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ON DELAY DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC BOUNDARY CONDITIONS STARTED FROM DIFFERENT POINTS

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Abstract. In this paper, we discuss the problems of existence of extremal solutions of delay differential equations which satisfy almost periodic boundary conditions. Some comparison results are given. Corresponding existence results are also formulated for differential equations having more delayed arguments.

1. Introduction

The investigation of the initial value problems of differential equations relative to the changes in the initial time was first discussed in papers [6,7], see also [1]. Such problems are important when we need to compare solutions of two problems started from different initial points. Corresponding results are also formulated for integro–differential problems, see [2,4]. This paper extends this topic on delay differential problems with almost periodic boundary conditions started from different points.

Let us introduce the operator F by

$$Fx(t;t_0) = f\,(t, x(t), x(\alpha(t-t_0)+t_0)).$$

Consider two problems

$$x'(t) = Fx(t;t_0), \qquad t \in J_1 = [t_0, t_0+T], \qquad x(t_0) = x(t_0+T)+\bar{k}, \tag{1}$$

$$x'(t) = Fx(t;\tau_0), \qquad t \in J_2 = [\tau_0, \tau_0+T], \qquad x(\tau_0) = x(\tau_0+T)+\bar{k} \tag{2}$$

for some fixed positive $T > 0$.

To obtain sufficient conditions under which differential problems have solutions someone can apply the monotone iterative method, for details, see [5]. Note that the application of this method to delay differential problems can be found, for example in [3,8,9]. In this paper, we discuss the problem of existence of extremal solutions to (1) and (2) using the monotone iterative technique. We assume that f satisfies a one sided Lipschitz condition with respect to the last two variables with corresponding Lipschitz functions. It is important to indicate that in such situation assumptions are less restricted in comparing with corresponding ones when in the place of functions we have Lipschitz constants. The problem when we have more delayed arguments is also discussed.

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2. Comparison results

Theorem 1 gives a comparison result between any two functions satisfying delay differential inequalities starting at different initial points.

THEOREM 1. *Assume that*

1° $f \in C(J_0 \times \mathbb{R}^2, \mathbb{R}),\ \alpha \in C(J, J),\ 0 \leqslant \alpha(t) \leqslant t$ *for* $J_0 = J_1 \cup J_2,\ \ J = [0, T],$

2° $v \in C^1(J_1, \mathbb{R}),\ w \in C^1(J_2, \mathbb{R}),\ \ v(t_0 + T) \leqslant w(\tau_0 + T)$ *and*

$$\begin{cases} v'(t) \leqslant Fv(t; t_0), & t \in J_1, \quad v(t_0) \leqslant v(t_0 + T) + \bar{k}, \\ w'(t) \geqslant Fw(t; \tau_0), & t \in J_2, \quad w(\tau_0) \geqslant w(\tau_0 + T) + \bar{k}, \end{cases}$$

3° $\eta = \tau_0 - t_0 > 0,$

4° $f(t, u_1, u_2)$ *is nondecreasing in* t *for each* $(u_1, u_2),$

5° $f(t, u_1, u_2)$ *is nondecreasing in* u_2 *for each* $(t, u_1),$

6° *there exists a nonnegative constant* L *such that*

$$f(t, u_1, v_1) - f(t, u_2, v_2) \leqslant L[u_1 - u_2 + v_1 - v_2]$$

for $t \in J_0,\ u_1, u_2, v_1, v_2 \in \mathbb{R},\ u_2 \leqslant u_1,\ v_2 \leqslant v_1.$

Then (a) $v(t) \leqslant w(t + \eta),\ t \in J_1,$ (b) $v(t - \eta) \leqslant w(t),\ t \in J_2.$

Proof. According to the assumptions on $\alpha,$ we see that $t_0 \leqslant \alpha(t - t_0) + t_0 \leqslant t$ for $t \in J_1$. Let $w_0(t) = w(t + \eta) + \epsilon \exp(3Lt),\ t \in J_1$ with $\epsilon > 0$. It yields $w_0(t) > w(t + \eta),\ t \in J_1$ and

$$w_0(t_0) > w(\tau_0) \geqslant w(\tau_0 + T) + \bar{k} \geqslant v(t_0 + T) + \bar{k} \geqslant v(t_0).$$

