)(k-4)}{(k+2)(k-2)} \tag{1.26}$$

and

$$1 - \frac{1}{\alpha} + \frac{k}{k-2} = 1 - \frac{k-1}{k-2} + \frac{k}{k-2} = \frac{k-1}{k-2}. \tag{1.27}$$

Therefore, in the case when $k \geqslant 4$, we have

$$\frac{k-1}{k-2} - \frac{(k+1)(k-4)}{(k+2)(k-2)} = \frac{2(2k+1)}{(k+2)(k-2)} \geqslant 0. \tag{1.28}$$

Moreover, in the case when $2 \leqslant k < 4$, we have

$$\frac{k-1}{k-2} - \frac{(k+1)(4-k)}{(k+2)(k-2)} = \frac{2(k^2-k-3)}{(k+2)(k-2)}. \tag{1.29}$$

Thus, if

$$2 \leqslant k \leqslant \frac{1+\sqrt{13}}{2} = 2.3027\cdots,$$

then we have

$$\frac{k-1}{k-2} \leqslant \frac{(k+1)(4-k)}{(k+2)(k-2)}. \tag{1.30}$$

Therefore

$$A(k,\alpha) = \begin{cases} \dfrac{k-1}{k-2} & \left(k \geqslant \dfrac{1+\sqrt{13}}{2}\right) \\[3mm] \dfrac{(k+1)(4-k)}{(k+2)(k-2)} & \left(2 \leqslant k < \dfrac{1+\sqrt{13}}{2}\right). \end{cases} \tag{1.31}$$

By the condition (1.23), we have

$$\alpha_0 = \frac{k-2}{k-1} \qquad \left(k \geqslant \frac{1+\sqrt{13}}{2}\right) \tag{1.32}$$

such that

$$f(z) \in \mathcal{M}\left(\frac{k-2}{k-1}\right) \qquad \left(k \geqslant \frac{1+\sqrt{13}}{2} = 2.3027\cdots\right). \tag{1.33}$$

Thus we have

$$\frac{5 - \sqrt{13}}{6} \leqslant \frac{k-2}{k-1} < 1 \qquad \left(\frac{5 - \sqrt{13}}{6} = 0.23241 \cdots \right). \tag{1.34}$$

When $0 < \alpha \leqslant \beta < 1$, we have the following inclusion relationship:

$$\mathscr{M}(\alpha) \supset \mathscr{M}(\beta), \tag{1.35}$$

which results from the definition of the class $\mathscr{M}(\alpha)$. Thus we conclude that

$$f(z) \in \mathscr{M}\left(\frac{k-2}{k-1} \right) \subset \mathscr{M}\left(\frac{5 - \sqrt{13}}{6} \right) \subset \mathscr{S}^*\left(\frac{5 - \sqrt{13}}{6} \right). \tag{1.36}$$

We now consider the following function:

$$f(z) = z + \frac{1}{2k} z^2 \quad (k \geqslant 2), \tag{1.37}$$

which immediately yields

$$zf'(z) = z + \frac{1}{k} z^2 \quad (k \geqslant 2). \tag{1.38}$$

Since, by definition,

$$f(z) \in \mathscr{N}(\alpha) \quad \Longleftrightarrow \quad zf'(z) \in \mathscr{M}(\alpha),$$

we finally obtain the following inclusion relationship:

$$f(z) \in \mathscr{N}\left(\frac{k-2}{k-1} \right) \subset \mathscr{N}\left(\frac{5 - \sqrt{13}}{6} \right) \subset \mathscr{K}\left(\frac{5 - \sqrt{13}}{6} \right)$$
$$\left(k \geqslant \frac{1 + \sqrt{13}}{2} = 2.3027 \cdots \right). \tag{1.39}$$

2. A set of coefficient inequalities

Our first coefficient inequality is contained in Theorem 1 below.

