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ERROR BOUNDS FOR AFFINE FRACTAL INTERPOLATION

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Abstract. Fractal interpolation constitutes an advance in the techniques of approximation in the sense that the functions used are not necessarily differentiable and show the rough aspect of real-world signals. We prove here that the affine fractal interpolation functions play, for non-smooth functions, a role similar to polynomials for smooth functions. The affine fractal interpolation operator is studied and its linearity and continuity is proven. A sufficient condition for the convergence of this type of interpolant as the step tends to zero is also given. As a consequence, the density of affine fractal functions in the space of continuous functions is deduced. The error of interpolation is bounded in two ways, in terms of the scale factors of the transformation and by means of the Lebesgue constant of the associated partition. Finally, a general method of data fitting is proposed and the validity and convergence of the procedure is proven as well.

1. Introduction

The reconstruction of an unknown function providing a set of data can be approached by means of fractal interpolation ([1], [2]). The power of that methodology allows us to generalize any other interpolant, both smooth and non-smooth ([7], [8], [9]). Another important fact is that this technique provides one of the few methods of non-differentiable interpolation ([10]). We prove here that fractal interpolation functions play, for non-smooth functions, a role similar to polynomials for smooth functions. The affine fractal interpolation operator is defined and studied and its linearity and continuity proven. Affine fractal functions become eigenfunctions (or fixed points) of this operator.

The error of interpolation is bounded in two ways, in terms of the scale factors of the transformation and by means of the Lebesgue constant of the associated partition. We give a sufficient condition for the convergence of this type of interpolants, when the step tends to zero. As a consequence, the density of affine fractal functions in the space of continuous functions on a compact interval is deduced. The sensitivity of the interpolants to small data errors is studied.

Another specific feature is the fact that the graph of these interpolants possesses a fractal dimension. This parameter constitutes a geometric characterization of the signal that can be used as a measure of the complexity of a phenomenon. The authors have used the fractal dimension of electroencephalographic recordings in order to describe the increase in the bioelectric complexity during several tests of attention in children

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([9]). From this practical point of view, we propose a general method of data fitting, and prove the validity and convergence of the procedure if the step is low enough.

2. Fractal interpolation functions

Let $t_0 < t_1 < ... < t_N$ be real numbers, and $I = [t_0, t_N]$ be the closed interval that contains them. Let a set of data points $\{(t_n, x_n) \in I \times R: \ n = 0, 1, 2, ..., N\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n, \ \ n \in \{1, 2, ..., N\}$ be contractive homeomorphisms such that

$$L_n(t_0) = t_{n-1}, \ L_n(t_N) = t_n, \tag{1}$$

$$|L_n(c_1) - L_n(c_2)| \leqslant l\, |c_1 - c_2| \ \ \forall\, c_1, c_2 \ \in I \tag{2}$$

for some $0 \leqslant l < 1$.

Let $-1 < \alpha_n < 1$; $n = 1, 2, ..., N$, $F = I \times [c, d]$ for some $-\infty < c < d < +\infty$ and N continuous mappings, $F_n : F \to R$ be given satisfying:

$$F_n(t_0, x_0) = x_{n-1}, \ \ F_n(t_N, x_N) = x_n, \ \ n = 1, 2, ..., N, \tag{3}$$

$$|F_n(t, x) - F_n(t, y)| \leqslant r|x - y|, \quad t \in I, \quad x, y \in R, \quad 0 \leqslant r < 1. \tag{4}$$

Now define functions $w_n(t, x) = (L_n(t), F_n(t, x)), \ \forall\, n = 1, 2, ..., N$.

THEOREM 2.1. [(Barnsley [2])] *The iterated function system (IFS)* $\{F, \ w_n : n = 1, 2, ..., N\}$ *defined above admits a unique attractor* G. G *is the graph of a continuous function* $f : I \to R$ *which obeys* $f(t_n) = x_n$ *for* $n = 0, 1, 2, ..., N$.

The previous function is called a fractal interpolation function (FIF) corresponding to $\{(L_n(t), F_n(t, x))\}_{n=1}^{N}$.

Let $\mathcal{G}$ be the set of continuous functions $f : [t_0, t_N] \to [c, d]$ such that $f(t_0) = x_0$; $f(t_N) = x_N$. $\mathcal{G}$ is a complete metric space with respect to the uniform norm

$$\|f\|_\infty = \sup\{|f(t)|; t \in I\}.$$

Define a mapping $T : \mathcal{G} \to \mathcal{G}$ by

$$(Tf)(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)) \quad \forall\, t \in [t_{n-1}, t_n], \ \ n = 1, 2, ..., N. \tag{5}$$

T is a contraction mapping on the metric space $(\mathcal{G}, \ \|\cdot\|_\infty)$

$$\|Tf - Tg\|_\infty \leqslant |\alpha|_\infty \|f - g\|_\infty \tag{6}$$

where $|\alpha|_\infty = \max\{|\alpha_n|; \ n = 1, 2, ..., N\}$. Since $|\alpha|_\infty < 1$, T possesses a unique fixed point on $\mathcal{G}$, that is to say, there is $f \in \mathcal{G}$ such that $(Tf)(t) = f(t) \ \ \forall\, t \in [t_0, t_N]$. This function is the FIF corresponding to w_n and it is the unique $f \in \mathcal{G}$ satisfying the functional equation ([1])

$$f(t) = Tf(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)), \tag{7}$$

with $n = 1, 2, ..., N$, and $t \in I_n = [t_{n-1}, t_n]$.

