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# GEOMETRIC MEANS OF TWO POSITIVE NUMBERS

RAGHIB M. ABU-SARIS AND MOWAFFAQ HAJJA

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*Abstract.* We introduce three families of means that encompass several classical means that arise naturally in geometric contexts. Within these families, we discuss dominance relations, closure under Gauss' compounding, behavior under equal increments, and other related issues.

## 1. Introduction

In the context of elementary geometry, it is natural to think of two given positive numbers $x$ and $y$ as representing a rectangle $R[x,y]$ with sides $x$ and $y$, and to think of their mean $z$ as the side of the square which shares a certain natural feature with that rectangle. Taking this common features to be the perimeter, area and the diagonal, one obtains the *arithmetic mean*, the *geometric mean* and what is known as the *root-mean square*, respectively. These are the classical means defined by

$$\frac{x+y}{2}, \quad \sqrt{xy}, \quad \sqrt{\frac{x^2+y^2}{2}}$$

and known since the Pythagoreans [16, page 75]. A less natural feature of the rectangle $R[x,y]$ is the total length $2(x+y+\sqrt{x^2+y^2})$ of the complete network that connects its vertices. The side $z$ of the square that shares this feature is the solution of the equation $2(x+y+\sqrt{x^2+y^2}) = 2(2z+\sqrt{2z^2})$, i.e., the mean defined by

$$\frac{1}{2\sqrt{2}}\left(x+y+\sqrt{x^2+y^2}\right).$$

If one fixes an angle $\theta$ with ( $\sin\theta/2 = s$, say) and replaces rectangles and squares above by parallelograms and rhombi having vertex angle $\theta$, one obtains the families of means given by

$$\alpha_s(x,y) = \frac{x+y+\sqrt{(x-y)^2+4xys^2}}{2(1+s)}, \quad \beta_s(x,y) = \frac{\sqrt{(x-y)^2+4xys^2}}{2s} \tag{1}$$

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Another natural way of defining a mean of $x$ and $y$ is to take any trapezoid $Z = Z[x,y]$ whose parallel bases have lengths $x$ and $y$ and to think of their mean as the base length of a rectangle having the same height and area as $Z$ [7, page 38]. This approach gives rise to the arithmetic mean again, since the area of a trapezoid with parallel sides $x$ and $y$ and height $h$ is given by

$$A(x,y) = \frac{x+y}{2}h. \tag{2}$$

However, if one repeats this procedure in the three-dimensional space by considering a frustum whose parallel bases have areas $x$ and $y$ and thinking of their mean as the base area of a parallelepiped having the same height and volume, one gets what is referred to in the literature as the *Heronian mean*, i.e., the mean given by

$$H(x,y) = \frac{x+\sqrt{xy}+y}{3}.$$

This is equivalent to saying that the volume of the frustum whose parallel bases have areas $x$ and $y$ and whose height $h$ is given by

$$V(x,y) = \frac{x+\sqrt{xy}+y}{3}h,$$

a fact amazingly known to the Ancient Egyptians nearly 4000 years ago. It is expressed (through a numerical example as was usual) in Problem 14 of the Moscow Papyrus discovered in 1893, a problem that so strongly impressed Eric Temple Bell that he called it "the greatest Egyptian Pyramid" [15, Lecture 2]. It is worth mentioning that the false formula $V(x,y) = (x+y)h/2$ inspired by (2) is the one that the Babylonians used for the volume of a frustum [15, page 12].

Going back to the trapezoid $Z = Z[x,y]$ whose parallel bases have lengths $x$ and $y$ one considers the family $\Omega$ of all the line segments parallel to the bases and intercepted by the sides, and then takes the mean $t$ of $x$ and $y$ to be the length of the segment $S$ in $\Omega$ that satisfies a certain natural condition. If $S$ is required to divide $Z$ into trapezoids $Z_1$ and $Z_2$ of the same height, then the length of $S$ is the arithmetic mean of $x$ and $y$. If one requires $Z_1$ and $Z_2$ to be similar, then one gets the geometric mean. If one requires them to have the same area, then one gets the root-mean square. If $S$ is to pass through the point of intersection of the diagonals of $Z$, then one gets the *harmonic mean* defined by

$$\frac{2xy}{x+y},$$

and if $S$ is to pass through the centroid of the region enclosed in the trapezoid, then one gets the *centroidal mean* defined by

$$\frac{2(x^2+xy+y^2)}{3(x+y)};$$

see [16, page 168] and [17].