THEOREM 1. *Let $0 < \alpha < 1$. If $f(z) \in \mathscr{A}$ satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty} (n - \alpha)|a_n| \leqslant \frac{1}{2}(1 - |1 - 2\alpha|) = \begin{cases} \alpha & \left(0 < \alpha \leqslant \frac{1}{2} \right) \\ 1 - \alpha & \left(\frac{1}{2} \leqslant \alpha < 1 \right), \end{cases} \tag{2.1}$$

then $f(z) \in \mathscr{M}(\alpha)$.

Proof. By virtue of the condition (1.10), we have to show that

$$\left| \frac{2\alpha f(z)}{zf'(z)} - 1 \right| < 1. \tag{2.2}$$

We first observe that

$$\left| \frac{2\alpha f(z) - zf'(z)}{zf'(z)} \right| = \left| \frac{1 - 2\alpha + \sum\limits_{n=2}^{\infty}(n - 2\alpha)a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} na_n z^{n-1}} \right|$$

$$\leqslant \frac{|1 - 2\alpha| + \sum\limits_{n=2}^{\infty}(n - 2\alpha)|a_n| \cdot |z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} n|a_n| \cdot |z|^{n-1}} \qquad (2.3)$$

$$< \frac{|1 - 2\alpha| + \sum\limits_{n=2}^{\infty}(n - 2\alpha)|a_n|}{1 - \sum\limits_{n=2}^{\infty} n|a_n|}.$$

Now, by using the coefficient inequality (2.1), we have

$$\frac{|1 - 2\alpha| + \sum\limits_{n=2}^{\infty}(n - 2\alpha)|a_n|}{1 - \sum\limits_{n=2}^{\infty} n|a_n|} \leqslant 1, \qquad (2.4)$$

which, in conjunction with (2.2) and (2.3), completes the proof of Theorem 1. $\square$

By means of Theorem 1, we introduce the subclass $\mathscr{M}^*(\alpha)$ of the class $\mathscr{M}(\alpha)$ consisting of all functions $f(z)$ which satisfy the coefficient inequality (2.1) for some α $(0 < \alpha < 1)$.

THEOREM 2. *Suppose that* $0 < \alpha < 1$. *If* $f(z) \in \mathscr{A}$ *satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty} n(n - \alpha)|a_n| \leqslant \frac{1}{2}(1 - |1 - 2\alpha|) = \begin{cases} \alpha & \left(0 < \alpha \leqslant \frac{1}{2}\right) \\ 1 - \alpha & \left(\frac{1}{2} \leqslant \alpha < 1\right), \end{cases} \qquad (2.5)$$

then $f(z) \in \mathscr{N}(\alpha)$.

Proof. The proof of Theorem 2 follows from Theorem 1 and the aforementioned fact that

$$f(z) \in \mathscr{N}(\alpha) \quad \Longleftrightarrow \quad zf'(z) \in \mathscr{M}(\alpha).$$

$\square$

By means of Theorem 2, we also introduce the subclass $\mathscr{N}^*(\alpha)$ of the class $\mathscr{N}(\alpha)$ consisting of all functions $f(z)$ which satisfy the coefficient inequality (2.5) for some α $(0 < \alpha < 1)$.

3.   Distortion bounds

For $f \in \mathscr{A}$, we define the integro-differential operators $I_k f(z)$ given by

$$I_{-1}f(z) = f'(z), \quad I_0 f(z) = f(z),$$

and

$$I_k f(z) = \int_0^z I_{k-1} f(t) dt \qquad (k \in \mathbb{N} := \{1, 2, 3, \cdots\}).$$

Then we find from (1.1) that

$$I_k f(z) = \frac{1}{(k+1)!} z^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n+k}. \tag{3.1}$$

THEOREM 3. If $f(z) \in \mathscr{M}^*(\alpha)$, then

$$\frac{1}{(k+1)!}|z|^{k+1} - \frac{1 - |1 - 2\alpha|}{(k+2)!\,(2-\alpha)}|z|^{k+2} \leqslant |I_k f(z)|$$
$$\leqslant \frac{1}{(k+1)!}|z|^{k+1} + \frac{1 - |1 - 2\alpha|}{(k+2)!\,(2-\alpha)}|z|^{k+2} \tag{3.2}$$

$$(z \in \mathbb{U}; \quad k \in \mathbb{N}_0 \cup \{-1\}; \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

