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AN ALTERNATIVE NOTE ON THE SCHUR–CONVEXITY OF THE EXTENDED MEAN VALUES

HUAN-NAN SHI, SHAN-HE WU AND FENG QI

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Abstract. The Schur-convex and Schur-concave properties with (x, y) in $(0,\infty)\times(0,\infty)$ for fixed (r, s) of the extended mean values $E(r, s; x, y)$ are researched again, some errors in [F. Qi, J. Sándor, S. S. Dragomir, and A. Sofo, *Notes on the Schur-convexity of the extended mean values*, Taiwanese J. Math. **9**, (3) (2005), 411–420. RGMIA Res. Rep. Coll. **5** (2002), no. 1, Art. 3, 19–27. Available online at *URL:* `http://rgmia.vu.edu.au/v5n1.html`.] are corrected.

1. Introduction

Let $\boldsymbol{x} = (x_0, \ldots, x_n)$ and $\boldsymbol{y} = (y_0, \ldots, y_n)$ denote two real $(n+1)$-tuples. $\boldsymbol{x}$ is said to be not greater than $\boldsymbol{y}$ (in symbols, $\boldsymbol{x} \leqslant \boldsymbol{y}$) if $x_i \leqslant y_i$ for $0 \leqslant i \leqslant n+1$. $\boldsymbol{x}$ is said to majorize $\boldsymbol{y}$ (in symbols, $\boldsymbol{x} \succ \boldsymbol{y}$) if

$$\sum_{i=0}^{k} x_{[i]} \geqslant \sum_{i=0}^{k} y_{[i]} \tag{1}$$

for $k = 0, 1, \ldots, n-1$ and

$$\sum_{i=0}^{n} x_i = \sum_{i=0}^{n} y_i, \tag{2}$$

where

$$x_{[0]} \geqslant x_{[1]} \geqslant \cdots \geqslant x_{[n]} \tag{3}$$

and

$$y_{[0]} \geqslant y_{[1]} \geqslant \cdots \geqslant y_{[n]} \tag{4}$$

are the decreasingly ordered components of $\boldsymbol{x}$ and $\boldsymbol{y}$. See [10, p. 75].

A function $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \leqslant \boldsymbol{y}$ implies $\psi(\boldsymbol{x}) \leqslant \psi(\boldsymbol{y})$. It has been proved in [21, p. 38, Proposition 4.3] that the function $\psi(\boldsymbol{x})$ is increasing

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if and only if $\nabla\psi(\boldsymbol{x}) \geqslant 0$ for $\boldsymbol{x} \in A$, where $A \subset \mathbb{R}^{n+1}$ is an open set, $\psi : A \to \mathbb{R}$ is differentiable, and

$$\nabla\psi(\boldsymbol{x}) = \left(\frac{\partial\psi(\boldsymbol{x})}{\partial x_0}, \dots, \frac{\partial\psi(\boldsymbol{x})}{\partial x_n}\right) \in \mathbb{R}^{n+1}. \tag{5}$$

A function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ is said to be Schur-convex if $\boldsymbol{x} \succ \boldsymbol{y}$ implies $g(\boldsymbol{x}) \geqslant g(\boldsymbol{y})$. A function f is Schur-concave if and only if $-f$ is Schur-convex. See [10, p. 332].

The extended mean values $E(r,s;x,y)$ were defined in [20] by

$$E(r,s;x,y) = \left[\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r}\right]^{1/(s-r)}, rs(r-s)(x-y) \neq 0; \tag{6}$$

$$E(r,0;x,y) = \left[\frac{1}{r} \cdot \frac{y^r - x^r}{\ln y - \ln x}\right]^{1/r}, r(x-y) \neq 0; \tag{7}$$

$$E(r,r;x,y) = \frac{1}{e^{1/r}}\left[\frac{x^{x^r}}{y^{y^r}}\right]^{1/(x^r-y^r)}, r(x-y) \neq 0; \tag{8}$$

$$E(0,0;x,y) = \sqrt{xy}, x \neq y; \tag{9}$$

$$E(r,s;x,x) = x, x = y;$$

where $x, y > 0$ and $(r,s) \in \mathbb{R}^2$.

