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ON RECIPROCAL POLYNOMIALS WITH ZEROS OF MODULUS ONE

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(*communicated by Z. Daróczy*)

Abstract. The purpose of this paper is to characterize reciprocal polynomials all of whose zeros are on the unit circle. Using this characterization we give bounds for the coefficients of such polynomials and show how other necessary conditions (inequalities for the coefficients) can be obtained from the characterization theorem.

1. Introduction

The aim of this paper is to characterize reciprocal polynomials all of whose zeros are on the unit circle. First, using elementary algebra, we give a factorization of reciprocal polynomials whose zeros are on the unit circle. In Section 2 we prove a Viéta–like formula for such reciprocal polynomials and give their characterization in terms of the coefficients. In Section 3 we find bounds for the coefficients of reciprocal polynomials whose zeros lie on the unit circle. In Section 4 we prove some other necessary conditions (inequalities) for the coefficients.

DEFINITION 1. A polynomial p_m of the form

$$p_m(z) = \sum_{j=0}^{m} a_j z^j \quad (z \in \mathbb{C})$$

where $m \in \mathbb{N}, a_m \neq 0, a_0, \dots, a_m \in \mathbb{C}$ and

$$a_j = a_{m-j} \; (j = 0, \dots, m)$$

is called a *reciprocal polynomial* of degree m.

If p_{2n} is a reciprocal polynomial of degree $2n$ with $n \in \mathbb{N}$ then

$$p_{2n}(z) = \sum_{j=0}^{2n} a_j z^j = z^n \left[a_{2n} \left(z^n + \frac{1}{z^n} \right) + \dots + a_{n+1} \left(z + \frac{1}{z} \right) + a_n \right].$$

This shows that if β is a zero of p_{2n} then so is $\frac{1}{\beta}$.

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LEMMA 1. *A complex reciprocal polynomial p_{2n} of degree $2n$ has all its zeros on the unit circle if and only if there are n real numbers $\alpha_1,\dots,\alpha_n$ in the interval $[-2,2]$ such that*

$$p_{2n}(z)=a_{2n}\prod_{k=1}^{n}(z^2-\alpha_k z+1). \tag{1}$$

Proof. If β is a zero of p_{2n} on the unit circle then $\frac{1}{\beta}=\bar{\beta}$ is also a zero, hence the polynomial has a factor of the form

$$(z-\beta)(z-\bar{\beta})=z^2-(\beta+\bar{\beta})+1=z^2-\alpha z+1$$

where $\alpha=\beta+\bar{\beta}=2\mathrm{Re}(\beta)$ with Re denoting the real part of a complex number. Clearly $|\alpha|=2|\mathrm{Re}(\beta)|\leqslant 2$. Collecting all zeros of p_{2n} into pairs $\left(\beta_k,\frac{1}{\beta_k}\right)$, $(k=1,\dots,n)$ and multiplying the corresponding factors we obtain the factorization (1) where $\alpha_k=\beta_k+\bar{\beta_k}$, $(k=1,\dots,n)$.

Conversely, if (1) holds then p_{2n} is clearly a complex reciprocal polynomial with all zeros are on the unit circle.

We remark that by the help of *Chebysev transformation* one can characterize reciprocal polynomials that have a given (even) number of zeros on the unit circle (see Lakatos [3] where a related result is proved for polynomials with real coefficients). From Lemma 1 it follows that *if all zeros of a complex monic reciprocal polynomial are on the unit circle then all of its coefficients are real.*

2. Characterization

From Lemma 1 it follows that all zeros of a monic reciprocal polynomial

$$p_{2n}(z)=\sum_{k=0}^{2n}A_{2n,k}z^k \quad (A_{2n,0}=1, A_{2n,k}=A_{2n,2n-k} \text{ for } k=0,1,\dots,2n)$$

of even degree $2n$ are on the unit circle if and only if it has the form

$$p_{2n}(z)=\prod_{k=1}^{n}(z^2-\alpha_k z+1).$$

where $\alpha_k\in[-2,2]$ $(k=1,2,\dots,n)$.

First we prove an identity (a Viéta–like formula for p_{2n}).