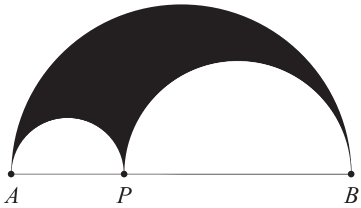


*Figure 1. Archimedes Arbelos*

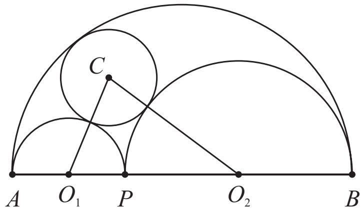


*Figure 2. The Incircle of the Arbelos*

The *arbelos* (or *shoemaker's knife*) of Archimedes, as shown in Figure 1, gives rise to other families of means. Starting with any positive numbers $x$ and $y$ one draws a line $APB$ where $AP$ and $PB$ have lengths $2x$ and $2y$, respectively, and three semicircles $S(AP), S(PB), S(AB)$ on the same side of $AB$ and having $AP$, $PB$ and $AB$ as diameters. The $(x,y)$-arbelos is the figure bounded by these three semicircles [14, page 27], [16, page 156], [6, page 148], and [9, page 228]. Its area can be easily seen to be $\pi ab$. Its inradius is the radius of its incircle, i.e., the circle tangent to the three semicircles (see Figure 2), and it is given by

$$\frac{xy(x+y)}{x^2+xy+y^2};$$

see [26, pages 61-72] for this and the other formulas that are used below. If $Q$ is the point on the semicircle $S(AB)$ such that $PQ$ is perpendicular to $AB$, then $PQ$ divides the arbelos into two regions whose incircles turn out to be equal of radius $xy/(x+y)$ each. This fact is attributed to Archimedes and is referred to in the literature as Archimedes' theorem, and the two circles may be called the *Archimedean circles*. It also turns out that if the incircle of the arbelos touches the semicircles $S(AP)$ and $S(PB)$ at $X$ and $Y$, then the circle through $P$, $X$ and $Y$ has the same radius as the Archimedean circles. So does the incircle of the triangle $CO_1O_2$, where $C$ is the center of the incircle of the arbelos and $O_1$ and $O_2$ are the centers of the semicircles $C(AP)$ and $C(PB)$. The semi-perimeter and area of the triangle $CO_1O_2$ are given by

$$\frac{(x+y)^3}{x^2+xy+y^2}, \ \frac{xy(x+y)^2}{x^2+xy+y^2},$$

respectively. One can now define the mean of $x$ and $y$ to be the number $z$ such that the $(z,z)$-arbelos has a certain natural feature in common with the $(x,y)$-arbelos. If this common feature is taken to be the area of the arbelos or the radius of the Archimedean circle, we obtain the geometric mean and the harmonic means, respectively. If it is taken to be the radius of the incircle, the semi-perimeter or the area of the triangle $CO_1O_2$, then we obtain the three means given by

$$\frac{3xy(x+y)}{2(x^2+xy+y^2)}, \ \frac{3(x+y)^3}{8(x^2+xy+y^2)}, \ \sqrt{\frac{xy(x+y)^2}{x^2+xy+y^2}}, \tag{3}$$

respectively.

Of course, there is no end to this line of thinking. There are, beside the rectangle, the $\theta$-parallelograms, the trapezoid, the frustum, the arbelos, and many other geometric objects can be described by two positive numbers. Considering ellipses and lemniscates, for example, would lead to the means that involve non-elementary elliptic integrals and that are beyond this note.

We also remark that most of examples above conform to what is sometimes referred to in the literature as Chisini's definition of means. According to Chisini, *$x$ is said to be the mean of $n$ numbers $x_1, \cdots, x_n$, with respect to a problem in which a function of them $f(x_1, \cdots, x_n)$ is of interest, if the function assumes the same value when all the $x_h$ are replaced by the mean value $x$* : $f(x_1, \cdots, x_n) = f(x, \cdots, x)$; see [19, page 56] and [22]. Other possible viewpoints of means are reflected in [11], [23] and [24].

