# APPROXIMATION RATE FOR MÜNTZ
# RATIONAL FUNCTIONS IN $L^p$ SPACES

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*Abstract.* Let $\{\lambda_n\}$ be a sequence of real numbers, satisfying $0 \leqslant \lambda_1 < \lambda_2 < \cdots$, and $\lambda_{n+1} - \lambda_n \geqslant Mn$ with $M$ a positive constant. This note gives a Jackson type theorem for Müntz rational approximation in $L^p$ spaces by using a new kind of Ditzian-Totik type modulus.

## 1. Introduction

Let

$$\Lambda_n = \{\lambda_1, \lambda_2, \cdots, \lambda_n\},$$

$$R_n(\Lambda_n) := \left\{ \frac{P(x)}{Q(x)} : P(x), Q(x) \in \text{Span}\{x^{\lambda_k}\}, \lambda_k \in \Lambda_n \right\},$$

where $\text{Span}\{x^{\lambda_k}\}$ is the class of all linear combinations of $\{x^{\lambda_k}\}$. For any $f \in L^p_{[0,1]}$, $1 \leqslant p < \infty$, define the best Müntz rational approximation for Müntz system $\{x^{\lambda_n}\}$ by

$$R_n(f)_p := \inf_{r \in R_n(\Lambda_n)} \|f - r\|_{L^p},$$

where $\| \cdot \|_{L^p}$ is the usual $L^p-$norm.

As we know, it is a very difficult subject to estimate the Müntz rational approximation rate. One reason lies in the construction of suitable Müntz rational operators which can be used to approximate functions, the other is the fact that Müntz rational function is surely not linear and is not even closed for addition. There are some progress in recent years on this topic (see [5]). Among them, we [4] established

THEOREM 1. *Given $M > 0$, if $0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, and*

$$\lambda_{n+1} - \lambda_n \geqslant Mn, \ n = 1, 2, \cdots.$$

*Then for any $f \in C_{[0,1]}$, there exists a $r(x) \in R_n(\Lambda)$ such that*

$$|f(x) - r(x)| \leqslant C_{M,N} \omega \left( f, \frac{1}{n^{N+1}} + \frac{\psi(x)}{n} \right), \ x \in [0,1], \tag{1}$$

*where $N$ is a given positive integer, $\psi(x) := x^{\frac{N}{N+1}}$ and $\omega(f,t)$ is the usual modulus of continuity of $f$.*

From the proof of Theorem 1, one can easily find that, in fact, $N$ can be any given real number $\geqslant 1$. At the same paper, we pointed out that the right term in (1) can also be replaced by the Ditzian-Totik type modulus $\omega_\psi(f, \frac{1}{n})$ (see [2] for definition of the Ditzian-Totik Modulus of continuity). All of them were the great improvements of Bak's well known results ([1]), where he obtained that $R_n(f, \Lambda) \leqslant C_M \omega(f, 1/n)$.

Recently, the second named author with his collaborator [3] generalized Bak's result to the $L^p$ spaces. In fact, they established the following

THEOREM 2. *Given $M > 0$, if $0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, and*

$$\lambda_{n+1} - \lambda_n \geqslant Mn, \ n = 1, 2, \cdots.$$

*Then for any $f \in L^p_{[0,1]}$, $1 \leqslant p < \infty$, there is a positive constant $C_M$ only depending on $M$ such that*

$$R_n(f, \Lambda)_{L^p} \leqslant C_M \omega \left(f, \frac{1}{n}\right)_{L^p}.$$

In this note, we consider to further improve the result of Theorem 2. We prove the following

THEOREM 3. *Given $M > 0$, if $0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, and*

$$\lambda_{n+1} - \lambda_n \geqslant Mn, \ n = 1, 2, \cdots.$$

*Then for any $f \in L^p_{[0,1]}$, $1 \leqslant p < \infty$, there is a positive constant $C_{M,N}$ only depending on $M$ and $N$ such that*

$$R_n(f, \Lambda)_{L^p} \leqslant C_{M,N} \omega_\psi \left(f, \frac{1}{n}\right)_{L^p},$$