LEMMA 2. *For every $n\in\mathbb{N}$ and $z,\alpha_1,\dots,\alpha_n\in\mathbb{C}$ we have the identity*

$$\prod_{k=1}^{n}(z^2-\alpha_k z+1)=\sum_{k=0}^{2n}A_{2n,k}z^k \tag{2}$$

where

$$\begin{cases} A_{2n,k} &= (-1)^k\sum_{l=0}^{\left[\frac{k}{2}\right]}\binom{n-k+2l}{l}\sigma_{k-2l}^{(n)} \quad \text{for } k=0,1,\dots,n \\ A_{2n,k} &= A_{2n,2n-k} \quad \text{for } k=n+1,n+2,\dots,2n \end{cases} \tag{3}$$

and $\sigma_k^{(n)} = \sigma_k^{(n)}(\alpha_1, \dots, \alpha_n)$ $(k = 1, \dots, n)$ *denotes the* k *th elementary symmetric polynomial of the variables* $\alpha_1, \alpha_2, \dots, \alpha_n$ *and* $\sigma_0^{(n)} = \sigma_0^{(n)}(\alpha_1, \dots, \alpha_n) := 1$.

Proof. We prove by induction with respect to n. For $n = 1$ equation (3) reduces to

$$A_{2,0} = \binom{2}{0}\sigma_0^{(1)} = 1, \quad A_{2,1} = -\binom{1}{0}\sigma_1^{(1)} = -\alpha_1, \quad A_{2,2} = A_{2,0} = 1$$

which is obviously true.

Assume (3) to hold. Multiplying p_{2n} by $z^2 - \alpha_{n+1}z + 1$ and comparing the coefficients we get that

$$\begin{cases} A_{2n+2,0} = & & & A_{2n,0}, \\ A_{2n+2,1} = & & -\alpha_{n+1}A_{2n,0} & +A_{2n,1}, \\ A_{2n+2,k} = & A_{2n,k-2} & -\alpha_{n+1}A_{2n,k-1} & +A_{2n,k} \quad (k = 2, \dots, n), \\ A_{2n+2,n+1} = & A_{2n,n-1} & -\alpha_{n+1}A_{2n,n} & +A_{2n,n-1}, \end{cases} \tag{4}$$

where, by (3), on the right hand side of the last equation we have replaced the (first) term $A_{2n,n+1}$ by $A_{2n,n-1}$. Thus by the induction assumption and the recurrence equation

$$\alpha_{n+1}\sigma_k^{(n)} = \begin{cases} \sigma_{k+1}^{(n+1)} - \sigma_{k+1}^{(n)} & \text{if} \quad k = 0, 1 \dots, n-1 \\ \sigma_{k+1}^{(n+1)} & \text{if} \quad k = n \end{cases} \tag{5}$$

we get from (4) that

$$A_{2n+2,0} = \binom{n}{0}\sigma_0^{(n)} = \binom{n+1}{0}\sigma_0^{(n+1)},$$

$$A_{2n+2,1} = -\alpha_{n+1}\binom{n}{0}\sigma_0^{(n)} - \binom{n-1}{0}\sigma_1^{(n)} = -\binom{n}{0}\sigma_1^{(n+1)},$$

$$A_{2n+2,k} = (-1)^{k-2}\sum_{l=0}^{\left[\frac{k-2}{2}\right]}\binom{n-k+2+2l}{l}\sigma_{k-2-2l}^{(n)} + (-1)^k\sum_{l=0}^{\left[\frac{k-1}{2}\right]}\binom{n-k+1+2l}{l}\alpha_{n+1}\sigma_{k-1-2l}^{(n)}$$

$$+ (-1)^k\sum_{l=0}^{\left[\frac{k}{2}\right]}\binom{n-k+2l}{l}\sigma_{k-2l}^{(n)}$$

$$= (-1)^k\left[\sum_{l=1}^{\left[\frac{k}{2}\right]}\binom{n-k+2l}{l-1}\sigma_{k-2l}^{(n)} + \sum_{l=0}^{\left[\frac{k-1}{2}\right]}\binom{n-k+1+2l}{l}\left(\sigma_{k-2l}^{(n+1)} - \sigma_{k-2l}^{(n)}\right)\right.$$