In the next sections, we will introduce families of means that encompass almost all the examples encountered above, and we will try to answer the questions that are most commonly asked about any family of means. These include questions of comparability, Gauss' compounding, behavior under equal increments of the variables and similar questions.

## 2. Conditions for internality

With the exception of the last mean in (3), each of the means $\mu(x,y)$ encountered above is of one of the forms

$$L(x,y) \pm \sqrt{Q(x,y)}, \ \frac{Q(x,y)}{L(x,y)}, \ \frac{C(x,y)}{Q(x,y)} \tag{4}$$

where $L(x,y), Q(x,y)$, and $C(x,y)$ are symmetric forms of degrees 1, 2 and 3 (respectively). However, for $\mu(x,y)$ to qualify as a mean, it is essential that $\mu(x,y)$ is *internal* in the sense that

$$\min\{x,y\} \leqslant \mu(x,y) \leqslant \max\{x,y\}$$

for all $x,y > 0$. (In fact, the literature on means seems to agree that internality is the only essential requirement for a function to be a mean; see [5, page 230].) The restrictions that internality places on the coefficients of $L$, $Q$, and $C$ are the subject of Theorem 1 below. Once internality is satisfied, our resulting means will also have the desirable properties of symmetry, continuity and 1-homogeneity.

We start with a few simple observations that we will freely use. It is easy to see that linear, quadratic and cubic symmetric forms in two variables are of the forms

$$L(x,y) = a(x+y), \quad Q(x,y) = b(x-y)^2 + cxy, \quad C(x,y) = (x+y)(b(x-y)^2 + cxy).$$

By considering the pairs $(x,y) = (1,0)$ and $(1,1)$, one sees that $Q(x,y) \geqslant 0$ for all $x,y \geqslant 0$ if and only if $b,c \geqslant 0$.

From these observations and making use of the property $\mu(1,1) = 1$ that is implied by internality, we see that the first type in (4) takes the form

$$\mathcal{M}_{a,b}(x,y) = (1+a)\frac{x+y}{2} - \operatorname{sgn}(a)\sqrt{b^2\left(\frac{x-y}{2}\right)^2 + a^2xy}$$

where sgn$(a)$ stands for the sign of $a$ if $a \neq 0$ and is allowed to take the two values $\pm 1$ if $a = 0$. If we agree that $a$ and $b$ have opposite signs, then $\mathcal{M}_{a,b}$ would take the form

$$\mathcal{M}_{a,b}(x,y) = (1+a)\frac{x+y}{2} - \operatorname{sgn}(a-b)\sqrt{b^2\left(\frac{x-y}{2}\right)^2 + a^2xy} \tag{5}$$

where it doesn't now matter what sgn$(0)$ is. The second type in (4) takes the form

$$\mathcal{G}_b(x,y) = \frac{b(x-y)^2 + 2xy}{x+y}, \ b \geqslant 0. \tag{6}$$

Finally, the third type in (4) is given by

$$\mu(x,y) = \frac{(x+y)\, Q_1(x,y)}{Q_2(x,y)},$$

where

$$Q_1(x,y) = b_1(x-y)^2 + c_1xy, \ \ Q_2(x,y) = b_2(x-y)^2 + c_2xy.$$

If $Q_1(x,y)$ and $Q_2(x,y)$ vanish at a common point $(x_0,y_0)$, $x_0,y_0 > 0$, then one is a multiple of the other. This is seen by considering the cases $x_0 = y_0$ and $x_0 \neq y_0$. Thus, if $\mu$ is to define a mean, it must be either the arithmetic mean or both $Q_1$ and $Q_2$ are positive for positive $x$ and $y$. Further, if $b_2 = 0$, then the behavior of $\mu$ near the point $(1,0)$ would imply $b_1 = 0$ and that $\mu$ is again the arithmetic mean. Thus one may assume that $b_2 = 1$. Therefore, $\mu$ takes the form $(x+y)Q_1(x,y)/2Q_2(x,y)$, with $b_2 = 1, c_1, c_2 \geqslant 0$ and $b_1 \geqslant 0$. The condition $\mu(1,1) = 1$ forces $c_1 = c_2$. We find it more convenient to write this in the equivalent form

$$\mathcal{N}_{b,c}(x,y) = \frac{x+y}{2}\,\frac{(1+b)(x-y)^2 + 2cxy}{(x-y)^2 + 2cxy}, \ b \geqslant -1, \ c \geqslant 0 \tag{7}$$

The arithmetic mean is not excluded as it corresponds to $b = 0$.