Consequently, according to Assumptions $2° - 4°, 6°,$ we get

$$\begin{aligned} w_0'(t) &= w'(t + \eta) + 3L\epsilon e^{3Lt} \\ &\geqslant f(t + \eta, w(t + \eta), w(\alpha(t + \eta - \tau_0) + \tau_0)) + 3L\epsilon \exp(3Lt) \\ &= f(t + \eta, w(t + \eta), w(\alpha(t - t_0) + \tau_0)) - f(t + \eta, w_0(t), w_0(\alpha(t - t_0) + t_0)) \\ &\quad + f(t + \eta, w_0(t), w_0(\alpha(t - t_0) + t_0)) + 3L\epsilon \exp(3Lt) \\ &\geqslant f(t, w_0(t), w_0(\alpha(t - t_0) + t_0)) + 3L\epsilon \exp(3Lt) \\ &\quad - L[w_0(t) - w(t + \eta) + w_0(\alpha(t - t_0) + t_0) - w(\alpha(t - t_0) + \tau_0)] \\ &= f(t, w_0(t), w_0(\alpha(t - t_0) + t_0)) + 3L\epsilon \exp(3Lt) \\ &\quad - L\epsilon\, [\exp(3Lt) + \exp(3L(\alpha(t - t_0) + t_0))] \\ &\geqslant f(t, w_0(t), w_0(\alpha(t - t_0) + t_0)) + L\epsilon \exp(3Lt) \\ &> f(t, w_0(t), w_0(\alpha(t - t_0) + t_0)). \end{aligned}$$

Now, we shall show that $v(t) < w_0(t),\ t \in J_1$. Assume that $v(t) < w_0(t),\ t \in J_1$ is false. Then there exists $t_1 \in (t_0, t_0 + T]$ such that $v(t_1) = w_0(t_1)$ and $v(t) < w_0(t),$ $t \in [t_0, t_1)$. Hence, $v'(t_1) \geqslant w_0'(t_1)$. In view of assumption $5°,$ we have

$$w_0'(t_1) \leqslant v'(t_1) \leqslant Fv(t_1; t_0) \leqslant Fw_0(t_1; t_0) < w_0'(t_1)$$

which is a contradiction. Hence, it follows that $v(t) < w_0(t),\ t \in J_1$. Letting $\epsilon \to 0$, we have $v(t) \leqslant w(t+\eta),\ t \in J_1$, so conclusion (a) is true. Relation (b) results from (a). The proof is therefore complete. □

Similarly, we can prove the following

THEOREM 2. *Let conditions* $1^\circ, 2^\circ, 5^\circ$ *and* 6° *of Theorem* 1 *hold and* $\eta = \tau_0 - t_0 < 0$. *Moreover assume that* $f(t,u_1,u_2)$ *is nonincreasing in* t *for each* (u_1,u_2).
Then conclusions (a) *and* (b) *of Theorem* 1 *hold.*

Indeed, instead of differential inequalities we can also consider integral inequalities to obtain similar results to that given in Theorems 1 or 2. In such case we extra need to assume that f is nondecreasing with respect to the second variable, for details see the next two theorems.

THEOREM 3. *Let Assumptions* $1^\circ, 3^\circ - 6^\circ$ *of Theorem* 1 *be satisfied. Assume that* $f(t,u_1,u_2)$ *is nondecreasing in* u_1 *for each* (t,u_2). *In addition assume that* $\Phi \in C(J_1,\mathbb{R}),\ \Psi \in C(J_2,\mathbb{R})$ *and*

$$\begin{cases} \Phi(t) \leqslant v(t_0) + \displaystyle\int_{t_0}^{t} F\Phi(s;t_0)ds, \quad t \in J_1, \ \ v(t_0) \leqslant v(t_0+T) + \bar{k} \\ \Psi(t) \geqslant w(\tau_0) + \displaystyle\int_{\tau_0}^{t} F\Psi(s;\tau_0)ds, \quad t \in J_2, \quad w(\tau_0) \geqslant w(\tau_0+T) + \bar{k}, \end{cases} \tag{3}$$

and

$$v(t_0+T) \leqslant w(\tau_0+T). \tag{4}$$

Then

$$(c)\ \Phi(t) \leqslant \Psi(t+\eta),\ \ t \in J_1, \quad (d) \quad \Phi(t-\eta) \leqslant \Psi(t),\ \ t \in J_2.$$

Proof. Define the functions v and w by

$$v(t) = v(t_0) + \int_{t_0}^{t} F\Phi(s;t_0)ds, \ \ t \in J_1, \qquad w(t) = w(\tau_0) + \int_{\tau_0}^{t} F\Psi(s;\tau_0)ds, \ \ t \in J_2.$$