$$\left. + \sum_{l=0}^{\left[\frac{k}{2}\right]}\binom{n-k+2l}{l}\sigma_{k-2l}^{(n)}\right]$$

where $\quad k = 2, 3, \dots, n,$

$$\begin{aligned}A_{2n+2,n+1} &= 2(-1)^{n-1}\sum_{l=0}^{\left[\frac{n-1}{2}\right]}\binom{2l+1}{l}\sigma_{n-1-2l}^{(n)} + (-1)^{n+1}\sum_{l=0}^{\left[\frac{n}{2}\right]}\binom{2l}{l}\alpha_{n+1}\sigma_{n-2l}^{(n)}\\ &= (-1)^{n+1}\left[\sum_{l=1}^{\left[\frac{n+1}{2}\right]}2\binom{2l-1}{l-1}\sigma_{n+1-2l}^{(n)} + \alpha_{n+1}\sigma_n^{(n)} + \sum_{l=1}^{\left[\frac{n}{2}\right]}\binom{2l}{l}\alpha_{n+1}\sigma_{n-2l}^{(n)}\right]\\ &= (-1)^{n+1}\left[\sum_{l=1}^{\left[\frac{n+1}{2}\right]}\binom{2l}{l}\sigma_{n+1-2l}^{(n)} + \sigma_{n+1}^{(n+1)} + \sum_{l=1}^{\left[\frac{n}{2}\right]}\binom{2l}{l}\left(\sigma_{n+1-2l}^{(n+1)} - \sigma_{n+1-2l}^{(n)}\right)\right],\end{aligned}$$

where in the first sum of $A_{2n+2,k}, A_{2n+2,n+1}$ we shifted the summation index and for the latter applied the identity

$$2\binom{2l-1}{l-1} = \binom{2l}{l} \quad (l = 1, 2, \dots).$$

Case $1:$ $k = 2s$ *is even*, $2 \leqslant k \leqslant n$.

As $\left[\frac{k-2}{2}\right] = \left[\frac{k-1}{2}\right] = s-1$, $\left[\frac{k}{2}\right] = s$ starting by the first term of the second sum of $A_{2n+2,k}$, we have

$$\begin{aligned}A_{2n+2,k} &= (-1)^k\left[\sum_{l=0}^{s}\binom{n+1-k+2l}{l}\sigma_{k-2l}^{(n+1)} - \binom{n+1}{s}\sigma_0^{(n+1)}\right]\\ &+(-1)^k\left[\sum_{l=1}^{s}\binom{n-k+2l}{l-1}\sigma_{k-2l}^{(n)} - \sum_{l=0}^{s-1}\binom{n-k+1+2l}{l}\sigma_{k-2l}^{(n)} + \sum_{l=0}^{s}\binom{n-k+2l}{l}\sigma_{k-2l}^{(n)}\right].\end{aligned}$$

The expression in the last bracket is

$$\begin{aligned}&\sum_{l=1}^{s}\left[\binom{n-k+2l}{l-1} - \binom{n-k+1+2l}{l} + \binom{n-k+2l}{l}\right]\sigma_{k-2l}^{(n)}\\ &\qquad - \binom{n-k+1}{0}\sigma_k^{(n)} + \binom{n-k+1+2s}{s}\sigma_{k-2s}^{(n)} + \binom{n-k}{0}\sigma_k^{(n)}\\ &\qquad = \binom{n+1}{s}\sigma_0^{(n)} = \binom{n+1}{s}\sigma_0^{(n+1)}\end{aligned}$$

as the sum of the three binomial coefficients is zero. Thus

$$A_{2n+2,k} = (-1)^k\sum_{l=0}^{s}\binom{n+1-k+2l}{l}\sigma_{k-2l}^{(n+1)} \tag{6}$$

proving (3) for $k = 2, 4, \dots, n$ if $n+1$ is odd.