3. A basis of affine fractal interpolation functions

The most widely studied fractal interpolation functions so far are defined by the IFS

$$\begin{cases} L_n(t) = a_n t + b_n \\ F_n(t,x) = \alpha_n x + q_n(t) \end{cases} \tag{8}$$

where

$$a_n = \frac{(t_n - t_{n-1})}{(t_N - t_0)} \quad \text{and} \quad b_n = \frac{(t_N t_{n-1} - t_0 t_n)}{(t_N - t_0)}. \tag{9}$$

α_n is called the vertical scaling factor of the transformation w_n and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is the scale vector of the IFS. If $q_n(t)$ is a line, the FIF is termed affine (AFIF). In this case, by equations (3) $q_n(t) = c_n t + d_n$, with

$$c_n = \frac{x_n - x_{n-1}}{t_N - t_0} - \alpha_n \frac{x_N - x_0}{t_N - t_0}, \tag{10}$$

$$d_n = \frac{t_N x_{n-1} - t_0 x_n}{t_N - t_0} - \alpha_n \frac{t_N x_0 - t_0 x_N}{t_N - t_0}. \tag{11}$$

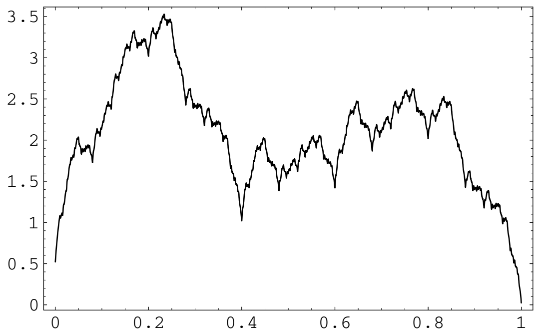

Figure 1. Graph of an affine fractal interpolation function with data $\{(0,0.5),(0.2,3),(0.4,1),(0.6,1.4),(0.8,2),(1,0)\}$ *and scale factors* $\alpha_n = 0.3 \quad \forall n = 1,2,\cdots,5$

In order to simplify the computations, we consider here the interval $I = [0,1]$ and a partition, for $N > 1$:

$$\Delta_N: \quad 0 < \frac{1}{N} < \frac{2}{N} < \ldots < \frac{N}{N} = 1. \tag{12}$$

In this case

$$a_n = \frac{1}{N} \quad \text{and} \quad b_n = \frac{n-1}{N}, \tag{13}$$

$$c_n = x_n - x_{n-1} - \alpha_n (x_N - x_0), \tag{14}$$

$$d_n = x_{n-1} - \alpha_n x_0, \tag{15}$$

and, if f is the associated function, by (7)

$$f(t) = Tf(t) = \alpha_n f \circ L_n^{-1}(t) + q_n \circ L_n^{-1}(t), \tag{16}$$

with $n = 1,2,...,N,$ and $t \in I_n = [t_{n-1}, t_n]$.

REMARK 1. If $\alpha_n = 0 \ \forall\, n = 1,2,...,N$, $f(t) = q_n \circ L_n^{-1}(t) \ \forall\, t \in I_n$ and f is piecewise linear with vertices (t_n, x_n) ([1]).

DEFINITION 1. Let us consider the data $\{(t_n, x_{kn})\}_{n=0}^N$ such that $t_n = \frac{n}{N}$ and $x_{kn} = \delta_{kn}$ where δ_{kn} is the delta of Kronecker. The $k-th$ affine fractal interpolation function f_{kN}^{α} with respect to the scale vector $\alpha \in \mathbb{R}^N$ and the partition Δ_N is defined by the equality

$$f_{kN}^{\alpha}(t_n) = x_{kn} \ \forall\, n = 0,1,...,N. \tag{17}$$

LEMMA 3.1. *Let us consider the function f_{kN}^{α} previously defined. The corresponding iterated function system is given by the mappings $\{(L_n, F_{kn})\}_{n=0}^N$ such that $L_n(t) = a_n t + b_n$ where*

$$a_n = \frac{1}{N}, \qquad b_n = \frac{n-1}{N} \tag{18}$$

and $F_{kn}(t,x) = \alpha_n x + q_{kn}(t)$ where

$$\begin{cases} q_{01}(t) = (-1+\alpha_1)(t-1) \\ q_{02}(t) = \alpha_2\ (t-1) \\ q_{03}(t) = \alpha_3\ (t-1) \\ \vdots \\ q_{0N}(t) = \alpha_N\ (t-1) \end{cases} \tag{19}$$

for $1 \leqslant k \leqslant N-1$:

$$\begin{cases} q_{k1}(t) = 0 \\ q_{k2}(t) = 0 \\ \vdots \\ q_{kk}(t) = t \\ q_{k(k+1)}(t) = -t+1 \\ \vdots \\ q_{kN}(t) = 0 \end{cases} \tag{20}$$

and

$$\begin{cases} q_{N1}(t) = \alpha_1(-t) \\ q_{N2}(t) = \alpha_2\ (-t) \\ \vdots \\ q_{NN}(t) = (-1+\alpha_N)\ (-t) \end{cases} \tag{21}$$

Proof. It is a trivial consequence of the definition of f_{kN}^{α} and the equalities (14) and (15). □