Leach and Sholander [6] showed that $E(r,s;x,y)$ are increasing with both r and s, or with both x and y. Later, the monotonicity of the extended mean values E was also researched in [2, 3, 5, 16, 18, 19] by using different ideas and simpler approaches.

Leach and Sholander [7] and Páles [9] respectively solved the problem of comparison of E. They found necessary and sufficient conditions for the parameters r, s and u, v in order that $E(r,s;x,y) \leqslant E(u,v;x,y)$ be satisfied for all positive x and y.

The logarithmic convexity of the extended mean values $E(r,s;x,y)$ with two parameters r and s was obtained in [12, 13].

The Schur-convexities of the extended mean values $E(r,s;x,y)$ with (r,s) and (x,y) were presented in [11, 14, 17] as follows.

THEOREM A ([11]) *For fixed (x,y) with $x > 0$, $y > 0$ and $x \neq y$, the extended mean values $E(r,s;x,y)$ are Schur-concave on $[0,+\infty) \times [0,+\infty)$ and Schur-convex on $(-\infty,0] \times (-\infty,0]$ with (r,s).*

THEOREM B. ([17]) *For given (r,s) with $r, s \notin (0, \frac{3}{2})$ (or $r, s \in (0,1]$, resp.), the extended mean values $E(r,s;x,y)$ are Schur-concave (or Schur-convex, resp.) with (x,y) on the domain $(0,\infty) \times (0,\infty)$.*

For more information on the extended mean values E, please refer to [1, 15] and the references therein.

The following two counterexamples of Theorem B tell us that there must exist some errors about the proof of the above Theorem B.

EXAMPLE 1. Let $(r,s) = (4,2)$. It is clear that $(4,2) \notin \left(0,\frac{3}{2}\right) \times \left(0,\frac{3}{2}\right)$. For $(2,2) \succ (1,3)$, directly calculating yields

$$E(4,2;1,3) = \left(\frac{4}{2} \cdot \frac{3^2-1^2}{3^4-1^4}\right)^{1/(2-4)} = \sqrt{5} > E(4,2;2,2) = 2.$$

This leads to a contradiction with Theorem B.

EXAMPLE 2. Take $(r,s) = (1,1)$. It is clear that $(r,s) \in (0,1] \times (0,1]$. For $(2,2) \succ (1,3)$, straightforward computation gives

$$E(1,1;1,3) = e^{-1/1}\left(\frac{1^{1^1}}{3^{3^1}}\right)^{1/(1^1-3^1)} = \frac{3\sqrt{3}}{e} < E(1,1;2,2) = 2.$$

This also leads to a contradiction with Theorem B.

These two contradictions motivate us to reconsider to find a new approach to prove the Schur-convexity of the extended mean values $E(r,s;x,y)$ and obtain the following

THEOREM 1. *For fixed* $(r,s) \in \mathbb{R}^2$,

(1) *if* $2 < 2r < s$ *or* $2 \leqslant 2s \leqslant r$, *then the extended mean values* $E(r,s;x,y)$ *is Schur-convex with* $(x,y) \in (0,\infty) \times (0,\infty)$,

(2) *if* $(r,s) \in \{r < s \leqslant 2r, 0 < r \leqslant 1\} \cup \{s < r \leqslant 2s, 0 < s \leqslant 1\} \cup \{0 < s < r \leqslant 1\} \cup \{0 < r < s \leqslant 1\} \cup \{s \leqslant 2r < 0\} \cup \{r \leqslant 2s < 0\}$, *then the extended mean values* $E(r,s;x,y)$ *is Schur-concave with* $(x,y) \in (0,\infty) \times (0,\infty)$.

2. Lemmas

In order to verify Theorem 1, the following lemmas are necessary.

LEMMA 1. ([21, p. 64]) *Let* $g{:}I \to \mathbb{R}$, $\phi{:}\mathbb{R}^n \to \mathbb{R}$ *and* $\psi(x) = \phi\big(g(x_1),\dots,g(x_n)\big)$.

(1) *If* g *is convex (concave) and* ϕ *is increasing and Schur-convex (Schur-concave), then* ψ *is Schur-convex (Schur-concave);*

(2) *If* g *is concave (convex) and* ϕ *is decreasing and Schur-convex (Schur-concave), then* ψ *is Schur-convex (Schur-concave).*

LEMMA 2. ([21, p. 63]) *Let* $A \subset \mathbb{R}^n$, $\phi_i : A \to \mathbb{R}$ *for* $1 \leqslant i \leqslant k$, $h : \mathbb{R}^k \to \mathbb{R}$ *and* $\psi(x) = h\big(\phi_1(x),\dots,\phi_k(x)\big)$.