If $n+1=2s$ is even then

$$\begin{aligned} A_{2n+2,n+1} &= (-1)^{n+1}\left[\sum_{l=1}^{s}\binom{2l}{l}\sigma_{n+1-2l}^{(n)}+\sigma_{n+1}^{(n+1)}+\sum_{l=1}^{s-1}\binom{2l}{l}\left(\sigma_{n+1-2l}^{(n+1)}-\sigma_{n+1-2l}^{(n)}\right)\right] \\ &= (-1)^{n+1}\left[\binom{2s}{s}\sigma_0^{(n)}+\sigma_{n+1}^{(n+1)}+\sum_{l=1}^{s-1}\binom{2l}{l}\sigma_{n+1-2l}^{(n+1)}\right] \\ &= (-1)^{n+1}\sum_{l=0}^{s}\binom{2l}{l}\sigma_{n+1-2l}^{(n+1)} \end{aligned}$$

completing the proof in Case 1.

Case 2*: $k=2s+1$ is odd*, $2\leqslant k\leqslant n$.

As $\left[\frac{k-2}{2}\right]=s-1, \left[\frac{k-1}{2}\right]=\left[\frac{k}{2}\right]=s$ starting by the second term of $A_{2n+2,k}$, we have

$$\begin{aligned} A_{2n+2,k} = (-1)^k\sum_{l=0}^{s}\binom{n+1-k+2l}{l}\sigma_{k-2l}^{(n+1)} + (-1)^k\Bigg[\sum_{l=1}^{s}\binom{n-k+2l}{l-1}\sigma_{k-2l}^{(n)} \\ -\sum_{l=0}^{s}\binom{n-k+1+2l}{l}\sigma_{k-2l}^{(n)}+\sum_{l=0}^{s}\binom{n-k+2l}{l}\sigma_{k-2l}^{(n)}\Bigg]. \end{aligned}$$

The expression in the bracket is

$$\begin{aligned} \sum_{l=1}^{s}\left[\binom{n-k+2l}{l-1}-\binom{n-k+1+2l}{l}+\binom{n-k+2l}{l}\right]\sigma_{k-2l}^{(n)} \\ -\binom{n-k+1}{0}\sigma_k^{(n)}+\binom{n-k}{0}\sigma_k^{(n)}=0. \end{aligned}$$

Thus (6) holds proving (3) for $k=3,5,\ldots,n$ if $n+1$ is even.

If $n+1=2s+1$ is odd then

$$\begin{aligned} A_{2n+2,n+1} &= (-1)^{n+1}\left[\sum_{l=1}^{s+1}\binom{2l}{l}\sigma_{n+1-2l}^{(n)}+\sigma_{n+1}^{(n+1)}+\sum_{l=1}^{s}\binom{2l}{l}\left(\sigma_{n+1-2l}^{(n+1)}-\sigma_{n+1-2l}^{(n)}\right)\right] \\ &= (-1)^{n+1}\left[\binom{2s+2}{s+1}\sigma_0^{(n)}+\sigma_{n+1}^{(n+1)}+\sum_{l=1}^{s}\binom{2l}{l}\sigma_{n+1-2l}^{(n+1)}\right] \\ &= (-1)^{n+1}\sum_{l=0}^{s+1}\binom{2l}{l}\sigma_{n+1-2l}^{(n+1)} \end{aligned}$$

completing the proof in Case 2.

THEOREM 1. (Characterization Theorem) *A complex monic reciprocal polynomial*

$$p_{2n}(z)=\sum_{k=0}^{2n}A_{2n,k}z^k \quad (A_{2n,k}\in\mathbb{C},\ A_{2n,0}=1,\ A_{2n,k}=A_{2n,2n-k}\, \textit{for}\ k=0,1,\ldots,2n$$

of even degree $2n$ $(n \in \mathbb{N})$ *has all of its zeros on the unit circle if and only if there exist real numbers* $\alpha_k \in [-2,2]$ $(k=1,\dots,n)$ *such that*

$$\begin{cases} A_{2n,k} &= (-1)^k \sum_{l=0}^{\left[\frac{k}{2}\right]} \binom{n-k+2l}{l} \sigma_{k-2l}^{(n)}(\alpha_1,\dots,\alpha_n) \quad \text{for } k=0,1,\dots,n \\ A_{2n,k} &= A_{2n,2n-k} \quad \text{for } k=n+1,n+2,\dots,2n \end{cases} \tag{7}$$

holds.