Obviously, $\Phi(t) \leqslant v(t),\ t \in J_1,\ \ \Psi(t) \geqslant w(t),\ t \in J_2$. It is easy to see that

$$\begin{cases} v'(t) = F\Phi(t;t_0) \leqslant Fv(t;t_0), \ \ t \in J_1, \\ w'(t) = F\Psi(t;\tau_0) \geqslant Fw(t;\tau_0), \ \ t \in J_2 \end{cases}$$

since $f(t,u_1,u_2)$ is nondecreasing in (u_1,u_2). Hence, the conclusions (a) and (b) of Theorem 1 are satisfied. It shows that (c) and (d) hold too. This completes the proof. □

THEOREM 4. *Let all assumptions of Theorem* 2 *hold with conditions* $(3)-(4)$ *in the place of Assumption* 2°. *Moreover assume that* $f(t,u_1,u_2)$ *is nondecreasing in* u_1 *for each* (t,u_2).
Then conclusions (c) *and* (d) *of Theorem* 3 *hold.*

3. Delay differential inequalities

This section deals with some useful delay differential inequalities. As we see later, such inequalities play an important role in the investigations of existence of solutions of problems (1) and (2).

LEMMA 1. *Let* $\alpha \in C(J,J)$, $0 \leqslant \alpha(t) \leqslant t$, *on* $J = [0,T]$. *Assume that* $L_1 \in C(J_1,\mathbb{R})$, $p \in C^1(J_1,\mathbb{R})$ *and*

$$\begin{cases} p'(t) \leqslant -L_1(t)p(t) - L_2(t)p(\alpha(t-t_0)+t_0), \ \ t \in J_1 = [t_0, t_0+T], \\ p(t_0) \leqslant 0, \end{cases} \tag{5}$$

where nonnegative function L_2 *is integrable on* J_1. *In addition assume that*

$$W(t_0) \leqslant 1, \tag{6}$$

where

$$W(c) = \int_c^{c+T} L_2(s) e^{\int_{\alpha(s-c)+c}^{s} L_1(r)dr} ds. \tag{7}$$

Then $p(t) \leqslant 0$ *on* J_1.

Proof. Note that the assertion holds if $L_2(t) = 0$ on J_1. Let $\int_{t_0}^{t_0+T} L_2(s)ds > 0$. Setting,

$$q(t) = e^{\int_{t_0}^{t} L_1(s)ds} p(t), \ \ t \in J_1,$$

we obtain

$$\begin{aligned} q'(t) &= e^{\int_{t_0}^{t} L_1(s)ds} \left[L_1(t)p(t) + p'(t)\right] \\ &\leqslant -e^{\int_{t_0}^{t} L_1(s)ds} L_2(t)p(\alpha(t-t_0)+t_0) \\ &= -L_2(t) e^{\int_{\alpha(t-t_0)+t_0}^{t} L_1(s)ds} q(\alpha(t-t_0)+t_0). \end{aligned}$$

Hence (5) takes the form

$$\begin{cases} q'(t) \leqslant -L_2(t) e^{\int_{\alpha(t-t_0)+t_0}^{t} L_1(s)ds} q(\alpha(t-t_0)+t_0), \ \ t \in J_1, \\ q(t_0) \leqslant 0. \end{cases} \tag{8}$$

We need to show that $q(t) \leqslant 0$ on J_1. Suppose that it is not true, then we can find $t_1 \in (t_0, t_0+T)$ such that $q(t_1) > 0$. Put

$$q(t_2) = \min_{[t_0,t_1]} q(t) \leqslant 0.$$

Integrating the differential inequality in (8) from t_2 to t_1 we have

$$\begin{aligned} q(t_1) - q(t_2) &\leqslant -\int_{t_2}^{t_1} L_2(s) e^{\int_{\alpha(s-t_0)+t_0}^{s} L_1(r)dr} q(\alpha(s-t_0)+t_0) ds \\ &\leqslant -\int_{t_0}^{t_0+T} L_2(s) e^{\int_{\alpha(s-t_0)+t_0}^{s} L_1(r)dr} ds \, q(t_2) \leqslant -q(t_2), \end{aligned}$$

by condition (6). It contradicts assumption that $q(t_1) > 0$. This proves that $q(t) \leqslant 0$ on J_1 and hence $p(t) \leqslant 0$ on J_1 too. The proof is complete. □

REMARK 1. If $t_0 = 0$, $L_1(t) = L_1 > 0$, $L_2(t) = L_2 > 0$ on J_1, then (6) takes the form

$$L_2 \int_0^T e^{L_1[t-\alpha(t)]} dt \leqslant 1.$$

Such condition appeared in paper [9].