LEMMA 3.2. *Let $\{(t_n, x_n)\}_{n=0}^N$ be data points and let $\{(L_n, F_n)\}_{n=0}^N$ with $F_n(t,x) = \alpha_n x + q_n(t)$ be the associated iterated function system. Let q_{kn} be the functions defined by (19), (20) and (21). The following equality holds for $1 \leqslant n \leqslant N$*

$$q_n(t) = \sum_{k=0}^N x_k q_{kn}(t) \quad \forall\, t \in I.$$

Proof. For $n = 1$,

$$\sum_{k=0}^N x_k q_{k1}(t) = x_0 q_{01}(t) + x_1 q_{11}(t) + x_N q_{N1}(t),$$

$$\sum_{k=0}^N x_k q_{k1}(t) = x_0(-1+\alpha_1)(t-1) + x_1 t + x_N \alpha_1(-t).$$

For $n \neq 1$, $n \neq N$,

$$\sum_{k=0}^N x_k q_{kn}(t) = x_0 q_{0n}(t) + x_{n-1} q_{(n-1)n}(t) + x_n q_{nn}(t) + x_N q_{Nn}(t),$$

$$\sum_{k=0}^N x_k q_{kn}(t) = x_0 \alpha_n (t-1) + x_{n-1}(-t+1) + x_n t + x_N \alpha_n(-t) = c_n t + d_n.$$

For $n = N$,

$$\sum_{k=0}^N x_k q_{kN}(t) = x_0 q_{0N}(t) + x_{N-1} q_{(N-1)N}(t) + x_N q_{NN}(t),$$

$$\sum_{k=0}^N x_k q_{kN}(t) = x_0 \alpha_N (t-1) + x_{N-1}(-t+1) + x_N(-1+\alpha_N)t.$$

In all the cases

$$\sum_{k=0}^N x_k q_{kn}(t) = c_n t + d_n,$$

where

$$c_n = (x_n - x_{n-1}) - \alpha_n(x_N - x_0),$$
$$d_n = x_{n-1} - \alpha_n x_0,$$

and

$$\sum_{k=0}^N x_k q_{kn}(t) = q_n(t). \qquad \square$$

PROPOSITION 3.3. *Let f_N^α be the fractal interpolation function associated to partition Δ_N and data points $\{(t_n, x_n)\}_{n=0}^N$ with scale vector α. f_N^α can be expressed as*

$$f_N^\alpha = \sum_{k=0}^N x_k f_{kN}^\alpha. \tag{22}$$

Proof. We will prove that $f_N^\alpha = \sum_{k=0}^N x_k f_{kN}^\alpha$ is the affine fractal interpolation function with respect to the data $\{(t_n, x_n)\}_{n=0}^N$ with scale vector α. If $t \in I_n = [t_{n-1}, t_n]$, let T be the mapping of the IFS associated to the data (5)

$$\begin{aligned} Tf_N^\alpha(t) &= T\left(\sum_{k=0}^N x_k f_{kN}^\alpha\right)(t) \\ &= F_n(L_n^{-1}(t), f_N^\alpha \circ L_n^{-1}(t)) \\ &= \alpha_n f_N^\alpha \circ L_n^{-1}(t) + q_n \circ L_n^{-1}(t), \end{aligned}$$

by Lemma 3.2

$$q_n = \sum_{k=0}^N x_k q_{kn},$$

and

$$\begin{aligned} T(f_N^\alpha)(t) &= \alpha_n \left(\sum_{k=0}^N x_k f_{kN}^\alpha\right) \circ L_n^{-1}(t) + \left(\sum_{k=0}^N x_k q_{kn}\right) \circ L_n^{-1}(t) \\ &= \sum_{k=0}^N x_k \left(\alpha_n f_{kN}^\alpha \circ L_n^{-1}(t) + q_{kn} \circ L_n^{-1}(t)\right). \end{aligned}$$

On the other hand, f_{kN}^α is the FIF associated to $\{(L_n, F_{kn})\}$ where $F_{kn}(t,x) = \alpha_n x + q_{kn}(t)$, therefore for $t \in I_n$,

$$f_{kN}^\alpha(t) = \alpha_n f_{kN}^\alpha \circ L_n^{-1}(t) + q_{kn} \circ L_n^{-1}(t),$$

and

$$T(f_N^\alpha) = \sum_{k=0}^N x_k f_{kN}^\alpha = f_N^\alpha.$$

f_N^α is the fixed point of the transformation T. By the uniqueness of the FIF, the result is obtained. $\square$

In the space of real-valued functions defined on $[0,1]$, consider the bilinear form provided by the partition Δ_N

$$\langle f, g \rangle = \sum_{n=0}^N f\left(\frac{n}{N}\right) g\left(\frac{n}{N}\right). \tag{23}$$

LEMMA 3.4. *The family* $\{f^{\alpha}_{kN}\}_{k=0}^{N}$ *is an orthonormal system with respect to the form* (23).

Proof.

$$\langle f^{\alpha}_{kN}, f^{\alpha}_{jN}\rangle = \sum_{n=0}^{N} f^{\alpha}_{kN}\left(\frac{n}{N}\right) f^{\alpha}_{jN}\left(\frac{n}{N}\right) = \sum_{n=0}^{N} \delta_{kn}\delta_{jn} = 0 \quad \text{if } j \neq k. \qquad \square$$

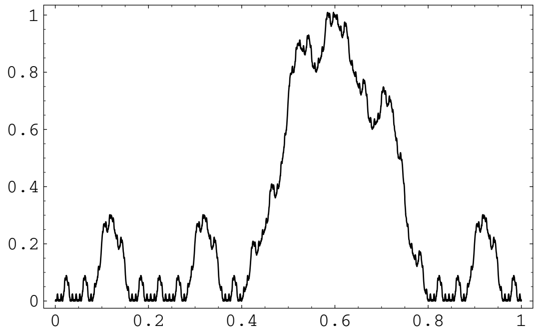

Figure 2. Basis function f^{α}_{35} *with scale* $\alpha_n = 0.3, \quad \forall n = 1, 2, \cdots, 5$

Let $\mathcal{B}^{\alpha}_{N}$ be the set of AFIF associated to the partition Δ_N with scale vector $\alpha \in \mathbb{R}^N$.