Proof. We begin by noting that

$$|I_k f(z)| = \left| \frac{1}{(k+1)!} z^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n+k} \right|$$
$$\leqslant \frac{1}{(k+1)!}|z|^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \cdot |z|^{n+k} \tag{3.3}$$
$$< \frac{1}{(k+1)!}|z|^{k+1} + |z|^{k+2} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n|.$$

Now it is easy to see that

$$\frac{1}{2}(k+2)!\,(2-\alpha) \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \leqslant \sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqslant \frac{1}{2}(1 - |1 - 2\alpha|), \tag{3.4}$$

which implies that

$$\sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \leqslant \frac{1 - |1 - 2\alpha|}{(k+2)!\,(2-\alpha)}. \tag{3.5}$$

Therefore, we have

$$|I_k f(z)| \leqslant \frac{1}{(k+1)!}|z|^{k+1} + \frac{1 - |1 - 2\alpha|}{(k+2)!\,(2-\alpha)}|z|^{k+2} \quad (z \in \mathbb{U}). \tag{3.6}$$

Also we can easily observe that

$$|I_k f(z)| \geqslant \frac{1}{(k+1)!}|z|^{k+1} - \frac{1 - |1 - 2\alpha|}{(k+2)!\,(2-\alpha)}|z|^{k+2} \quad (z \in \mathbb{U}). \tag{3.7}$$

$\square$

By setting $k = -1, 0, 1$ in Theorem 3, we deduce Corollary 1 below.

COROLLARY 1. *If* $f(z) \in \mathscr{M}^*(\alpha)$, *then*

$$1 - \frac{1 - |1 - 2\alpha|}{2 - \alpha}|z| \leqslant |f'(z)| \leqslant 1 + \frac{1 - |1 - 2\alpha|}{2 - \alpha}|z| \qquad (k = -1), \qquad (3.8)$$

$$|z| - \frac{1 - |1 - 2\alpha|}{2(2 - \alpha)}|z|^2 \leqslant |f(z)| \leqslant |z| + \frac{1 - |1 - 2\alpha|}{2(2 - \alpha)}|z|^2 \qquad (k = 0), \qquad (3.9)$$

and

$$\frac{1}{2}|z|^2 - \frac{1 - |1 - 2\alpha|}{6(2 - \alpha)}|z|^3 \leqslant |I_1 f(z)| \leqslant \frac{1}{2}|z|^2 + \frac{1 - |1 - 2\alpha|}{6(2 - \alpha)}|z|^3 \qquad (k = 1). \quad (3.10)$$

For $f \in \mathscr{A}$, we consider again the following integro-differential operators:

$$I_{-2}f(z) = f''(z) = \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}, \quad I_{-1}f(z) = f'(z), \quad I_0 f(z) = f(z),$$

and

$$I_k f(z) = \int_0^z I_{k-1} f(t) dt \qquad (k \in \mathbb{N}).$$

Next we state and prove the following result.

THEOREM 4. *If* $f(z) \in \mathscr{N}^*(\alpha)$, *then*

$$2|a_2| - \frac{1}{2}(1 - |1 - 2\alpha|)|z| \leqslant |I_{-2}f(z)| \leqslant 2|a_2| + \frac{1}{2}(1 - |1 - 2\alpha|)|z| \qquad (3.11)$$

and

$$\frac{1}{(k+1)!}|z|^{k+1} - \frac{1 - |1 - 2\alpha|}{(k+2)! \, (2-\alpha)}|z|^{k+2} \leqslant |I_k f(z)|$$

$$\leqslant \frac{1}{(k+1)!}|z|^{k+1} + \frac{1 - |1 - 2\alpha|}{(k+2)! \, (2-\alpha)}|z|^{k+2} \quad (z \in \mathbb{U}; \quad k \in \mathbb{N}_0 \cup \{-1\}). \tag{3.12}$$