REMARK 2. Let $L_1(t) \geqslant 0$ on J_1 and

$$\int_{t_0}^{t_0+T} L_2(s) e^{\int_{t_0}^{s} L_1(r)dr} ds \leqslant 1.$$

Then condition (6) holds. Note that in this case α is absent. If $L_1(t) = 0$, $t \in J_1$, then (6) holds if $\int_{t_0}^{t_0+T} L_2(s)ds \leqslant 1$.

REMARK 3. Assume that $L_1(t) = L_1 > 0$, $L_2(t) = L_2 > 0$, $t \in J_0$ and

$$L_2 \left[e^{L_1 T} - 1\right] \leqslant L_1.$$

Then

$$\max(W(t_0), W(\tau_0)) \leqslant 1,$$

where W is defined as in (7), see also [3].

4. Application of the monotone iterative technique

In this section, we discuss the monotone iterative method to delay differential equations subject to almost periodic boundary conditions started from different points.

THEOREM 5. *Assume that*

1° $f \in C(J_0 \times \mathbb{R}^2, \mathbb{R})$ *for* $J_0 = J_1 \cup J_2$, *and* $\alpha \in C(J, J)$, $0 \leqslant \alpha(t) \leqslant t$ *for* $t \in J$,

2° $\eta = \tau_0 - t_0 > 0$,

3° $y_0 \in C^1(J_1, \mathbb{R})$, $z_0 \in C^1(J_2, \mathbb{R})$ *and*

$$y_0'(t) \leqslant Fy_0(t;t_0), \quad t \in J_1, \quad y_0(t_0) \leqslant y_0(t_0+T) + \bar{k},$$
$$z_0'(t) \geqslant Fz_0(t;\tau_0), \quad t \in J_2, \quad z_0(\tau_0) \geqslant z_0(\tau_0+T) + \bar{k},$$

and $y_0(t) \leqslant z_0(t+\eta)$, $t \in J_1$,

4° *there exist functions* L_1, L_2 *such that* $L_1 \in C(J_0, \mathbb{R})$, $L_2 \in C(J_0, \mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$ *and*

$$f(t, u_1, v_1) - f(t, \bar{u}_1, \bar{v}_1) \geqslant -L_1(t)[u_1 - \bar{u}_1] - L_2(t)[v_1 - \bar{v}_1] \tag{9}$$

for $y_0(t) \leqslant \bar{u}_1 \leqslant u_1 \leqslant z_0(t+\eta)$, $y_0(\alpha(t-t_0)+t_0) \leqslant \bar{v}_1 \leqslant v_1 \leqslant z_0(\alpha(t-t_0)+\tau_0)$, $t \in J_1$; *note that if* $t \in J_2$, *then relation (9) holds for* $y_0(t-\eta) \leqslant \bar{u}_1 \leqslant u_1 \leqslant z_0(t)$, $y_0(\alpha(t-\tau_0)+t_0) \leqslant \bar{v}_1 \leqslant v_1 \leqslant z_0(\alpha(t-\tau_0)+\tau_0)$,

5° $\max(W(t_0), W(\tau_0)) \leqslant 1$, *where* W *is defined by* (7),

6° f *is nondecreasing with respect to the first variable.*

Then there exist monotone sequences $\{y_n, z_n\}$ *such that* $y_n(t) \to y(t)$, $t \in J_1$, $z_n(t) \to z(t)$, $t \in J_2$ *uniformly and*

$$y_0(t) \leqslant y(t) \leqslant z(t+\eta) \leqslant z_0(t+\eta), \quad t \in J_1,$$
$$y_0(t-\eta) \leqslant y(t-\eta) \leqslant z(t) \leqslant z_0(t), \quad t \in J_2.$$

The function y *is the minimal solution of problem* (1) *in the sector* $[y_0, z_0]_0$, *while* z *is the maximal solution of* (2) *in the sector* $[y_0, z_0]_1$, *where*

$$[y_0, z_0]_i = \{u \in C^1(J_{i+1}, \mathbb{R}) : y_0(t) \leqslant u(t+i\eta) \leqslant z_0(t+\eta),\ t \in J_1\},\ i = 0, 1.$$