THEOREM 3.5. *The system* $\{f^{\alpha}_{kN}\}_{k=0}^{N}$ *is a basis of the linear space* $\mathcal{B}^{\alpha}_{N}$.

Proof. By Proposition 3.3, $\{f^{\alpha}_{kN}\}_{k=0}^{N}$ spans $\mathcal{B}^{\alpha}_{N}$. The independence is a consequence of Lemma 3.4, because if

$$\sum_{k=0}^{N} \lambda_k f^{\alpha}_{kN} = 0$$

then

$$0 = \langle \sum_{k=0}^{N} \lambda_k f^{\alpha}_{kN}, f^{\alpha}_{jN}\rangle = \sum_{k=0}^{N} \lambda_k \langle f^{\alpha}_{kN}, f^{\alpha}_{jN}\rangle$$

$$= \lambda_j \langle f^{\alpha}_{jN}, f^{\alpha}_{jN}\rangle = \lambda_j \sum_{n=0}^{N} \left(f^{\alpha}_{jN}\left(\frac{n}{N}\right)\right)^2 = \lambda_j \sum_{n=0}^{N} \delta^2_{jn} = \lambda_j. \qquad \square$$

REMARK 2 $\mathcal{B}^{\alpha}_{N}$ is a space of finite dimension (see also [6]). As a consequence, if $\mathcal{B}^{\alpha}_{N}$ is considered as subspace of $\mathcal{C}[0,1]$ with the uniform norm, $\mathcal{B}^{\alpha}_{N}$ is a closed subset.

Barnsley's Theorem assures that the following general problem of finite interpolation has a solution for $X = \mathcal{B}^{\alpha}_{N}$ and $l_i(f) = f(t_i)$:

"Let X be a linear space of dimension $N+1$ and let $l_0, l_1, \ldots, l_N$ be $N+1$ given linear functionals defined on X. For a given set of values $x_0, x_1, \ldots, x_N$, find an element of X, say f, such that $l_i(f) = x_i$ for $i = 0, 1, \ldots, N$"

The next result is provided in [4].

THEOREM 3.6. ([4]) *Let X be a linear space of dimension $N+1$ and let X^* be its algebraic conjugate space. The general problem of finite interpolation possesses a solution for arbitrary $x_0, x_1, \ldots, x_N$ if and only if the operators $\{l_i\}_{i=0}^N$ are linearly independent in X^*.*

From this result one can deduce that the elements $l_i \in (\mathcal{B}_N^\alpha)^*$ defined by $l_i(f) = f(t_i)$ are linearly independent and consequently form a basis of the dual space $(\mathcal{B}_N^\alpha)^*$.

In [10] Chen gives some sufficient conditions for the nowhere differentiability of an affine fractal interpolation function.

THEOREM 3.7. ([10]) *If $|\alpha_n| \geqslant \frac{1}{N}$ and $x_n - x_{n-1} \neq \frac{x_N - x_0}{N}$ for all $n \in \{1, 2, \ldots, N\}$, then the associated AFIF is nowhere differentiable in the interval $(0,1)$.*

The former condition is fulfilled by f_{0N}^α and f_{NN}^α whenever $\min_n |\alpha_n| \geqslant \frac{1}{N}$, $(N \geqslant 2)$. In consequence, at least the basis functions are nowhere differentiable for $k=0$ and $k=N$ in this case.

DEFINITION 2. Let g be a real-valued function defined in $[0,1]$, the operator of affine fractal interpolation $\mathcal{A}_N^\alpha$ is defined by the equality

$$\mathcal{A}_N^\alpha(g)(t) = \sum_{k=0}^{N} g\left(\frac{k}{N}\right) f_{kN}^\alpha(t) \tag{24}$$

$\forall\, t \in [0,1]$.

LEMMA 3.8. *Let f and g be defined in $[0,1]$ such that $f(t_k) = g(t_k)$ $\forall\, t_k \in \Delta_N$ for some N, then*

$$\mathcal{A}_N^\alpha(f) = \mathcal{A}_N^\alpha(g).$$

Proof. $\forall t \in [0,1]$

$$\mathcal{A}_N^\alpha(f)(t) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right) f_{kN}^\alpha(t) = \sum_{k=0}^{N} g\left(\frac{k}{N}\right) f_{kN}^\alpha(t) = \mathcal{A}_N^\alpha(g)(t) \qquad \square$$

The set $\Delta = \cup_{N \in \mathbb{N}} \Delta_N$ is dense in the interval I.