A complex monic reciprocal polynomial $p_{2n+1}(z)$ *of odd degree* $2n+1$ $(n \in \mathbb{N})$ *has all of its zeros on the unit circle if and only if* $p_{2n+1}(z) = (z+1)p_{2n}(z)$ *where for the coefficients of* p_{2n} (7) *holds.*

Proof. For even degree polynomials this follows from Lemmas 1, 2. For odd degree polynomials $z=-1$ is always a zero thus we have $p_{2n+1}(z) = (z+1)p_{2n}(z)$ where $p_{2n}(z)$ is also reciprocal, thus the first part of the statement applies.

3. Coefficient estimate

Using the Characterization Theorem we can obtain bounds for the coefficients of a reciprocal polynomial whose zeros are on the unit circle.

THEOREM 2. (Coefficient estimate) *If all zeros of the complex monic reciprocal polynomial*

$$p_{2n}(z) = \sum_{k=0}^{2n} A_{2n,k} z^k \quad (A_{2n,k} \in \mathbb{C},\ A_{2n,0}=1,\ A_{2n,k} = A_{2n,2n-k}\, \textit{for}\ k=0,1,\dots,2n$$

of even degree $2n$ $(n \in \mathbb{N})$ *are on the unit circle, but there are no zeros of* p_{2n} *in the arcs*

$$\{\ e^{iu}\ :\ -\beta \leqslant u \leqslant \beta\ \}\ \textit{and}\ \{\ e^{iu}\ :\ \pi-\beta \leqslant u \leqslant \pi+\beta\ \}$$

where $0 \leqslant \beta \leqslant \frac{\pi}{2}$ *then*

$$|A_{2n,k}| = |A_{2n,2n-k}| \leqslant \sum_{l=0}^{\left[\frac{k}{2}\right]} \binom{n-k+2l}{l} \binom{n}{k-2l} (2\cos\beta)^{k-2l} \quad (k=0,1,\dots,n). \tag{8}$$

Proof. We have $|\alpha_k| \leqslant 2\cos\beta$ hence using the characterization theorem and the estimate

$$|\sigma_k^{(n)}(\alpha_1,\dots,\alpha_n)| \leqslant \binom{n}{k} (2\cos\beta)^k$$

we get

$$|A_{2n,k}| \leqslant \sum_{l=0}^{\left[\frac{k}{2}\right]} \binom{n-k+2l}{l} |\sigma_{k-2l}^{(n)}(\alpha_1,\dots,\alpha_n)| \leqslant \sum_{l=0}^{\left[\frac{k}{2}\right]} \binom{n-k+2l}{l} \binom{n}{k-2l} (2\cos\beta)^{k-2l}$$

for $k=0,1,\dots,n$ proving (8).

REMARK. A $\beta > 0$ (with the property in Theorem 2) exists if ± 1 is not a zero of p_{2n}. In the case when $\beta = 0$ (8) goes over into

$$|A_{2n,k}| = |A_{2n,2n-k}| \leqslant \binom{2n}{k} \quad (k = 0, 1, \ldots, n) \tag{9}$$

as applying the binomial theorem, expanding $1 + 2x + x^2$ by the polynomial theorem, rearranging the sum according to the powers of x and comparing the coefficients of x^k in

$$\begin{aligned} \sum_{k=0}^{2n} \binom{2n}{k} x^k &= (1+x)^{2n} = (1+2x+x^2)^n = \sum_{s,l \geqslant 0,\, s+l \leqslant n} \frac{2^s x^{s+2l} n!}{s!\, l!\, (n-s-l)!} \\ &= \sum_{k=0}^{2n} \left(\sum_{l=0}^{\left[\frac{k}{2}\right]} \frac{2^{k-2l} n!}{(k-2l)!\, l!\, (n-k+l)!} \right) x^k = \sum_{k=0}^{2n} \left(\sum_{l=0}^{\left[\frac{k}{2}\right]} \binom{n-k+2l}{l} \binom{n}{k-2l} 2^{k-2l} \right) x^k \end{aligned}$$

we conclude that

$$\sum_{l=0}^{\left[\frac{k}{2}\right]} \binom{n-k+2l}{l} \binom{n}{k-2l} 2^{k-2l} = \binom{2n}{k}.$$

The estimate (9) also *obviously follows* from the usual Viéta formulae.