Proof. Let

$$\begin{cases} y'_{n+1}(t) = Fy_n(t;t_0) - L_1(t)[y_{n+1}(t) - y_n(t)] \\ \qquad\qquad - L_2(t)[y_{n+1}(\alpha(t-t_0)+t_0) - y_n(\alpha(t-t_0)+t_0)], \quad t \in J_1, \\ y_{n+1}(t_0) = y_n(t_0+T) + \bar{k}, \end{cases}$$

$$\begin{cases} z'_{n+1}(t) = Fz_n(t;\tau_0) - L_1(t)[z_{n+1}(t) - z_n(t)] \\ \qquad\qquad - L_2(t)[z_{n+1}(\alpha(t-\tau_0)+\tau_0) - z_n(\alpha(t-\tau_0)+\tau_0)], \quad t \in J_2, \\ z_{n+1}(\tau_0) = z_n(\tau_0+T) + \bar{k} \end{cases}$$

for $n = 0, 1, \ldots,$. Note that the sequences $\{y_n, z_n\}$ are well defined.

We first show that

$$y_0(t) \leqslant y_1(t) \leqslant z_0(t+\eta), \quad t \in J_1. \tag{10}$$

Put $p(t) = y_0(t) - y_1(t)$, $t \in J_1$. Then $p(t_0) \leqslant 0$, and

$$\begin{aligned} p'(t) &\leqslant Fy_0(t;t_0) - Fy_0(t;t_0) + L_1(t)[y_1(t) - y_0(t)] \\ &\quad + L_2(t)[y_1(\alpha(t-t_0)+t_0) - y_0(\alpha(t-t_0)+t_0)] \\ &= -L_1(t)p(t) - L_2(t)p(\alpha(t-t_0)+t_0) \end{aligned}$$

for $t \in J_1$. By Lemma 1, $p(t) \leqslant 0$, $t \in J_1$, so $y_0(t) \leqslant y_1(t)$, $t \in J_1$. Let $p(t) = y_1(t) - z_0(t+\eta)$, $t \in J_1$, so $p(t_0) \leqslant 0$. In view of Assumptions $3^\circ, 4^\circ$ and 6°, we have

$$\begin{aligned} p'(t) &\leqslant Fy_0(t;t_0) - Fz_0(t+\eta;\tau_0) - L_1(t)[y_1(t) - y_0(t)] \\ &\quad - L_2(t)[y_1(\alpha(t-t_0)+t_0) - y_0(\alpha(t-t_0)+t_0)] \\ &\leqslant L_1(t)[z_0(t+\eta) - y_0(t)] + L_2(t)[z_0(\alpha(t+\eta-\tau_0)+\tau_0) - y_0(\alpha(t-t_0)+t_0)] \\ &\quad - L_1(t)[y_1(t) - y_0(t)] - L_2(t)[y_1(\alpha(t-t_0)+t_0) - y_0(\alpha(t-t_0)+t_0)] \\ &= -L_1(t)p(t) - L_2(t)p(\alpha(t-t_0)+t_0). \end{aligned}$$

It yields, $y_1(t) \leqslant z_0(t+\eta)$, $t \in J_1$, by Lemma 1, so (10) holds.

In the next step we show that

$$y_0(t-\eta) \leqslant z_1(t) \leqslant z_0(t),\ t \in J_2. \tag{11}$$

Put $p(t) = z_1(t) - z_0(t),\ t \in J_2,$ so $p(\tau_0) \leqslant 0.$ In view of $3^\circ,$ we have

$$\begin{aligned} p'(t) &\leqslant z_1'(t) - Fz_0(t;\tau_0) \\ &= -L_1(t)p(t) - L_2(t)p(\alpha(t-\tau_0)+\tau_0). \end{aligned}$$

Because $W(\tau_0) \leqslant 1,$ it yields $z_1(t) \leqslant z_0(t),\ t \in J_2,$ by Lemma 1. Now let $p(t) = y_0(t-\eta) - z_1(t),\ t \in J_2,$ so $p(\tau_0) \leqslant 0,$ and