Proof. We note that, for $k \in \mathbb{N}_0 \cup \{-1\}$,

$$|I_k f(z)| = \left| \frac{1}{(k+1)!}z^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}a_n z^{n+k} \right|$$

$$\leqslant \frac{1}{(k+1)!}|z|^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \cdot |z|^{n+k} \tag{3.13}$$

$$< \frac{1}{(k+1)!}|z|^{k+1} + |z|^{k+2} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n|.$$

Since, for $f(z) \in \mathscr{N}^*(\alpha)$,

$$(k+2)! \, (2-\alpha) \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \leqslant \sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leqslant \frac{1}{2}(1 - |1 - 2\alpha|), \quad (3.14)$$

we find that

$$\sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \leqslant \frac{1-|1-2\alpha|}{2 \cdot (k+2)! \, (2-\alpha)}. \tag{3.15}$$

Therefore, we have

$$\frac{1}{(k+1)!}|z|^{k+1} - \frac{1-|1-2\alpha|}{2 \cdot (k+2)! \, (2-\alpha)}|z|^{k+2} \leqslant |I_k f(z)|$$

$$\leqslant \frac{1}{(k+1)!}|z|^{k+1} + \frac{1-|1-2\alpha|}{2 \cdot (k+2)! \, (2-\alpha)}|z|^{k+2} \quad (z \in \mathbb{U}; \quad k \in \mathbb{N}_0 \cup \{-1\}). \tag{3.16}$$

In the exceptional case of (3.16) when $k = -2$, we have

$$2|a_2| - \frac{1}{2}(1-|1-2\alpha|)|z| \leqslant |I_{-2}f(z)| \leqslant 2|a_2| + \frac{1}{2}(1-|1-2\alpha|)|z| \quad (z \in \mathbb{U}). \tag{3.17}$$

$\square$

By setting $k = -1, 0, 1$ in Theorem 4, we deduce the following corollary.

COROLLARY 2. *If $f(z) \in \mathcal{N}^*(\alpha)$, then*

$$1 - \frac{1-|1-2\alpha|}{2(2-\alpha)}|z| \leqslant |f'(z)| \leqslant 1 + \frac{1-|1-2\alpha|}{2(2-\alpha)}|z| \qquad (k = -1), \tag{3.18}$$

$$|z| - \frac{1-|1-2\alpha|}{4(2-\alpha)}|z|^2 \leqslant |f(z)| \leqslant |z| + \frac{1-|1-2\alpha|}{4(2-\alpha)}|z|^2 \qquad (k = 0), \tag{3.19}$$

and

$$\frac{1}{2}|z|^2 - \frac{1-|1-2\alpha|}{12(2-\alpha)}|z|^3 \leqslant |I_1 f(z)| \leqslant \frac{1}{2}|z|^2 + \frac{1-|1-2\alpha|}{12(2-\alpha)}|z|^3 \quad (k = 1). \tag{3.20}$$

4. Inclusion relationships between the function classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$

Using the coefficient inequalities for the classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$, we now derive Theorem 5 below.

THEOREM 5. *The following inclusion relationships hold true for the class $\mathcal{M}^*(\alpha)$:*

(A) $\mathcal{M}^*(\alpha) \subset \mathcal{M}^*(1-\alpha) \quad \left(0 < \alpha \leqslant \frac{1}{2}\right).$

(B) $\mathcal{M}^*(\alpha) \subset \mathcal{M}^*\left(1 - \frac{1}{3-2\alpha}\right) \quad \left(\frac{1}{2} \leqslant \alpha < 1\right).$

(C) $\mathcal{M}^*(\alpha) \subset \mathcal{M}^*(\beta) \quad \left(0 < \alpha \leqslant \beta \leqslant \frac{1}{2}\right).$

(D) $\mathcal{M}^*(\beta) \subset \mathcal{M}^*(\alpha) \quad \left(\frac{1}{2} \leqslant \alpha \leqslant \beta < 1\right).$