*where $\psi(x)$ defined as in Theorem 1.*

Let $\varphi(x) = \sqrt{x}$, then $\omega_\psi(f, 1/n) \leqslant \omega_\varphi(f, 1/n) \leqslant \omega(f, 1/n)$. Therefore, Theorem 3 is an improvement of Theorem 2. For example, Let $f(x) = \log x$, then

$$\omega_\varphi(f, 1/n)_{L^p} \sim \begin{cases} \dfrac{1}{n}, & 1 \leqslant p < 2 \\[2mm] \dfrac{(\log n)^{\frac{1}{2}}}{n}, & p = 2 \\[2mm] \left(\dfrac{1}{n}\right)^{\frac{2}{p}}, & p > 2, \end{cases}$$

and

$$\omega_\psi(f, 1/n)_{L^p} \sim \begin{cases} \dfrac{1}{n}, & 1 \leqslant p < N+1 \\[2mm] \dfrac{(\log n)^{\frac{1}{N+1}}}{n}, & p = N+1 \\[2mm] \left(\dfrac{1}{n}\right)^{\frac{N+1}{p}}, & p > N+1. \end{cases}$$

This indicates that, in many special cases, the approximation rate $R_n(f, \Lambda)_{L^p}$ can be essentially improved.

Through out the paper, $C_\mu$ (or $C_{\mu,\nu}$) denotes a constant only depending on $\mu$ (or $\mu$ and $\nu$), their values may be different even in the same line.

## 2. Proofs

For any given $N \geqslant 1$, define $x := x(\theta) : [0, 1] \to [0, 1]$ by

$$x(\theta) = \theta^{N+1}, \ \theta \in [0, 1].$$

Let $x_k = x(k/n), \ k = 0, 1, \cdots, n$.

LEMMA 1. *Let* $x \in [x_{j-1}, x_j], \ j = 2, 3, \cdots, n$, *then*

$$|x - x_k| \leqslant \frac{(|j - k| + 1)^{N+1}}{n} \psi(u), \ u \in [x, x_k] \ or \ [x_k, x] \tag{2}$$

*for* $k = 1, 2, \cdots, n$. *For* $x \in [0, x_1]$ *and* $k = 1, 2, \cdots, n$, *it holds that*

$$|x - x_k| \leqslant C_N \frac{(k + 1)^{N+1}}{n^{N+1}}. \tag{3}$$

*Proof.* Since $x'(\theta)$ is nondecreasing in $[0, 1]$, then

$$|x - x_k| \leqslant x'(\theta_k)(\theta_k - \theta) \leqslant C_N \frac{|j - k| + 1}{n} \psi(x) \left( \frac{x'(\theta_k)}{x'(\theta)} \right)$$

$$\leqslant C_N \frac{|j - k| + 1}{n} \psi(x) \left( \frac{k}{j} \right)^N \leqslant C_N \frac{(|j - k| + 1)^{N+1}}{n} \psi(x)$$

for $k \geqslant j$. For $k \leqslant j - 1$, we have

$$|x - x_k| \leqslant x'(\theta)(\theta_k - \theta) \leqslant C_N \frac{|j - k| + 1}{n} \psi(x).$$

Thus, we obtain that

$$|x - x_k| \leqslant C_N \frac{(|j - k| + 1)^{N+1}}{n} \psi(x). \tag{4}$$

Similarly, we have

$$|x - x_k| \leqslant C_N \frac{(|j - k| + 1)^{N+1}}{n} \psi(x_k). \tag{5}$$

Therefore, (2) is finished by (4) and (5). By applying (2), for $x \in [0, x_1]$ and $k = 1, 2, \cdots, n$, we conclude that

$$|x - x_k| \leqslant x_1 + |x_1 - x_k| \leqslant \frac{1}{n^{N+1}} + C_N \frac{(k + 1)^{N+1}}{n} \psi(x_1)$$

$$\leqslant C_N \frac{(k + 1)^{N+1}}{n^{N+1}}.$$

LEMMA 2. *Suppose* $f \in L_{[0,1]}^p$ $1 \leqslant p < \infty$, *define*

$$K_\psi(f,t) := \inf \left\{ \|f - g\|_{L^p} + t\|\psi g'\|_{L^p} + t^{N+1}\|g'\|_{L^p} : g \in \text{A.C.}_{[0,1]} \right\}.$$

*Then there exists a constant* $C_N$ *and* $t_0$ *such that*

$$C_N^{-1}\omega_\psi(f,t)_{L^p} \leqslant K_\psi(f,t) \leqslant C_N\omega_\psi(f,t)_{L^p}.$$

*Proof.* There is a slightly difference from the proof of Theorem 3.1.2 of [2]. In fact, in our case, use $0 = \beta(1) < \beta = \beta(0) = \frac{N}{N+1} < 1$ instead of $\beta = \beta(0) = \beta(1) < 1$. The proof of Lemma 2 can be proceeded by repeating that of [2].