THEOREM 1. *Let $\mathcal{M}_{a,b}$, $\mathcal{G}_b$ and $\mathcal{N}_{b,c}$ be as given in (5), (6) and (7) above, and let*

$$\mathscr{D}_{\mathcal{M}} = \{(a,b) : ab \leqslant 0, \ -1 \leqslant a+b \leqslant 1\} \tag{8}$$

$$\mathscr{D}_{\mathcal{G}} = [0,1] \tag{9}$$

$$\mathscr{D}_{\mathcal{N}} = \{(b,c) : -1 \leqslant b \leqslant 1, \ c \geqslant 1\} \cup \{(b,c) : b^2 + (c-1)^2 \leqslant 1\} \tag{10}$$

*(i) $\mathcal{M}_{a,b}$ is internal if and only if $(a,b) \in \mathscr{D}_{\mathcal{M}}$.*
*(ii) $\mathcal{G}_b$ is internal if and only if $b \in \mathscr{D}_{\mathcal{G}}$.*
*(iii) $\mathcal{N}_{b,c}$ is internal if and only if $(b,c) \in \mathscr{D}_{\mathcal{N}}$.*
*(iv) With the exception of*

$$\mathcal{M}_{0,1}(a,b) = \max\{a,b\} \text{ and } \mathcal{M}_{0,-1}(a,b) = \min\{a,b\},$$

*all the means $\mu$ above are strict in the sense that*

$$\min\{a,b\} < \mu(a,b) < \max\{a,b\} \qquad \forall\, a,b > 0 \text{ with } a \neq b.$$

*Proof.*

$(i)$ We first note that if $\mathcal{M}_{a,b}$ is internal, then by multiplying

$$\min\{x,y\} \leqslant \mathcal{M}_{a,b}(x,y) \leqslant \max\{x,y\}$$

by $-1$ and adding $x+y$, we obtain

$$\max\{x,y\} \geqslant x+y-\mathcal{M}_{a,b}(x,y) \geqslant \min\{x,y\}.$$

Since $x+y-\mathcal{M}_{a,b}(x,y)$ is nothing but $\mathcal{M}_{-a,b}$, it follows that $\mathcal{M}_{-a,b}$ is internal. Also, if $\mathcal{M}_{a,b}$ is internal for all $a \neq 0$, then internality for $a=0$ would follow by taking limits since the pointwise limit of a sequence of internal functions is clearly internal. Therefore, it is enough to establish internality for $a<0$.

Thus assume $a<0$. The internality of $\mathcal{M}_{a,b}$ at the pair $(1,0)$ is equivalent to the condition $0 \leqslant (1+a+b)/2 \leqslant 1$, which simplifies into the desired condition

$$-1 \leqslant a+b \leqslant 1 \tag{11}$$

Conversely, suppose that (11) holds. Then

$$-1-a \leqslant b \leqslant 1-a \;\; (\text{and } a<0,\ b \geqslant 0) \tag{12}$$

From homogeneity and symmetry of $\mathcal{M}_{a,b}$, it is enough to establish internality for $x=1$ and $y \geqslant 1$. Thus we are to show that

$$y-a-ay-1 \;\geqslant\; \sqrt{b^2(y-1)^2+4a^2y} \quad \forall\, y \geqslant 1 \tag{13}$$

$$1-a-ay-y \;\leqslant\; \sqrt{b^2(y-1)^2+4a^2y} \quad \forall\, y \geqslant 1 \tag{14}$$

From $a<0$, we know that $y-a-ay-1 = -a(y+1)+(y-1) > 0$. Therefore

$$\begin{aligned}
(13) &\iff (y-a-ay-1)^2 - b^2(y-1)^2 - 4a^2y \geqslant 0 \quad \forall\, y \geqslant 1\\
&\iff (y-1)\left(((1-a)^2-s^2)y + (b^2-(a+1)^2)\right) \geqslant 0 \quad \forall\, y \geqslant 1\\
&\iff f(y) := ((1-a)^2-b^2)y + (b^2-(a+1)^2) \geqslant 0 \quad \forall\, y \geqslant 1.
\end{aligned}$$

But the last statement follows from (12) since $f(1) = -4a$ and $f'(1) = (1-a)^2-b^2$ are both non-negative.