Cartwright and Steiger [1], Corollary 1.1 found sharp constraints for the coefficients of an arbitrary complex monic polynomial all of whose zeros are of the same modulus. Their result however does not seem to be easily specialized for our case.

4. Further necessary conditions

In the sequel we show how further necessary conditions can be obtained by the help of our characterization theorem. Let $n \in \mathbb{N}$ be fixed and

$$S_k(\alpha_1, \ldots, \alpha_n) = \sum_{j=1}^{n} \alpha_j^k \quad (k \in \mathbb{N}).$$

S_k being a symmetric polynomial of the α_j 's can be expressed by help of the elementary symmetric polynomials $\sigma_1, \ldots \sigma_n$ (where, for the sake of simplicity we omitted the upper subscript $^{(n)}$ from the σ_k 's) according the equations (see [8], p. 232)

$$\begin{aligned} S_1 - \sigma_1 &= 0 \\ S_2 - \sigma_1 S_1 + 2\sigma_2 &= 0 \\ S_3 - \sigma_1 S_2 + \sigma_2 S_1 - 3\sigma_3 &= 0 \\ &\vdots \\ S_{n-1} - \sigma_1 S_{n-2} + \cdots + (-1)^{n-2}\sigma_{n-2} S_1 + (-1)^{n-1}(n-1)\sigma_{n-1} &= 0. \end{aligned}$$

Therefore

$$\begin{aligned}
\sigma_1 &= S_1\\
\sigma_2 &= 1/2\left(S_1^2 - S_2\right)\\
\sigma_3 &= 1/6\left(S_1^3 - 3S_1S_2 + 2S_3\right)\\
\sigma_4 &= 1/24\left(S_1^4 - 6S_1^2S_2 + 8S_1S_3 + 3S_2^2 - 6S_4\right)\\
\sigma_5 &= 1/120\left(S_1^5 - 10S_1^3S_2 + 20S_1^2S_3 + 15S_1S_2^2 - 30S_1S_4 - 20S_2S_3 + 24S_5\right)\ldots.
\end{aligned}$$

From (7)

$$\begin{aligned}
A_{2n,0} &= \sigma_0 & A_{2n,3} &= -\left[\sigma_3 + \tbinom{n-1}{1}\sigma_1\right]\\
A_{2n,1} &= -\sigma_1 & A_{2n,4} &= \sigma_4 + \tbinom{n-2}{1}\sigma_2 + \tbinom{n}{2}\sigma_0\\
A_{2n,2} &= \sigma_2 + \tbinom{n}{1}\sigma_0 & A_{2n,5} &= -\left[\sigma_5 + \tbinom{n-3}{1}\sigma_3 + \tbinom{n-1}{2}\sigma_1\right]\ldots.
\end{aligned}$$

which shows that σ_k 's can be obtained as linear combinations of the coefficients $A_{2n,k}$.

We can obtain further necessary conditions in the following ways.

1. *Using well-known inequalities for the* σ_k 's.

Newton's inequality (see Mitrinović [6] p. 95) states that for $k = 1, 2, \ldots, n-1$

$$E_k^2 \geqslant E_{k-1}E_{k+1} \tag{10}$$

with equality if and only if $\alpha_1 = \cdots = \alpha_n$, where

$$E_k = E_k(\alpha_1, \ldots \alpha_n) = \frac{\sigma_k(\alpha_1, \ldots \alpha_n)}{\binom{n}{k}}.$$

Maclaurin's theorem (see [6] p. 97) reads as follows. For $1 < k < l < n$ we have

$$E_l^{1/l} \leqslant E_k^{1/k} \tag{11}$$

with equality if and only if $\alpha_1 = \cdots = \alpha_n$.

Several other inequalities can be found for the E_k 's in the paper of Niculescu [7] and in the references there.

2. *Using the estimate* $|\sigma_k^{(n)}(\alpha_1, \ldots, \alpha_n)| \leqslant \binom{n}{k}2^k$ or (under the conditions of Theorem 2) the inequality $|\sigma_k^{(n)}(\alpha_1, \ldots, \alpha_n)| \leqslant \binom{n}{k}(2\cos\beta)^k$.