$$\begin{aligned} p'(t) \leqslant{}& Fy_0(t-\eta;t_0) - Fz_0(t;\tau_0) + L_1(t)[z_1(t) - z_0(t)] \\ &+ L_2(t)[z_1(\alpha(t-\tau_0)+\tau_0) - z_0(\alpha(t-\tau_0)+\tau_0)] \\ \leqslant{}& L_1(t)[z_0(t) - y_0(t-\eta)] + L_2(t)[z_0(\alpha(t-\tau_0)+\tau_0) - y_0(\alpha(t-\eta-t_0)+t_0)] \\ &+ L_1(t)[z_1(t) - z_0(t)] + L_2(t)[z_1(\alpha(t-\tau_0)+\tau_0) - z_0(\alpha(t-\tau_0)+\tau_0)] \\ ={}& -L_1(t)p(t) - L_2(t)p(\alpha(t-\tau_0)+\tau_0), \end{aligned}$$

by Assumptions $3^\circ, 4^\circ$ and 6°. In view of Lemma 1, $y_0(t-\eta) \leqslant z_1(t),\ t \in J_2,$ so relation (11) holds.

Note that

$$\begin{cases} y_0(t) \leqslant y_1(t) \leqslant z_0(t+\eta),\ t \in J_1, \\ y_0(t) \leqslant z_1(t+\eta) \leqslant z_0(t+\eta),\ t \in J_1, \end{cases} \tag{12}$$

and

$$\begin{cases} y_0(t-\eta) \leqslant y_1(t-\eta) \leqslant z_0(t),\ t \in J_2, \\ y_0(t-\eta) \leqslant z_1(t) \leqslant z_0(t),\ t \in J_2 \end{cases} \tag{13}$$

result from (10) and (11), by changing of variables.

In the next step we need to prove that

$$y_1'(t) \leqslant Fy_1(t;t_0),\ \ t \in J_1, \qquad y_1(t_0) \leqslant y_1(t_0+T) + \bar{k}. \tag{14}$$

In view of (12) and Assumption 4°, we have

$$\begin{aligned} y_1'(t) ={}& Fy_0(t;t_0) - Fy_1(t;t_0) + Fy_1(t;t_0) - L_1(t)[y_1(t) - y_0(t)] \\ &- L_2(t)[y_1(\alpha(t-t_0)+t_0) - y_0(\alpha(t-t_0)+t_0)] \\ \leqslant{}& L_1(t)[y_1(t) - y_0(t)] + L_2(t)[y_1(\alpha(t-t_0)+t_0) - y_0(\alpha(t-t_0)+t_0)] \\ &- L_1(t)[y_1(t) - y_0(t)] - L_2(t)[y_1(\alpha(t-t_0)+t_0) - y_0(\alpha(t-t_0)+t_0)] \\ &+ Fy_1(t;t_0) = Fy_1(t;t_0), \end{aligned}$$

and $y_1(t_0) = y_0(t_0+T) + \bar{k} \leqslant y_1(t_0+T) + \bar{k}.$ It proves that (14) holds. Similarly, we can show that

$$z_1'(t) \geqslant Fz_1(t;\tau_0),\ \ t \in J_2, \qquad z_1(\tau_0) \geqslant z_1(\tau_0+T) + \bar{k}. \tag{15}$$

Let $q(t) = y_1(t) - z_1(t+\eta),\ t \in J_1,$ so $q(t_0) \leqslant 0.$ Hence

$$\begin{aligned} q'(t) \leqslant{}& Fy_0(t;t_0) - Fz_1(t+\eta;\tau_0) - L_1(t)[y_1(t) - y_0(t)] \\ &- L_2(t)[y_1(\alpha(t-t_0)+t_0) - y_0(\alpha(t-t_0)+t_0)] \\ \leqslant{}& L_1(t)[z_1(t+\eta) - y_0(t)] + L_2(t)[z_1(\alpha(t+\eta-\tau_0)+\tau_0) - y_0(\alpha(t-t_0)+t_0)] \\ &- L_1(t)[y_1(t) - y_0(t)] - L_2(t)[y_1(\alpha(t-t_0)+t_0) - y_0(\alpha(t-t_0)+t_0)] \\ ={}& -L_1(t)q(t) - L_2(t)q(\alpha(t-t_0)+t_0), \end{aligned}$$

by (15) and Assumptions $4°, 6°$. Obviously, $y_1(t) \leqslant z_1(t+\eta),\ t \in J_1,$ by Lemma 1.