To prove (14), it is enough to restrict ourselves to the values of $y$ for which $1-a-ay-y \geqslant 0$. This happens if $(a \leqslant -1)$ or $(a > -1$ and $y \leqslant (1-a)/(1+a)$. Under these assumptions,

$$\begin{aligned}
(14) &\iff b^2(y-1)^2 + 4a^2y - (1-a-ay-y)^2 \geqslant 0 \quad \forall\, y \geqslant 1\\
&\iff (y-1)((b^2-(1+a)^2)y - (b^2-(1-a)^2)) \geqslant 0 \quad \forall\, y \geqslant 1\\
&\iff g(y) := (b^2-(1+a)^2)y + ((1-a)^2-b^2) \geqslant 0 \quad \forall\, y \geqslant 1.
\end{aligned}$$

If $a \leqslant -1$, then it follows from (12) that both $b^2-(a+1)^2$ and $(1-a)^2-b^2$ are non-negative and the last statement follows. It remains to prove the last statement when

$$0 > a > -1 \text{ and } 1 \leqslant y \leqslant \frac{1-a}{1+a}.$$

In this case, $1-a^2$ and hence $b^2+1-a^2$ are both non-negative, as well as $1+a$. Thus

$$g(1) = -4a \geqslant 0 \quad \text{and} \quad g\left(\frac{1-a}{1+a}\right) = \frac{-2a(b^2+1-a^2)}{1+a} \geqslant 0,$$

and it follows that $g(y) \geqslant 0$ for all $y$ such that $1 \leqslant y \leqslant (1-a)/(1+a)$, as desired. This establishes the internality of $\mathcal{M}_{a,b}$ for $a<0$, and hence for all $a$.

$(ii)$ The internality of $\mathcal{G}_b$ at $(1,0)$ implies that $0 \leqslant b \leqslant 1$. Conversely, suppose that $0 \leqslant b \leqslant 1$. Since $\mathcal{G}_b$ increases with $b$, and since $\mathcal{G}_0$ and $\mathcal{G}_1$ are internal, being nothing but the well-known harmonic and Gini means, it follows that $\mathcal{G}_b$ is internal for all $b \in [0,1]$.

$(iii)$ By symmetry and homogeneity, the internality of $\mathcal{N}_{b,c}$ is equivalent to the condition that $1 \leqslant \mathcal{N}_{b,c}(1,y) \leqslant y \ \forall\, y \geqslant 1$. This simplifies into the conditions

$$F(y) := (1-b)y^2 + 2(c-1)y + (1+b) \geqslant 0 \quad \forall\, y \geqslant 1 \tag{15}$$

$$G(y) := (1+b)y^2 + 2(c-1)y + (1-b) \geqslant 0 \quad \forall\, y \geqslant 1 \tag{16}$$

Clearly (15) and (16) imply that

$$-1 \leqslant b \leqslant 1.$$

Also, if $\Delta$ denotes the discriminant, then $\Delta(F) = \Delta(G) = 4(b^2+(c-1)^2-1)$ and both (15) and (16) vacuously hold if $(b,c)$ lies inside the circle $b^2+(c-1)^2 \leqslant 1$ of the $(b,c)$-plane. Thus we assume that $b^2+(c-1)^2 > 1$. Then (15) is equivalent to saying that the larger root of $F$ is $\leqslant 1$, i.e.,

$$\sqrt{b^2+(c-1)^2-1} \leqslant c-b.$$

This in turn is equivalent to $(c-b \geqslant 0 \text{ and } (1-b)c \geqslant 0)$ and hence simply to the condition

$$c-b \geqslant 0.$$

Similarly, (16) is equivalent to the condition

$$c+b \geqslant 0.$$

Therefore $\mathcal{N}_{b,c}$ is internal if and only if $(b,c)$ lies in the strip $-1 \leqslant b \leqslant 1$ and above the lower half of the circle $b^2+(c-1)^2=1$, as desired.