3. *We can also use the inequalities*

$$\begin{aligned}
-2^k n \leqslant\ & S_k \leqslant 2^k n \qquad \text{if } k \text{ is odd},\\
0 \leqslant\ & S_k \leqslant 2^k n \qquad \text{if } k \text{ is even}
\end{aligned} \tag{12}$$

and, by this, we can estimate σ_k and also $A_{2n,k}$.

We show examples for the above methods.

THEOREM 3. *If all zeros of the complex monic reciprocal polynomial*

$$p_{2n}(z) = \sum_{k=0}^{2n} A_{2n,k} z^k \quad (A_{2n,k} \in \mathbb{C},\, A_{2n,0} = 1,\ A_{2n,k} = A_{2n,2n-k}\, \textit{for}\ k = 0, 1, \ldots, 2n$$

of degree $2n$ $(n \in \mathbb{N})$ *are on the unit circle then* $A_{2n,k}$ $(k = 0, 1, \ldots, 2n)$ *are real and*

$$\left(\frac{A_{2n,2} - n}{\binom{n}{2}}\right)^2 \geqslant \frac{A_{2n,1}\left(A_{2n,3} - (n-1)A_{2n,1}\right)}{\binom{n}{1}\binom{n}{3}}, \tag{13}$$

$$\left(\frac{-A_{2n,3} + (n-1)A_{2n,1}}{\binom{n}{3}}\right)^{1/3} \leqslant \left(\frac{A_{2n,2} - n}{\binom{n}{2}}\right)^{1/2}, \tag{14}$$

$$-\binom{n}{3}2^3 \leqslant -A_{2n,3} + (n-1)A_{2n,1} \leqslant \binom{n}{3}2^3, \tag{15}$$

$$-n \leqslant A_{2n,2}. \tag{16}$$

In (13) *and* (14) *we have equality if and only if* $p_{2n}(z) = (z^2 - \alpha z + 1)^n$ *with* $\alpha \in [-2, 2]$.

In (15) *we have equality on the left if* $p_{2n}(z) = (z+1)^{2n}$ *and on the right if* $p_{2n}(z) = (z-1)^{2n}$.

In (16) *we have equality if and only if* n *is even and* $p_{2n}(z) = (z^2 - 1)^n$.

Proof. We have

$$\sigma_1 = -A_{2n,1}, \quad \sigma_2 = A_{2n,2} - n, \quad \sigma_3 = -A_{2n,3} + (n-1)A_{2n,1} \tag{17}$$

hence by the inequality (10) for $k = 2$, and by (11) for $k = 2, l = 3$ we get (13), (14) respectively. Equality holds in these if and only if $\alpha_1 = \cdots = \alpha_n = \alpha \in [-2, 2]$ which gives the extremal polynomial $(z^2 - \alpha z + 1)^n$.

To prove (15) use (9) for $k = 3$, apply the last equation of (17) and the second proposed method. Equality holds on the left side if $\alpha_1 = \cdots = \alpha_n = -2$ and on the right side if $\alpha_1 = \cdots = \alpha_n = 2$.

Finally by (12) $0 \leqslant S_1^2 \leqslant 4n^2$, $0 \leqslant S_2 \leqslant 4n$ hence

$$-2n \leqslant \sigma_2 = 1/2\left(S_1^2 - S_2\right) \text{ or by (17) } -2n \leqslant A_{2n,2} - n$$

proving (16). Equality holds if and only if $S_1 = \sum_{k=1}^{n} \alpha_k = 0, S_2 = \sum_{k=1}^{n} \alpha_k^2 = 4n$ i.e. if and only if $n = 2m$ is even and half of the α_k's equals 2, the other half equal -2, which gives the extremal polynomial $p_{2n}(z) = (z^2 - 2z + 1)^m (z^2 + 2z + 1)^m = (z^2 - 1)^{2m} = (z^2 - 1)^n$. We remark that for odd n the coefficient of z^{n-2} of p_{2n} is also $-n$ but p_{2n} is not reciprocal.

We remark that there are several *sufficient conditions* (see Lakatos [2], [3], Schinzel [9], Lakatos and Losonczi [4], [5]) for the coefficients of reciprocal and self-inversive polynomials to have all their zeros on the unit circle.

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