Let $t = \frac{1}{n}$, then for sufficient large $n$, by Lemma 2, we obtain

LEMMA 3. *Suppose that* $f \in L_{[0,1]}^p$, $1 \leqslant p < \infty$, *then there exists a* $g \in \text{A.C.}_{[0,1]}$ *such that*

$$\|f - g\|_{L^p} \leqslant C_N\omega_\psi\left(f,\frac{1}{n}\right)_{L^p}, \tag{6}$$

$$\|\psi g'\|_{L^p} \leqslant C_N n\omega_\psi\left(f,\frac{1}{n}\right)_{L^p}, \tag{7}$$

$$\|g'\|_{L^p} \leqslant C_N n^{N+1}\omega_\psi\left(f,\frac{1}{n}\right)_{L^p}. \tag{8}$$

Set

$$\Delta\lambda_1 = \lambda_1, \ \Delta\lambda_k = \lambda_k - \lambda_{k-1}, \ k = 1, 2, \cdots,$$

$$P_j(x) := P_{jn}(x) = x^{\lambda_j} \prod_{l=1}^{j} x_l^{-\Delta\lambda_l}, \ j = 1, 2, \cdots, n.$$

For any $f \in L_{[0,1]}^p$ $1 \leqslant p < \infty$, taking $g$ satisfies Lemma 3, define

$$L_n(g,x) := \sum_{k=1}^{n} g(x_k)r_k(x),$$

where

$$r_k(x) := \frac{P_k(x)}{\sum\limits_{j=1}^{n} P_j(x)}.$$

It is obvious that $L_n(g,x) \in R_N(\Lambda)$. For $r_k(x)$, it holds that

LEMMA 4. ([4]) *For any* $x \in [x_{j-1}, x_j]$, $1 \leqslant j \leqslant n$, *we have for* $k = 1, 2, \cdots, n$,

$$r_k(x) \leqslant C_{M,N}e^{-C_M(N+1)|k-j|}. \tag{9}$$

*Proof of Theorem* 3. By the construction of $L_n(g,x)$ and (6), we have

$$\|L_n(g) - f\|_{L^p} \leqslant \|L_n(g) - g\|_{L^p} + \|f - g\|_{L^p}$$

$$\leqslant \|L_n(g) - g\|_{L^p} + C_N \omega_\psi \left(f, \frac{1}{n}\right)_{L^p}.$$

Therefore, we only need to prove

$$\|L_n(g) - g\|_{L^p} \leqslant C_{M,N} \omega_\psi \left(f, \frac{1}{n}\right)_{L^p}. \tag{10}$$

We will always assume that $1 < p < \infty$, the case of $p = 1$ can be treated in the same way. Since $\sum_{k=1}^{n} r_k(x) = 1$, then by Hölder inequality, we observe that

$$\|L_n(g) - g\|_{L^p}^p \leqslant C_{M,N}^p \left\{ \int_0^{x_1} \sum_{k=1}^{n} r_k^{p/2}(x) \left| \int_x^{x_k} g'(u)du \right|^p dx \right.$$

$$\left. + \sum_{j=2}^{n} \sum_{k=1}^{n} \int_{x_{j-1}}^{x_j} r_k^{p/2}(x) \left| \int_x^{x_k} g'(u)du \right|^p dx \right\}$$

$$:= I_1 + I_2. \tag{11}$$

By (3), (9) and using Hölder inequality again, we get

$$I_1 \leqslant C_{M,N}^p \int_0^{x_1} \sum_{k=1}^{n} r_k^{p/2}(x) \left| \int_x^{x_k} |g'(u)|^p du \right| |x - x_k|^{p-1} dx$$

$$\leqslant C_{M,N}^p \|g'\|_{L^p}^p \sum_{k=1}^{n} e^{-C_M(N+1)kp/2} \frac{(k+1)^{(p-1)(N+1)}}{n^{(p-1)(N+1)}} x_1$$

$$\leqslant C_{M,N}^p \frac{1}{n^{p(N+1)}} \|g'\|_{L^p}^p \left( \sum_{k=1}^{n} e^{-C_M(N+1)k/2} (k+1)^{N+1} \right)^p$$

$$\leqslant C_{M,N}^p \frac{1}{n^{p(N+1)}} \|g'\|_{L^p}^p. \tag{12}$$

For any $k$, $1 \leqslant k \leqslant n$, and $x \in [x_{j-1}, x_j]$, $j = 2, 3, \cdots, n$, take

$$x^* = \begin{cases} x_{j-1}, & \text{for } k \geqslant j, \\ x_j, & \text{for } k \leqslant j - 1. \end{cases}$$