LEMMA 3.9. *If f and g are functions defined on I such that*

$$\lim_{N \to \infty} \mathcal{A}_N^\alpha(f)(t) = g(t) \ \ \forall\, t \in \Delta$$

then

$$f(t) = g(t) \ \ \forall\, t \in \Delta.$$

Proof. If $t \in \Delta$, then $t \in \Delta_{N_0}$ for some N_0, $t = \frac{n}{N_0}$ and

$$\mathcal{A}_{N_0}^\alpha(f)(t) = f(t).$$

If the sequence of partitions $\{\Delta_{mN_0}\}$ is considered $t = \frac{n}{N_0} = \frac{mn}{mN_0} \in \Delta_{mN_0}$ and $\mathcal{A}_{mN_0}^\alpha(f)(t) = f(t)$

$$f(t) = \lim_{m \to \infty} \mathcal{A}_{mN_0}^\alpha(f)(t) = g(t). \qquad \square$$

LEMMA 3.10. *If, in addition to the hypotheses of Lemma 3.9, $f, g \in \mathcal{C}[0,1]$, then $f = g$.*

Proof. It is a consequence of the density of Δ in $[0,1]$ and the continuity of f and g. If $t \in [0,1]$ one can find a sequence $\bar{t}_m \in \Delta$ such that $\bar{t}_m \to t$ as $m \to \infty$. In this case $f(\bar{t}_m) \to f(t)$ and $g(\bar{t}_m) \to g(t)$. But $f(\bar{t}_m) = g(\bar{t}_m)$ and as a consequence $f(t) = g(t)$. □

PROPOSITION 3.11. *If g is uniform limit of $\mathcal{A}_N^\alpha(f)$ with $f \in \mathcal{C}[0,1]$ as $N \to \infty$, then $f = g$.*

Proof. In this case g is continuous because $\mathcal{A}_N^\alpha(f)$ are so. The hypotheses of Lemma 3.10 are fulfilled and the result is obtained. □

4. Error of interpolation

Let $\mathcal{C}[0,1]$ be the space of continuous functions defined on $I = [0,1]$, endowed with the uniform norm

$$\|f\|_\infty = \sup\{|f(t)|;\ t \in [0,1]\}$$

PROPOSITION 4.1. *$\mathcal{A}_N^\alpha$ is a linear operator of $\mathcal{C}[0,1]$.*

Proof. $\mathcal{A}_N^\alpha(g)$ is a continuous function and $\forall\, t \in [0,1]$

$$\mathcal{A}_N^\alpha(f+g)(t) = \sum_{k=0}^{N} (f+g)\left(\frac{k}{N}\right) f_{kN}^\alpha(t) = \mathcal{A}_N^\alpha(f)(t) + \mathcal{A}_N^\alpha(g)(t). \qquad \square$$

PROPOSITION 4.2. *$\mathcal{A}_N^\alpha$ is a bounded operator.*

Proof. If $g \in \mathcal{C}[0,1]$, $\mathcal{A}_N^\alpha(g)$ is an affine fractal interpolation function associated to $\{(t_n, x_n = g(t_n))\}$. In [5] Feng and Xie prove, for functions of this type,

$$\|\mathcal{A}_N^\alpha(g)\|_\infty \leqslant \frac{1+|\alpha|_\infty}{1-|\alpha|_\infty} \max_{0\leqslant n\leqslant N}\{|x_n|\} \leqslant \frac{1+|\alpha|_\infty}{1-|\alpha|_\infty} \|g\|_\infty,$$

where $|\alpha|_\infty = \max_{1\leqslant n\leqslant N}\{|\alpha_n|\}$. As a consequence

$$\|\mathcal{A}_N^\alpha\| \leqslant \frac{1+|\alpha|_\infty}{1-|\alpha|_\infty}. \qquad \square$$

PROPOSITION 4.3. *If g^α is an affine fractal interpolation function associated to the partition Δ_N with scale vector $\alpha \in \mathbb{R}^N$, then*

$$\mathcal{A}_N^\alpha(g^\alpha) = g^\alpha.$$

Proof.

$$\mathcal{A}_N^\alpha(g^\alpha) = \sum_{k=0}^{N} g^\alpha\left(\frac{k}{N}\right) f_{kN}^\alpha = \sum_{k=0}^{N} x_k f_{kN}^\alpha.$$

In Proposition 3.3 we have proved that $\sum_{k=0}^N x_k f_{kN}^\alpha$ is an AFIF associated to the data points $\{(t_k, x_k)\}_{k=0}^N$. By the uniqueness of this kind of functions

$$g^\alpha = \mathcal{A}_N^\alpha(g^\alpha). \qquad \square$$

REMARK 3. The AFIF with scale vector α are fixed points or eigenfunctions of the operator $\mathcal{A}_N^\alpha$ and $1 \in \sigma_p(\mathcal{A}_N^\alpha)$ where $\sigma_p(\mathcal{A}_N^\alpha)$ is the point spectrum of $\mathcal{A}_N^\alpha$. $\mathcal{A}_N^\alpha$ is not a contraction since, if it were, there would be a unique fixed point.

PROPOSITION 4.4.

$$\mathcal{A}_N^\alpha \circ \mathcal{A}_N^\alpha = \mathcal{A}_N^\alpha.$$

Proof. It is a consequence of Proposition 4.3, since $\mathcal{A}_N^\alpha(g)$ is an affine fractal interpolation function associated to the partition Δ_N with scale vector $\alpha \in \mathbb{R}^N$ and therefore a fixed point of $\mathcal{A}_N^\alpha$

$$\mathcal{A}_N^\alpha(\mathcal{A}_N^\alpha(g)) = \mathcal{A}_N^\alpha(g). \qquad \square$$

PROPOSITION 4.5. $\forall f, g \in \mathcal{C}[0,1]$

$$\|\mathcal{A}_N^\alpha(f) - \mathcal{A}_N^\alpha(g)\|_\infty \leqslant \frac{1+|\alpha|_\infty}{1-|\alpha|_\infty}\|f-g\|_\infty.$$

Proof. By the linearity and boundness of the operator

$$\|\mathcal{A}_N^\alpha(f) - \mathcal{A}_N^\alpha(g)\|_\infty \leqslant \|\mathcal{A}_N^\alpha(f-g)\|_\infty \leqslant \frac{1+|\alpha|_\infty}{1-|\alpha|_\infty}\|f-g\|_\infty. \qquad \square$$

REMARK 4. As explained in Remark 1, $\mathcal{A}_N^0$ is the operator that maps each function g to the piecewise linear function with vertices $(t_n, g(t_n))$.