Combining this with (12) and (13), we have

$$y_0(t) \leqslant y_1(t) \leqslant z_1(t+\eta) \leqslant z_0(t+\eta), \quad t \in J_1,$$
$$y_0(t-\eta) \leqslant y_1(t-\eta) \leqslant z_1(t) \leqslant z_0(t), \quad t \in J_2.$$

Now, it is easy to show, by mathematical induction, that

$$y_0(t) \leqslant y_1(t) \leqslant \cdots \leqslant y_n(t) \leqslant z_n(t+\eta) \leqslant \cdots \leqslant z_1(t+\eta) \leqslant z_0(t+\eta),\ t \in J_1,$$
$$y_0(t-\eta) \leqslant y_1(t-\eta) \leqslant \cdots \leqslant y_n(t-\eta) \leqslant z_n(t) \leqslant \cdots \leqslant z_1(t) \leqslant z_0(t),\ t \in J_2.$$

By standard arguments, $y_n(t) \to y(t),\ t \in J_1,\ z_n(t) \to z(t),\ t \in J_2$ uniformly. Indeed, $y \in C^1(J_1, \mathbb{R}), z \in C^1(J_2, \mathbb{R})$ are solutions of problems (1) and (2), respectively.

It remains to show that y is the minimal solution of problem (1) in the sector $[y_0, z_0]_0$. Assume that there exists another solution u of (1) such that $y_0(t) \leqslant u(t) \leqslant z_0(t+\eta),\ t \in J_1$. Put $p(t) = y_1(t) - u(t),\ t \in J_1,$ so $p(t_0) \leqslant 0$. In view of Assumption $4°$, we see that

$$\begin{aligned} p'(t) &= Fy_0(t;t_0) - Fu(t;t_0) - L_1(t)[y_1(t) - y_0(t)] \\ &\quad - L_2(t)[y_1(\alpha(t-t_0)+t_0) - y_0(\alpha(t-t_0)+t_0)] \\ &\leqslant -L_1 p(t) - L_2 p(\alpha(t-t_0)+t_0). \end{aligned}$$

This and Lemma 1 prove that $y_1(t) \leqslant u(t) \leqslant z_0(t+\eta),\ t \in J_1$. By induction, we can show that $y_n(t) \leqslant u(t) \leqslant z_0(t+\eta)$ for $t \in J_1$ and all natural n. Now, if $n \to \infty$, then

$$y_0(t) \leqslant y(t) \leqslant u(t) \leqslant z_0(t+\eta),\ t \in J_1.$$

It proves that y is the minimal solution of (1) in the sector $[y_0, z_0]_0$. By a similar way, we can show that z is the maximal solution of (2) in the sector $[y_0, z_0]_1$.

This completes the proof. □

THEOREM 6. *Let Assumptions* $1°, 3° - 5°$ *of Theorem* 5 *be satisfied. Moreover, assume that* $\eta = \tau_0 - t_0 < 0$, *and* f *is nonincreasing with respect to the first variable. Then the assertion of Theorem* 5 *holds.*

5. Generalizations

In this section we consider two boundary–value problems of the form

$$x'(t) = \mathscr{F}x(t;t_0), \quad t \in J_1 = [t_0, t_0+T], \quad x(t_0) = x(t_0+T) + \bar{k}, \tag{16}$$

$$x'(t) = \mathscr{F}x(t;\tau_0), \quad t \in J_2 = [\tau_0, \tau_0+T], \quad x(\tau_0) = x(\tau_0+T) + \bar{k}, \tag{17}$$

where

$$\mathscr{F}x(t;t_0) = f(t, x(t), x(\alpha_1(t-t_0)+t_0), \cdots, x(\alpha_r(t-t_0)+t_0)).$$

In this general case we can formulate similar results to the corresponding ones of this paper but we only formulate corresponding results to Theorems 5 and 6 without any proof.

THEOREM 7. *Assume that*

1° $f \in C(J_0 \times \mathbb{R}^{r+1}, \mathbb{R})$ *for* $J_0 = J_1 \cup J_2$, *and* $\alpha_i \in C(J, J)$, $0 \leqslant \alpha_i(t) \leqslant t$ *for* $t \in J$ *and* $i = 1, 2, \ldots, r$,