We verify that

$$\frac{\psi(x_j)}{\psi(u)} \leqslant (|j - k| + 1)^N, \ u \in [x^*, x_k], \text{ or } [x_k, x^*]. \tag{13}$$

In fact, if $x^* = x_j$, that is, $k \leqslant j - 1$, then

$$\frac{\psi(x_j)}{\psi(u)} \leqslant \frac{\psi(x_j)}{\psi(x_k)} = \left(\frac{j}{k}\right)^N = \left(1 + \frac{j-k}{k}\right)^N \leqslant (|j - k| + 1)^N.$$

If $x^* = x_{j-1}$, then

$$\frac{\psi(x_j)}{\psi(u)} \leqslant \frac{\psi(x_j)}{\psi(x_{j-1})} \leqslant \left(1 + \frac{1}{j-1}\right)^N \leqslant (|j-k|+1)^N.$$

In a similar way to (12), we deduce that

$$I_2 \leqslant C_{M,N}^p \sum_{j=2}^{n} \sum_{k=1}^{n} \int_{x_{j-1}}^{x_j} r_k^{p/2}(x) \left| \int_{x^*}^{x_k} g'(u)du \right|^p dx$$

$$\leqslant C_{M,N}^p \sum_{j=2}^{n} \sum_{k=1}^{n} \int_{x_{j-1}}^{x_j} r_k^{p/2}(x) \left| \int_{x^*}^{x_k} |g'(u)\psi(u)| \frac{\psi(x_j)}{\psi(u)} du \right|^p \psi(x_j)^{-p} dx$$

$$\leqslant C_{M,N}^p \sum_{j=2}^{n} \sum_{k=1}^{n} \int_{x_{j-1}}^{x_j} r_k^{p/2}(x)(|j-k|+1)^{Np} \psi(x_j)^{-p} |x_k - x^*|^{p-1} \left| \int_{x^*}^{x_k} |g'(u)\psi(u)|^p du \right| dx$$

$$\leqslant C_{M,N}^p \sum_{j=2}^{n} \sum_{k=1}^{n} e^{-C_M(N+1)p|j-k|/2} (|j-k|+1)^{Np} \psi(x_j)^{-p} |x_k - x^*|^p \left| \int_{x^*}^{x_k} |g'(u)\psi(u)|^p du \right|$$

$$\leqslant C_{M,N}^p \frac{1}{n^p} \sum_{j=2}^{n} \sum_{k=1}^{n} e^{-C_M(N+1)p|j-k|/2} (|j-k|+1)^{2(N+1)p} \left(\frac{\psi(x^*)}{\psi(x_j)}\right)^p \left| \int_{x^*}^{x_k} |g'(u)\psi(u)|^p du \right|$$

$$\leqslant C_{M,N}^p \frac{1}{n^p} \sum_{j=2}^{n} \sum_{k=1}^{n} e^{-C_M(N+1)p|j-k|/2} (|j-k|+1)^{2(N+1)p} \left| \int_{x^*}^{x_k} |g'(u)\psi(u)|^p du \right|$$

$$\leqslant C_{M,N}^p \frac{1}{n^p} \sum_{m=1}^{n} \sum_{|j-k|=m} e^{-C_M(N+1)p|j-k|/2} (|j-k|+1)^{2(N+1)p} \left| \int_{x^*}^{x_k} |g'(u)\psi(u)|^p du \right|$$

$$\leqslant C_{M,N}^p \frac{1}{n^p} \sum_{m=1}^{n} e^{-C_M(N+1)mp/2} m^{2(N+1)p+1} \int_0^1 |g'(u)\psi(u)|^p du$$

$$\leqslant C_{M,N}^p \frac{1}{n^p} \|\psi g'\|_{L^p}^p \left( \sum_{m=1}^{n} e^{-C_M(N+1)m/2} m^{3(N+1)} \right)^p$$

$$\leqslant C_{M,N}^p \frac{1}{n^p} \|\psi g'\|_{L^p}^p \tag{14}$$

Combining (11), (12) with (14), and by (7), (8), we already have

$$\|L_n(g) - g\|_{L^p}^p \leqslant C_{M,N}^p \left( \frac{1}{n^{p(N+1)}} \|g'\|_{L^p}^p + \frac{1}{n^p} \|\psi g'\|_{L^p}^p \right) \leqslant C_{M,N}^p \left( \omega_\psi \left(f, \frac{1}{n}\right)_{L^p} \right)^p.$$

Thus, (12) is completed.

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