PROPOSITION 4.6. *The sequence of operators $\mathcal{A}_N^0$ of $\mathcal{C}[0,1]$ converges to the identity as N tends to ∞.*

Proof. $\mathcal{A}_N^0(g)$ is a piecewise linear function with vertices $(t_n, g(t_n) = x_n)$. Let us denote $\bar{g} = \mathcal{A}_N^0(g)$. In the interval $[t_{n-1}, t_n]$

$$\bar{g}(t) = x_n + \frac{x_n - x_{n-1}}{t_n - t_{n-1}}(t - t_n). \tag{25}$$

Then $\forall\, t \in I_n$,

$$|\bar{g}(t) - g(t)| = |x_n + \frac{x_n - x_{n-1}}{t_n - t_{n-1}}(t - t_n) - g(t)| \leqslant |x_n - x_{n-1}| + |x_n - g(t)|. \tag{26}$$

Let $w(h)$ be the modulus of continuity of g defined as

$$w(h) = \sup_{|t-t'|\leqslant h} |g(t) - g(t')|.$$

Then $\forall\, t \in I_n$,

$$|x_n - x_{n-1}| = |g(t_n) - g(t_{n-1})| \leqslant w\left(\frac{1}{N}\right),$$

$$|x_n - g(t)| = |g(t_n) - g(t)| \leqslant w\left(\frac{1}{N}\right).$$

The inequality (26) implies

$$|\bar{g}(t) - g(t)| \leqslant 2\, w\left(\frac{1}{N}\right) \quad \forall\, t \in I_n$$

and

$$\|\mathcal{A}_N^0(g) - g\|_\infty \leqslant 2\, w\left(\frac{1}{N}\right). \tag{27}$$

If g is uniformly continuous, $w(h) \to 0$ when $h \to 0$ ([3]). In consequence,

$$\|\mathcal{A}_N^0(g) - g\|_\infty \to 0$$

as $N \to \infty$, from which the result is deduced. □

THEOREM 4.7. (Error bound of affine fractal interpolation) *If* $g \in \mathcal{C}[0,1]$ *the following inequality holds*

$$\|\mathcal{A}_N^\alpha(g) - g\|_\infty \leqslant 2\, w\left(\frac{1}{N}\right) + \frac{2|\alpha|_\infty}{1-|\alpha|_\infty}\|g\|_\infty. \tag{28}$$

Proof.

$$\|\mathcal{A}_N^\alpha(g) - g\|_\infty \leqslant \|\mathcal{A}_N^\alpha(g) - \mathcal{A}_N^0(g)\|_\infty + \|\mathcal{A}_N^0(g) - g\|_\infty. \tag{29}$$

In [9] we proved that if g^α is the AFIF associated to the data $\{(t_k, g(t_k))\}_{k=0}^N$, and $\bar{g} = \mathcal{A}_N^0(g)$ is the polygonal with vertices $\{(t_k, g(t_k))\}_{k=0}^N$

$$\|g^\alpha - \bar{g}\|_\infty \leqslant \frac{2|\alpha|_\infty}{1-|\alpha|_\infty} \max_{0\leqslant k\leqslant N}\{|g(t_k)|\}.$$

As a consequence, the first term of (29) satisfies

$$\|\mathcal{A}_N^\alpha(g) - \mathcal{A}_N^0(g)\|_\infty \leqslant \frac{2|\alpha|_\infty}{1-|\alpha|_\infty}\|g\|_\infty.$$

The second is bounded in Proposition 4.6 by (27)

$$\|\mathcal{A}_N^0(g) - g\|_\infty \leqslant 2\, w\left(\frac{1}{N}\right),$$

and the result is deduced. □

As a consequence one has the following result.

THEOREM 4.8. (Sufficient condition of convergence). *If the scale factors are chosen so that*

$$|\alpha|_\infty = \mathcal{O}((\ln N)^{-p})$$

with $p > 0$*, or an infinitesimal of higher order, then the sequence of affine fractal interpolation functions of* g *at* Δ_N *tends to the function* $g \in \mathcal{C}[0,1]$ *as* N *tends to infinity.*

Proof. It is a consequence of Theorem 4.7. As g is uniformly continuous, $w(\frac{1}{N}) \to 0$ as N tends to infinity ([3]) and the second term of (28) does as well due to the proposed hypothesis. □

Let $\mathcal{B} = \cup_{N\in\mathbb{N},\alpha\in\mathcal{J}}\mathcal{B}_N^\alpha$ be the set of affine fractal interpolation functions with scale factors α such that $|\alpha|_\infty < 1$ on any finite partition of $I = [0,1]$.

THEOREM 4.9. $\mathcal{B}$ *is dense in* $\mathcal{C}[0,1]$.