2° $\eta = \tau_0 - t_0 > 0$,

3° $y_0 \in C^1(J_1, \mathbb{R})$, $z_0 \in C^1(J_2, \mathbb{R})$ *and*

$$y_0'(t) \leqslant \mathscr{F} y_0(t; t_0), \quad t \in J_1, \quad y_0(t_0) \leqslant y_0(t_0 + T) + \bar{k},$$
$$z_0'(t) \geqslant \mathscr{F} z_0(t; \tau_0), \quad t \in J_2, \quad z_0(\tau_0) \geqslant z_0(\tau_0 + T) + \bar{k},$$

and $y_0(t) \leqslant z_0(t + \eta)$, $t \in J_1$,

4° *there exist functions* $L_0 \in C(J_0, \mathbb{R})$ *and* $L_i \in C(J_0, \mathbb{R}_+)$, $i = 1, 2, \cdots, r$ *such that*

$$f(t, v_0, v_1, \cdots, v_r) - f(t, \bar{v}_0, \bar{v}_1, \cdots, \bar{v}_r) \geqslant -\sum_{i=0}^{r} L_i(t)[v_i - \bar{v}_i] \tag{18}$$

for $y_0(t) \leqslant \bar{v}_0 \leqslant v_0 \leqslant z_0(t + \eta)$, $y_0(\alpha_i(t - t_0) + t_0) \leqslant \bar{v}_i \leqslant v_i \leqslant z_0(\alpha(t - t_0) + \tau_0)$, $t \in J_1$, $i = 1, 2, \cdots, r$; *if* $t \in J_2$, *then relation (18) holds for* $y_0(t - \eta) \leqslant \bar{v}_0 \leqslant v_0 \leqslant z_0(t)$, $y_0(\alpha_i(t - \tau_0) + t_0 \leqslant \bar{v}_i \leqslant v_i \leqslant z_0(\alpha_i(t - \tau_0) + \tau_0)$, $i = 1, 2, \cdots, r$,

5° $\max(\tilde{W}(t_0), \tilde{W}(\tau_0)) \leqslant 1$, *where*

$$\tilde{W}(c) = \int_c^{c+T} \sum_{i=1}^{r} L_i(t) e^{\int_{\alpha_i(t-c)+c}^{t} L_0(s)ds} dt \leqslant 1$$

6° f *is nondecreasing with respect to the first variable.*

Then there exist monotone sequences $\{y_n, z_n\}$ *such that* $y_n(t) \to y(t)$, $t \in J_1$, $z_n(t) \to z(t)$, $t \in J_2$ *uniformly and*

$$y_0(t) \leqslant y(t) \leqslant z(t + \eta) \leqslant z_0(t + \eta), \quad t \in J_1,$$
$$y_0(t - \eta) \leqslant y(t - \eta) \leqslant z(t) \leqslant z_0(t), \quad t \in J_2.$$

The function y *is the minimal solution of problem* (16) *in the sector* $[y_0, z_0]_0$, *while* z *is the maximal solution of* (17) *in the sector* $[y_0, z_0]_1$.

THEOREM 8. *Let Assumptions* $1°, 3° - 5°$ *of Theorem* 7 *be satisfied. Moreover, assume that* $\eta = \tau_0 - t_0 < 0$, *and* f *is nonincreasing with respect to the first variable. Then the assertion of Theorem* 7 *holds.*

EXAMPLE 1. Consider two problems

$$\begin{cases} x'(t) = Fx(t; 1), & t \in J_1 = [1, 2], \\ x(1) = x(2) \end{cases} \tag{19}$$

$$\begin{cases} x'(t) = Fx(t; 0), & t \in J_2 = [0, 1], \\ x(0) = x(1), \end{cases} \tag{20}$$

with

$$Fx(t;c) = -ax(t) - bt\,x(0.5(t-c)+c) - a, \quad a,b > 0.$$

Here $t_0 = 1,\ \tau_0 = 0,$ so $\eta < 0.$ Assume that

$$b\max[\frac{2}{a}e^{\frac{a}{2}}(1-\frac{2}{a}) + \frac{4}{a^2}, \frac{4}{a}e^{\frac{a}{2}}(1-\frac{1}{a}) + \frac{2}{a}(\frac{2}{a}-1)] \leqslant 1 \tag{21}$$

Put $y_0(t) = -1,\ t \in J_1,\ \ z_0(t) = 0,\ t \in J_2.$ Then assumption 3° of Theorem 5 holds. Assumption 4° of Theorem 5 is also satisfied with $L_1(t) = a,\ L_2(t) = bt,\ t \in [0,\,2].$ Note that, in view of (21), all assumptions of Theorem 6 holds, so problem (19) has the minimal solution in the sector $[y_0, z_0]_0$ and problem (20) has the maximal solution in the sector $[y_0, z_0]_1$.

For example if $a = \frac{1}{2},$ then condition (21) holds if

$$b \leqslant \min[\frac{1}{16-12e^{\frac{1}{4}}}, \frac{1}{12-8e^{\frac{1}{4}}}] \approx 0.58.$$

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