$(iv)$ This easily follows by examining the proofs above. □

Note that the means $\beta_s$ and $\alpha_s$ that we have introduced in (1) are nothing but $\mathcal{M}_{-1,1/s}$ and $\mathcal{M}_{-s/(1+s),1/(1+s)}$. Using Theorem 1 $(i)$ above, one sees that $\beta_s$ is internal if and only if $s \geqslant 1/2$ (i.e., $\theta \geqslant 60^\circ$), while $\alpha_s$ is internal for all $\theta$.

## 3. Questions of comparability

In the family $\mathcal{G} = \{\mathcal{G}_b : 0 \leqslant b \leqslant 1\}$, comparability is trivial since $\mathcal{G}_b$ increases with $b$. In Theorems 2 and 3, we answer the questions of comparability in the families $\mathcal{M} = \{\mathcal{M}_{a,b} : (a,b) \in \mathcal{D}_{\mathcal{M}}\}$ and $\mathcal{N} = \{\mathcal{N}_{b,c} : (b,c) \in \mathcal{D}_{\mathcal{N}}\}$, where $\mathcal{D}_{\mathcal{M}}$ and $\mathcal{D}_{\mathcal{N}}$ are the domains of internality of these families, as given in (8) and (10). We will refer to means in $\mathcal{M}$, $\mathcal{G}$ and $\mathcal{N}$ as root-quadratic, rational-quadratic and rational-cubic means (respectively).

We start with comparability within $\mathcal{M}$. For $(a,b),(A,B) \in \mathscr{D}_{\mathcal{M}}$, we set

$$d = \frac{a^2 - b^2}{a^2} \quad \text{if} \quad a \neq 0; \quad D = \frac{A^2 - B^2}{A^2} \quad \text{if} \quad A \neq 0. \tag{17}$$

We will study conditions on $(a,b),(A,B) \in \mathscr{D}_{\mathcal{M}}$ under which $\mathcal{M}_{A,B} = \mathcal{M}_{a,b}$ and conditions under which $\mathcal{M}_{A,B} \geqslant \mathcal{M}_{a,b}$ for all $x, y \geqslant 0$. Under the harmless assumptions

$$\frac{x+y}{2} = 1, \quad \frac{x-y}{2} = t, \quad -1 \leqslant t \leqslant 1\,,$$

the mean $\mathcal{M}_{a,b}$ defined in (5) can be re-written in the form

$$\begin{aligned} g_{a,b}(t) &= 1 + a - \operatorname{sgn}(a-c)\sqrt{a^2 + (b^2 - a^2)t^2} \\ &= \begin{cases} 1 + b\,|\,t\,| & \text{if} \quad a = 0 \\ 1 + a - a\sqrt{1 - dt^2} & \text{if} \quad a \neq 0. \end{cases} \end{aligned}$$

Its MacLaurin series is given by

$$g_{a,b}(t) = 1 + \frac{1}{2}adt^2 + \frac{1}{8}ad^2t^4 + O(t^6)\,, \ \text{ if } a \neq 0. \tag{18}$$

We first note that $g_{a,b}$ is linear on $t \in [0,1]$ if and only if $a = 0$ or $a + b = 0$, and that $g_{a,b} = 1$ for all $t$ (i.e., $\mathcal{M}_{a,b}$ is the arithmetic mean) if and only if $a + b = 0$.

Suppose now that $g_{a,b} = g_{A,B}$. We neglect the trivial case when $a + b = 0$. Also, if $a = 0$, then $g_{a,b}$ is linear (for $t \in [0,1]$) and hence $g_{A,B}$ is linear and $A = 0$ and $b = B$. Otherwise, using the MacLaurin's series (18), we see that $ad = AD$ and $ad^2 = AD^2$ and therefore $a = A$ and $d = D$ (and hence $b = B$). Thus we conclude that $g_{a,b} = g_{A,B}$ if and only if $(a,b) = (A,B)$ or $a + b = A + B = 0$ in which case we have the arithmetic mean.