Proof. It is a consequence of Theorem 4.7, since for each $g \in \mathcal{C}[0,1]$ and each $\varepsilon > 0$, some sufficiently small $\alpha \in \mathcal{J}$ and $\frac{1}{N}$ can be chosen so that the difference between $\mathcal{A}_N^\alpha(g)$ and g be lower than ε. □

REMARK 5. The same result is proved in the reference [9] by means of other procedures.

REMARK 6. It is important to emphasize the parallelism existing between fractal interpolation functions for non-differentiable functions, and polynomials for smooth functions.

In the following, a bound of the interpolation error by means of the Lebesgue constant of Δ_N is obtained.

DEFINITION 3. ([4]) A scheme of interpolation nodes which is fixed a priori (independently of the functions to be approximated),

$$K = \begin{pmatrix} t_{10} & t_{11} & & & \\ t_{20} & t_{21} & t_{22} & & \\ t_{30} & t_{31} & t_{32} & t_{33} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is called a node matrix.

DEFINITION 4. ([4]) The Lebesgue function of K of order $N+1$ is

$$\Lambda_N(t;K) = \sum_{i=0}^{N} |\varphi_{i,N}(t)|,$$

where φ_{iN} is the i-th interpolant polynomial respect to the partition $t_{0N}, t_{1N}, \cdots t_{NN}$ defined by

$$\varphi_{iN}(t_{jN}) = \delta_{ij}.$$

The Lebesgue constant of K of order $N+1$ is

$$\Lambda_N(K) = \|\Lambda_N(t;K)\|_\infty.$$

Let $\mathcal{P}_N[0,1]$ be the set of polynomials of degree at most N in the interval [0,1]. The fact that $\mathcal{P}_N[0,1]$ has a finite dimension allows the existence of a minimum distance from a function $f \in \mathcal{C}[0,1]$ to this subspace

$$d_N^*(f) = d(f, \mathcal{P}_N[0,1]) \tag{30}$$

THEOREM 4.10. ([4]). *For a function $f \in \mathcal{C}[a,b]$ and a sequence of polynomial interpolants $\{p_N(f)\}_{N=1}^{\infty}$ with respect to the node matrix K, the following inequality holds*

$$\|f - p_N(f)\|_\infty \leqslant d_N^*(f)(1 + \Lambda_N(K)), \qquad N = 1, 2, \ldots$$

The Lebesgue constant (which depends on the node matrix K but not on f) gives a measure of the separation of the interpolation error from the minimum error $d_N^*(f)$.

THEOREM 4.11. *If $g \in \mathcal{C}[0,1]$, then*

$$\|\mathcal{A}_N^\alpha(g) - g\|_\infty \leqslant (d_N^*(g) + d_N^*(\mathcal{A}_N^\alpha(g)))(1 + \Lambda_N(K))$$

with $\Lambda_N(K)$ being the Lebesgue constant of the node matrix corresponding to partitions Δ_N.

Proof. If $p_N(g)$ is the polynomial interpolant of g with respect to Δ_N

$$\mathcal{A}_N^\alpha(g) - g = \mathcal{A}_N^\alpha(g) - p_N(\mathcal{A}_N^\alpha(g)) + p_N(\mathcal{A}_N^\alpha(g)) - p_N(g) + p_N(g) - g.$$

But $p_N(g) = p_N(\mathcal{A}_N^\alpha(g))$ because the functions g and $\mathcal{A}_N^\alpha(g)$ agree at the points of Δ_N and

$$\|\mathcal{A}_N^\alpha(g) - g\|_\infty \leqslant \|\mathcal{A}_N^\alpha(g) - p_N(\mathcal{A}_N^\alpha(g))\|_\infty + \|g - p_N(g)\|_\infty,$$
$$\|\mathcal{A}_N^\alpha(g) - g\|_\infty \leqslant (d_N^*(\mathcal{A}_N^\alpha(g)) + d_N^*(g))(1 + \Lambda_N(K)),$$

according to Theorem 4.10. □

The behaviour of $d_N^*(g)$ for functions g with specific properties is treated in [3]. The generalized Jackson's theorem states the inequality

$$d_N^*(g) \leqslant c\, w\left(\frac{1}{N}\right),$$

where c is a constant and w the modulus of continuity of g.

5. Fitting real data

In this paragraph we propose a general method of real data fitting, and prove the validity and convergence of the procedure if the sampling frequency is sufficiently large.

Let $\{(t_n, x_n)\}_{n=0}^{J}$ be a subset of the data, that are considered equidistant here, $t_n = t_0 + nh$, $t_0 = 0$ and $t_J = 1$. These values are used as interpolation nodes, and we consider the intermediate points of the signal $\bar{t_j} \in I_n = [t_{n-1}, t_n]$, $j = 1, 2, ..., m-1$ as target points to define the fit. If $\bar{t_j}$ are also equidistant

$$\bar{t_j} = \frac{(m-j)t_{n-1} + jt_n}{m} \quad \text{and} \quad L_n^{-1}(\bar{t_j}) = \frac{(m-j)t_0 + jt_N}{m} = \frac{j}{m}. \tag{31}$$

