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FUNCTIONAL INEQUALITIES FOR INCOMPLETE GAMMA AND RELATED FUNCTIONS

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Abstract. For the incomplete gamma function and related functionse we establish functional inequalities of the type $f(x)f(y) - f(x+y) \geqslant 0$ for all $x, y \in (0, \infty)$.

1. Introduction and main results

The incomplete gamma function is defined by [2]

$$\Gamma(a;x) = \int_x^\infty t^{a-1}e^{-t}, \quad a > 0, \tag{1.1}$$

for real x. When $x = 0$ we find

$$\Gamma(a;0) = \int_0^\infty t^{a-1}e^{-t} = \Gamma(a).$$

The incomplete gamma function plays a fundamental role in probability theory, statistics, physics and many other disciplines which uses mathematical techniques. Gautschi's survey article [4] covers many of the important results on $\Gamma(a;x)$ and its applications. The aim of this paper is to establish functional inequalities of the type

$$f(x)f(y) - f(x+y) \geqslant 0, \quad \text{for every } x, y \geqslant 0, \tag{1.2}$$

for certain classes of functions f. The first result is the following.

THEOREM 1.1. *For a fixed $a > 0$, the function*

$$f(x) := \Gamma(a;x)/\Gamma(a) \tag{1.3}$$

satisfies (1.2) *when* $a \geqslant 1$. *If* $a \leqslant 1$ *then inequality in* (1.2) *is reversed.*

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The proof will be given in §2 together with the proof of the following corollary.

COROLLARY 1.2. *Let $\gamma(a;x)$ be the incomplete gamma function defined by*

$$\gamma(a;x) = \int_0^x t^{a-1}\, e^{-t} dt. \tag{1.4}$$

Then the function

$$g(x) := \gamma(a;x)/\Gamma(a),$$

satisfies

$$g(x)g(y) - g(x) - g(y) + g(x+y) > 0, \tag{1.5}$$

when $a \geqslant 1$. If $0 < a < 1$ then the inequality in (1.5) is reversed. Moreover the equality in (1.5) holds for all $x > 0$ if $a = 1$.

The above corollary follows from Theorem 1.1 when replacing $\Gamma(a;x)$ by $\Gamma(a) - \gamma(a;x)$.

This work was motivated by the inequality [5, p. 291]

$$\operatorname{erf}(x)\operatorname{erf}(y) - \operatorname{erf}(x) - \operatorname{erf}(y) + \operatorname{erf}(x+y) \geqslant 0, \tag{1.6}$$

where erf is the error function

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

The inequality (1.6) has been recently extended to n variables by Alzer in [1].

Let u be a positive monotone twice continuously differentiable function on $(0,\infty)$ and assume that $e^{-t}u(t) \in L_1((0,\infty))$. In §2 we shall prove the following generalization of Theoem 1.1.

THEOREM 1.3. *Let*

$$h(x) := \int_x^\infty e^{-t}\, u(t)\, dt, \quad f(x) := \frac{h(x)}{h(0)}. \tag{1.7}$$

If $u(x+y)/u(x)$ is nonincreasing in x on $(0,\infty)$ for every $y > 0$, then f satisfies (1.2). If $u(x+y)/u(x)$ is nondecreasing in x on $(0,\infty)$ for every $y > 0$, then f satisfies the reversed inequality

$$f(x)f(y) - f(x+y) \leqslant 0. \tag{1.8}$$

It is clear that Theorem 1.1 corresponds to the choice $u(x) = x^{a-1}$.

As an application of Theorem 1.3 consider the case $u(x) = 2x^{a-1}\sinh(\lambda x)$ for $a \geqslant 1$ and $0 < \lambda < 1$. In this case

$$\frac{u(x+y)}{u(x)} = \left(1 + \frac{y}{x}\right)^{a-1} [\cosh(\lambda y) + \sinh(\lambda y)\, \coth(\lambda x)],$$

hence is decreasing in x on $(0,\infty)$. Moreover

$$h(x) = (1-\lambda)^{-a}\Gamma(a;(1-\lambda)x) - (1+\lambda)^{-a}\Gamma(a;(1+\lambda)x),$$

hence

$$f(x) = \frac{(1+\lambda)^a\Gamma(a;(1-\lambda)x) - (1-\lambda)^a\Gamma(a;(1+\lambda)x)}{\Gamma(a)\left[(1+\lambda)^a-(1-\lambda)^a\right]}. \tag{1.9}$$

Thus f satisfies (1.2) for $a \geqslant 1$. On the other hand the choice $u(x) = 2x^{a-1}\cosh(\lambda x)$ makes

$$\frac{u(x+y)}{u(x)} = \left(1+\frac{y}{x}\right)^{a-1} [\cosh(\lambda y) + \sinh(\lambda y)\,\tanh(\lambda x)],$$

which is an increasing function of x for $x > 0$ and $0 < a < 1$. In this case

$$f(x) = \frac{(1+\lambda)^a\Gamma(a;(1-\lambda)x) + (1-\lambda)^a\Gamma(a;(1+\lambda)x)}{\Gamma(a)\left[(1+\lambda)^a+(1-\lambda)^a\right]} \tag{1.10}$$

and f satisfies (1.8).

The problem of finding a q-analogue, [3] of the results of this paper remains open.

2. Proofs

In this section we give proofs of Theorems 1.1 and 1.3.

Proof of Theorem 1.1. Let f be as in (1.3). For fixed $y > 0$ define the function F by

$$F(x) = f(x)f(y) - f(x+y).$$

Formulas (1.1) and (1.3) yield

$$f(0) = 1, \qquad \lim_{x\to\infty} f(x) = 0,$$

hence F has the property

$$F(0) = 0, \qquad \lim_{x\to\infty} F(x) = 0. \tag{2.1}$$

Thus, we apply Rolle's theorem to the function F on $[0,\infty)$, and conclude that there exists a point $c \in (0,\infty)$ such that $f'(c) = 0$. On the other hand

$$F'(x) = f(y)e^{-x}x^{a-1}H(x), \quad H(x) := \left[\frac{e^{-y}}{f(y)}\left(1+\frac{y}{x}\right)^{a-1} - 1\right]. \tag{2.2}$$

Thus $H(c) = 0$. The case $a = 1$ is trivial because $F(x) \equiv 0$ in this case. If $a > 1$, the function $H(x)$ strictly decreases with x, hence $H(x) > 0$ on $(0,c)$. Moreover $F'(x)$, being a positive multiple of $H(x)$, vanishes only at $x = c$. Since F' has the same sign as H, F is strictly increasing on $(0,c)$ and strictly decreasing on (c,∞). This fact together with (2.1) implies $F(x) > 0$ for $x > 0$ when $a > 1$. When $0 < a < 1$ the function F' in (2.2) is negative for $x < c$ and positive for $x > c$. As in the case

$a > 1$, we argue that $F'(x) = 0$ if and only if $x = c$, and $F(x) < 0$ on $(0,\infty)$. This completes the proof. □

Proof of Theorem 1.3. The proof is similar to the proof of Theorem 1.1. Fix $y > 0$, and define F by

$$F(x) = f(x)f(y) - f(x+y)$$

so that $F(0) = 0$ and $F(+\infty) = 0$, hence $F'(c) = 0$ for some $c > 0$. A calculation gives

$$\begin{aligned} F'(x) &= f(y)e^{-x}u(x)H(x), \\ H(x) &:= \left[\frac{e^{-y}u(x+y)}{f(y)u(x)} - 1\right]. \end{aligned} \tag{2.3}$$

The rest of the proof is as before.

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