Next, taking the three cases $(a = A = 0)$, $(a = 0, A \neq 0)$ and $(a \neq 0, A \neq 0)$, it is easy to see that $g_{a,b}(t)$ and $g_{A,B}(t)$ intersect on $[0,1]$ at $t = 0$ and at most one other point given by

$$t^2 = \begin{cases} 4aA(A-a)(AD-ad)/(DA^2 - da^2)^2 & \text{if} \quad a = 0, A \neq 0 \\ (2bA)^2/(b^2 + DA^2)^2 & \text{if} \quad a \neq 0, A \neq 0. \end{cases}$$

Thus if $g_{A,B} \neq g_{a,b}$, then

$$\begin{aligned} g_{A,B}(t) \geqslant g_{a,b}(t) \forall\, t \in [-1,1] &\Longleftrightarrow g_{A,B}(t) \geqslant g_{a,b}(t) \ \ \forall\, t \in [0,1] \\ &\Longleftrightarrow g_{A,B}(0^+) > g_{a,b}(0^+) \quad \text{and} \quad g_{A,B}(1) \geqslant g_{a,b}(1). \end{aligned}$$

This settles the question of dominance in the family of root-quadratic means. We summarize the above in the following theorem.

THEOREM 2. *Let* $(a,b),(A,B) \in \mathscr{D}_{\mathcal{M}}$.

*(i)* $\mathcal{M}_{a,b}$ *is the arithmetic mean if and only if* $a + b = 0$.

*(ii) If* $a + b \neq 0$ *and if* $\mathcal{M}_{a,b} = \mathcal{M}_{A,B}$, *then* $(a,b) = (A,B)$.

*(iii) If* $\mathcal{M}_{A,B} \neq \mathcal{M}_{a,b}$, *then* $\mathcal{M}_{A,B} \geqslant \mathcal{M}_{a,b}$ *if and only if the following two conditions holds:*

$$(1) \quad g_{A,B}(1) \geqslant g_{a,b}(1) \qquad (2) \quad g_{A,B}(0^+) > g_{a,b}(0^+).$$

Note that $g_{A,B}(1) \geqslant g_{a,b}(1)$ if and only if

$$A + B \geqslant a + b$$

and that $g_{A,B}(0^+) > g_{a,b}(0^+)$ if and only if any of the following conditions hold :

$(i)$ $A = 0,\ B \geqslant 0$ and $A + B > a + b$

$(ii)$ $a = 0,\ b \leqslant 0$ and $A + B > a + b$

$(iii)$ $(AD > ad)$ or $(AD = ad$ and $AD^2 > ad^2)$, where $d$ and $D$ are as defined in (17).

THEOREM 3. *Let* $(b,c) \in \mathscr{D}_{\mathcal{N}}$.

$(i)$ $\mathcal{N}_{b,c}$ *is the arithmetic mean if and only if* $b = 0$.

$(ii)$ *If* $b \neq 0$, *then* $\mathcal{N}_{B,C} = \mathcal{N}_{b,c}$ *if and only if* $B = b$ *and* $C = c$.

$(iii)$ $\mathcal{N}_{B,C} \geqslant \mathcal{N}_{b,c}$ *if and only if* $B \geqslant b$ *and* $B - b + Bc - bC \geqslant 0$.

*Proof.*

$(i)$ is trivial. To prove $(ii)$ and $(iii)$, it is direct to check that

$$\mathcal{N}_{B,C}(1,y) - \mathcal{N}_{b,c}(1,y) = \frac{(y+1)(y-1)^2}{2} H(y),$$

where $H(y) = (B-b)y^2 + 2(Bc - bC)y + (B-b)$. Thus $\mathcal{N}_{B,C} \geqslant \mathcal{N}_{b,c}$ if and only if $H(y) \geqslant 0$ for all $y \geqslant 1$. That this is equivalent to the given conditions follows from considering the leading coefficient of $H$ and from the fact that $H(1) = H'(1) = 2(B - b + Bc - bC)$. $\square$

## 4. Gauss' compounding of root-quadratic and rational-quadratic means

Referring to the family $\mathcal{M}$ described in (5), we note that

$$\begin{aligned} \mathcal{M}_{-1,0} &= G, \quad \text{the geometric mean,} \\ \mathcal{M}_{0,0} &= A, \quad \text{the arithmetic mean,} \\ \mathcal{M}_{1,0} &= H, \quad \text{the Heronian mean,} \\ \mathcal{M}_{0,-1} &= \min, \quad \text{the minimum mean,} \\ \mathcal{M}_{0,1} &= \max, \quad \text{the maximum mean.} \end{aligned}$$

We describe below the arithmetic-geometric mean of Gauss obtained by iterating $A$ and $G$ and we will consider iterations of other pairs [7, pages 359–367], [13], [12].