(1) *If each of* ϕ_i *for* $1 \leqslant i \leqslant k$ *is Schur-convex and* h *is increasing (decreasing), then* ψ *is Schur-convex (Schur-concave);*

(2) *If each of* ϕ_i *for* $1 \leqslant i \leqslant k$ *is Schur-concave and* h *is increasing (decreasing), then* ψ *is Schur-concave (Schur-convex).*

REMARK 1. These two lemmas can also be found in [8, p. 61] and [10, p. 334].

LEMMA 3. ([17]) *Let* f *be a continuous function and* p *a positive continuous weight on* I. *Then the weighted arithmetic mean of function* f *with weight* p *defined by*

$$F(x,y) = \begin{cases} \dfrac{\int_x^y p(t)f(t)\,\mathrm{d}t}{\int_x^y p(t)\,\mathrm{d}t}, & x \neq y, \\ f(x), & x = y \end{cases} \tag{10}$$

is Schur-convex (Schur-concave) on I^2 *if and only if inequality*

$$\frac{\int_x^y p(t)f(t)\,\mathrm{d}t}{\int_x^y p(t)\,\mathrm{d}t} \leqslant \frac{p(x)f(x)+p(y)f(y)}{p(x)+p(y)} \tag{11}$$

holds (reverses) for $(x,y) \in I^2$.

REMARK 2. A corresponding result for the special case $p \equiv 1$ of Lemma 3 has been showed in [1, Theorem 21, p. 384] and [4].

LEMMA 4. *Let* f *be a continuous function and* p *a positive continuous weight on* I. *Then the function* $F(x,y)$ *defined by* (10) *is increasing (decreasing) on* I^2 *if* f *is increasing (decreasing) on* I.

Proof. Direct calculation yields

$$\begin{aligned}\frac{\partial F(x,y)}{\partial y} &= -\frac{p(y)}{\left[\int_x^y p(t)\,\mathrm{d}t\right]^2}\int_x^y p(t)f(t)\,\mathrm{d}t + \frac{p(y)f(y)}{\int_x^y p(t)\,\mathrm{d}t}\\ &= \frac{p(y)}{\left[\int_x^y p(t)\,\mathrm{d}t\right]^2}\int_x^y p(t)\left[f(y)-f(t)\right]\mathrm{d}t.\end{aligned} \tag{12}$$

Hence, if f is increasing (decreasing), then $\frac{\partial F(x,y)}{\partial y}$ is positive (negative), this means that the function $F(x,y)$ is increasing (decreasing) with $y \in I$.

Since $F(x,y) = F(y,x)$ is symmetric, if f is increasing (decreasing), then $F(x,y)$ is also increasing (decreasing) with $x \in I$. The proof is complete.

REMARK 3. A special case of Lemma 4 for $p \equiv 1$ has been proved in [1, Theorem 5, p. 374] and [19].

3. Proof of Theorem 1

The extended mean values $E(r,s;x,y)$ can be expressed for $r(s-r) \neq 0$ as

$$E(r,s;x,y) = \left(\frac{1}{y^r - x^r}\int_{x^r}^{y^r} t^{s/r-1}\,\mathrm{d}t\right)^{1/(s-r)} \triangleq \left[\phi(x^r,y^r)\right]^{1/(s-r)}. \tag{13}$$

In $(0,\infty)$, the function $f(t) = t^{s/r-1}$ is increasing and convex if $\frac{s}{r} - 1 \geqslant 1$, decreasing and convex if $\frac{s}{r} - 1 \leqslant 0$, and increasing and concave if $0 < \frac{s}{r} - 1 \leqslant 1$. Utilizing Lemma 3 for a special case $p \equiv 1$ and Lemma 4, it follows that the function $\phi(x,y)$ defined in (13) is increasing and Schur-convex in $(0,\infty)\times(0,\infty)$ if $0 < 2r \leqslant s$ or $s \leqslant 2r < 0$, decreasing and Schur-convex in $(0,\infty)\times(0,\infty)$ if $0 < s < r$ or both $r \leqslant s$ and $r < 0$, and increasing and Schur-concave in $(0,\infty)\times(0,\infty)$ if $r < s \leqslant 2r$ or $0 > r > s \geqslant 2r$.