Proof. (A) For $0 < \alpha \leqslant \dfrac{1}{2}$ and $\dfrac{1}{2} \leqslant \beta < 1$,
we consider the maximum value of β such that

$$\sum_{n=2}^{\infty} \frac{n - \beta}{1 - \beta}|a_n| \leqslant \sum_{n=2}^{\infty} \frac{n - \alpha}{\alpha}|a_n| \leqslant 1. \qquad (4.1)$$

Thus we need to find the maximum value of β such that

$$\beta \leqslant \frac{n(1 - \alpha) - \alpha}{n - 2\alpha} \qquad (n \in \mathbb{N}\setminus\{1\}). \qquad (4.2)$$

By taking the derivative of the right-hand side of (4.2) with respect to n, it is easily seen that the right-hand side of (4.2) is monotonically decreasing for n. Thus, upon letting $n \to \infty$, we have $\beta = 1 - \alpha$. Noting also that

$$\frac{1}{2} \leqslant \beta < 1 \qquad \text{for} \qquad 0 < \alpha \leqslant \frac{1}{2},$$

we have

$$\mathscr{M}^*(\alpha) \subset \mathscr{M}^*(1 - \alpha) \qquad \left(0 < \alpha \leqslant \frac{1}{2}\right),$$

which evidently completes the proof of (A).

The proofs of (B), (C), and (D) are much akin to the proof of (A). $\square$

Finally, we consider some relationships between the function classes $\mathscr{M}^*(\alpha)$ and $\mathscr{N}^*(\alpha)$.

THEOREM 6. *Each of the following assertions holds true:*
(A) *If* $f(z) \in \mathscr{N}^*(\alpha)$ *for* $0 < \alpha \leqslant \dfrac{1}{2}$, *then* $f(z) \in \mathscr{M}^*\left(\dfrac{4 - 4\alpha}{4 - 3\alpha}\right)$.
(B) *If* $f(z) \in \mathscr{N}^*(\alpha)$ *for* $\dfrac{1}{2} \leqslant \alpha < 1$, *then* $f(z) \in \mathscr{M}^*\left(\dfrac{2 - 2\alpha}{5 - 3\alpha}\right)$.
(C) *If* $f(z) \in \mathscr{N}^*(\alpha)$ *for* $0 < \alpha \leqslant \dfrac{1}{2}$, *then* $f(z) \in \mathscr{M}^*\left(\dfrac{2\alpha}{4 - \alpha}\right)$.
(D) *If* $f(z) \in \mathscr{N}^*(\alpha)$ *for* $\dfrac{1}{2} \leqslant \alpha < 1$, *then* $f(z) \in \mathscr{M}^*\left(\dfrac{2}{3 - \alpha}\right)$.

Proof. (A) Let $0 < \alpha \leqslant \dfrac{1}{2}$ and $\dfrac{1}{2} \leqslant \beta < 1$.
We consider the maximum value of β such that

$$\sum_{n=2}^{\infty} \frac{n - \beta}{1 - \beta}|a_n| \leqslant \sum_{n=2}^{\infty} \frac{n(n - \alpha)}{\alpha}|a_n| \leqslant 1. \qquad (4.3)$$

This means that

$$\beta \leqslant \frac{n^2 - 2n\alpha}{n^2 - n\alpha - \alpha} \quad (n \in \mathbb{N}\setminus\{1\}). \qquad (4.4)$$

If we take the derivative of the right-hand side of (4.4) with respect to n, then the numerator becomes

$$n^2\alpha - 2n\alpha + 2\alpha^2 \geqslant 0 \qquad \left(0 < \alpha \leqslant \frac{1}{2}; \quad n \in \mathbb{N}\backslash\{1\}\right). \tag{4.5}$$

Therefore, the right-hand side of (4.4) is monotonically increasing for n.
Thus, by setting $n = 2$, we have

$$\beta = \frac{4 - 4\alpha}{4 - 3\alpha}.$$

It is easy to see that

$$\frac{1}{2} \leqslant \beta < 1 \qquad \text{for} \qquad 0 < \alpha \leqslant \frac{1}{2},$$

which obviously completes the proof of (A).

The proofs of (B), (C) and (D) would run parallel to the proof of (A). $\quad\square$

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