The value of the FIF at the point $\bar{t}_j$ is given by the equation (16). Replacing the value of the function f in $L_n^{-1}(\bar{t}_j)$ by the value of a chosen interpolant of the data, f_Δ, (with interpolation nodes t_n)

$$f(\bar{t}_j) \simeq \alpha_n f_\Delta \circ L_n^{-1}(\bar{t}_j) + q_n \circ L_n^{-1}(\bar{t}_j) \tag{32}$$

where

$$q_n \circ L_n^{-1}(\bar{t}_j) = c_n \frac{j}{m} + d_n,$$

according to (14) and (15)

$$q_n \circ L_n^{-1}(\bar{t}_j) = \frac{(m-j)x_{n-1} + jx_n}{m} - \alpha_n \frac{(m-j)x_0 + jx_N}{m}$$

and by (31) and (32)

$$f(\bar{t}_j) \simeq \alpha_n \left(f_\Delta \left(\frac{j}{m} \right) - \frac{(m-j)x_0 + jx_N}{m} \right) + \frac{(m-j)x_{n-1} + jx_n}{m}$$

$$\bar{x}_j = f(\bar{t}_j) \simeq \alpha_n u(j) + v_1(j).$$

Now we can compute α_n by means of least squares approximation

$$\min\ E(\alpha_n) = \sum_{j=1}^{m-1} (\alpha_n u(j) + v_1(j) - \bar{x}_j)^2.$$

Differentiating with respect to α_n, if $v(j) = v_1(j) - \bar{x}_j$, the following value of α_n is obtained

$$\alpha_n = \frac{-\sum_{j=1}^{m-1} v(j)u(j)}{\sum_{j=1}^{m-1} u(j)^2} \tag{33}$$

where

$$u(j) = f_\Delta\left(\frac{j}{m}\right) - \frac{(m-j)x_0 + jx_N}{m},$$

$$-v(j) = \bar{x}_j - \frac{(m-j)x_{n-1} + jx_n}{m}.$$

The point $(\bar{t}_j, \frac{(m-j)x_{n-1}+jx_n}{m})$ lies on the polygonal $\overline{g}$ with vertices (t_n, x_n). In Proposition 4.6, the inequality (27) gives an upper bound of the distance between $\overline{g}$ and the original continuous function g. The bound tends to zero as N tends to infinity, and thus $v(j)$. $u(j)$ does not depend on the interpolation step h. As a consequence, $\alpha \to 0$ if $h \to 0$. This fact allows us to obtain a step h low enough to get $|\alpha|_\infty < 1$.

5.1. Bounding the errors

By (33)

$$\alpha_n = -\frac{u.v}{|u|_2^2}$$

where $u = (u(1), u(2), \ldots, u(m-1))$ and $v = (v(1), v(2), \ldots, v(m-1))$.

Applying Schwartz's inequality

$$|\alpha_n| \leqslant \frac{|v|_2}{|u|_2} \,,$$

where $|u|_2 = \sqrt{\sum_{j=1}^{m-1} u(j)^2}$. The point $(\bar{t_j}, \frac{(m-j)x_{n-1}+jx_n}{m})$ lies on the line passing trough (t_{n-1}, x_{n-1}), (t_n, x_n) and therefore, using (27)

$$|v(j)| \leqslant 2w\left(\frac{1}{J}\right),$$

$$|v|_2 \leqslant 2w\left(\frac{1}{J}\right)\sqrt{m-1},$$

and

$$|\alpha|_\infty \leqslant \frac{2w(\frac{1}{J})\sqrt{m-1}}{|u|_2} = C.$$

If $C < 1$,

$$\frac{|\alpha|_\infty}{1-|\alpha|_\infty} \leqslant \frac{C}{1-C}.$$

Theorem 4.7 bounds the distance between the original function g and the defined AFIF f. Using the former inequalities one has

$$\|f-g\|_\infty \leqslant 2w\left(\frac{1}{J}\right) + \frac{2C}{1-C}\,\|g\|_\infty = 2w\left(\frac{1}{J}\right)\left(1+\frac{2\sqrt{m-1}}{|u|_2(1-C)}\,\|g\|_\infty\right).$$

The convergence of f towards g as J tends to infinity is guaranteed if g is continuous.

6. The condition of function and parameters

The number of condition describes the sensitivity of the approximants to perturbations of the data. Let $x_0, x_1, \ldots, x_N$ unperturbed data points and $\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_N$ the perturbed data. Let us denote $|x-\tilde{x}|_\infty = \max_{0\leqslant n\leqslant N}\{|\tilde{x}_n - x_n|\}$. The condition of parameters measures the variation of the constants defining the interpolant when the data are changed. The equalities (14) and (15) provide

$$|c_n-\tilde{c}_n| = |x_n-\tilde{x}_n-(x_{n-1}-\tilde{x}_{n-1})+\alpha_n((x_N-\tilde{x}_N)-(x_0-\tilde{x}_0)| \leqslant 2|x-\tilde{x}|_\infty(1+|\alpha|_\infty)$$

and the condition of this parameter is

$$k_c \leqslant 2(1+|\alpha|_\infty)\,.$$

For the second coefficient of the line q_n

$$|d_n-\tilde{d}_n| = |x_{n-1}-\tilde{x}_{n-1}-\alpha_n(x_0-\tilde{x}_0)| \leqslant |x-\tilde{x}|_\infty(1+|\alpha|_\infty)$$

and consequently

$$k_d \leqslant 1+|\alpha|_\infty.$$

The condition of function values k_f is obtained from the norm of $\mathcal{A}_N^\alpha$

$$\|\mathcal{A}_N^\alpha(f) - \mathcal{A}_N^\alpha(\tilde{f})\|_\infty \leqslant \|\mathcal{A}_N^\alpha\| \, \|f - \tilde{f}\|_\infty ,$$

then

$$k_f \leqslant \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty} .$$

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