In $(0,\infty)$, the function $g(t) = t^r$ is increasing and convex if $r \geqslant 1$, decreasing and convex if $r < 0$, and increasing and concave if $r \in (0,1]$. From Lemma 1, it is deduced that the function

$$\psi(x,y) = \phi\big(g(x),g(y)\big) = \phi\big(x^r,y^r\big) \tag{14}$$

is increasing and Schur-convex in $(0,\infty)\times(0,\infty)$ if $1\leqslant r\leqslant \frac{s}{2}$, decreasing and Schur-convex in $(0,\infty)\times(0,\infty)$ if $0<s<r\leqslant 1$ or $s\leqslant 2r<0$, and increasing and Schur-concave in $(0,\infty)\times(0,\infty)$ if $r<s\leqslant 2r$ and $0<r\leqslant 1$.

Further, since the function $h(t)=t^{1/(s-r)}$ is increasing when $s>r$ and decreasing when $s<r$ on $(0,\infty)$, then from Lemma 2 and formula (13), it is deduced that the function $h\big(\psi(x,y)\big)=E(r,s;x,y)$ is increasing and Schur-convex with $(x,y)\in(0,\infty)\times(0,\infty)$ if $1\leqslant r\leqslant\frac{s}{2}$ and increasing and Schur-concave with $(x,y)\in(0,\infty)\times(0,\infty)$ if $\{r<s\leqslant 2r, 0<r\leqslant 1\}\cup\{0<s<r\leqslant 1\}\cup\{s\leqslant 2r<0\}$.

Since the extended mean values $E(r,s;x,y)=E(s,r;x,y)$, then $E(r,s;x,y)$ is also increasing and Schur-convex with $(x,y)\in(0,\infty)\times(0,\infty)$ if $1\leqslant s\leqslant\frac{r}{2}$ and increasing and Schur-concave with $(x,y)\in(0,\infty)\times(0,\infty)$ if $\{s<r\leqslant 2s, 0<s\leqslant 1\}\cup\{0<r<s\leqslant 1\}\cup\{r\leqslant 2s<0\}$.

If $r=0$ and $s\neq 0$, the extended mean values $E(r,s;x,y)$ can be rewritten as

$$E(0,s;x,y)=\left[\frac{1}{\ln y-\ln x}\int_{\ln x}^{\ln y} t^{st}\,\mathrm{d}t\right]^{1/s}\triangleq\big[\theta(\ln x,\ln y)\big]^{1/s}.$$

Since the function $f(t)=e^{st}$ is decreasing and convex in $(0,\infty)$ for $s<0$, from Lemma 3 and Lemma 4, it follows that the function $\theta(x,y)$ is decreasing and Schur-convex in $(0,\infty)\times(0,\infty)$ for $s<0$. Furthermore, since $g(t)=\ln t$ is increasing and convex in $(0,\infty)$, by Lemma 1, it is found that $\theta(\ln x,\ln y)$ for $s<0$ is increasing and Schur-convex in $(0,\infty)$. Since the function $h(t)=\frac{1}{t}$ is decreasing for $t>0$, by Lemma 2, it is deduced that $h[\psi(x,y)]=E(0,s;x,y)$ is increasing and Schur-concave in $(x,y)\in(0,\infty)\times(0,\infty)$. The proof is complete.

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H.-N. Shi
Department of Electronic Information
Teachers' College
Beijing Union University
Beijing City, 100011
China
e-mail: shihuannan@yahoo.com.cn
e-mail: sfthuannan@buu.com.cn

Sh.-H. Wu
Department of Mathematics
Longyan University, Longyan City
Fujian Province, 364012
China
e-mail: wushanhe@yahoo.com.cn
e-mail: shanhewu@sohu.com

F. Qi
Research Institute of Mathematical Inequality Theory
Henan Polytechnic University
Jiaozuo City
Henan Province, 454010
China
e-mail: qifeng@hpu.edu.cn
e-mail: fengqi618@member.ams.org
URL: http://rgmia.vu.edu.au/qi.html