Starting with any positive numbers $x_0 = x$ and $y_0 = y$, (with $x \leqslant y$, say), one considers the iteration given by

$$x_{n+1} = G(x_n, y_n)\ , \quad y_{n+1} = A(x_n, y_n).$$

It is clear that the sequence $x_n$ is increasing and that $y_n$ is decreasing and that

$$0 \leqslant y_{n+1} - x_{n+1} \leqslant \frac{1}{2}(y_n - x_n).$$

Therefore they converge to the same limit. This common limit is denoted by $(A \otimes G)(x,y)$ and is referred to in the literature as the *Gauss arithmetic-geometric* mean (or the *Gauss compound* of $A$ and $G$). It was discovered by Lagrange but it was Gauss who discovered its relation with certain elliptic integrals (and in particular with the length of the lemniscate) [13]. Since $(A \otimes G)(a,b)$ is transcendental whenever $a$ and $b$ are distinct positive algebraic numbers (e.g. when $(a,b) = (1, \sqrt{2})$) [12], it follows that $A \otimes G$ does not belong to the family $\mathcal{M}$ described above. Thus the question whether the compound of two elements in $\mathcal{M}$ belongs to $\mathcal{M}$ (when it exists) may turn out to be hard to answer. However, certain subfamilies of $\mathcal{M}$ can easily be seen to be closed under compounding. For example, compounds in the subfamily of $\mathcal{M}$ obtained by taking $a = 0$ can be easily computed using the theory of elementary difference equations. However, we shall make use of the following very useful theorem, which is a result of combining Theorems 8.2 and Theorem 8.3 of [5, pages 244-245], attention being made to the sentence that preceded Theorem 8.2.

THEOREM 4. *If $\mu$ and $\nu$ are symmetric means and if at least one of them is strict, then $\mu \otimes \nu$ exists and it is the unique mean $\Phi$ satisfying*

$$\Phi(\mu(x,y), \nu(x,y)) = \Phi(x,y) \quad \forall\, x,y > 0.$$

THEOREM 5. *Suppose that $-1 \leqslant b,c \leqslant 1$, and suppose that $(b,c)$ is neither of the trivial pairs $(-1,1)$ and $(1,-1)$. Then $\mathcal{M}_{0,b} \otimes \mathcal{M}_{0,c} = \mathcal{M}_{0,p}$, where*

$$p = \frac{b+c}{2- \mid b-c \mid}.$$

*In particular,*

$$\mathcal{M}_{0,b} \otimes \mathcal{M}_{0,-b} = \mathcal{M}_{0,0}, \text{ the arithmetic mean.}$$

*Proof.* Recalling that

$$\mathcal{M}_{0,b}(x,y) = \frac{x+y}{2} + \frac{b \mid x-y \mid}{2},$$

it is routine to check that

$$\mathcal{M}_{0,r}(\mathcal{M}_{0,c}, \mathcal{M}_{0,d}) = \mathcal{M}_{0,p}, \text{ where } p = \mathcal{M}_{0,r}(c,d) = \frac{c+d}{2} + r\frac{\mid c-d \mid}{2} \tag{19}$$

Solving $r = p$ and using Theorem 4, we get the desired result. □

Note that (19) above is nothing but [5, Exercise 3(b), page 253]. Also, the last statement in Theorem 5 above says that if $a = 0$ then $\mathcal{M}_{a,b} \otimes \mathcal{M}_{-a,-b}$ is the arithmetic mean. This is indeed true for all $a$ as the next theorem shows.

THEOREM 6. *If $(a,b) \in \mathscr{D}_{\mathcal{M}} - \{(0,1),(0,-1)\}$, then*

$$\mathcal{M}_{a,b} \otimes \mathcal{M}_{-a,-b} = \mathcal{M}_{0,0}, \quad \textit{the arithmetic mean.}$$

*Proof.* This follows again from Theorem 4 and the easily proved fact that

$$\mathcal{M}_{0,0}(\mathcal{M}_{a,b}(x,y),\mathcal{M}_{-a,-b}(x,y)) = \mathcal{M}_{0,0}(x,y)$$

□

THEOREM 7. *The only solutions of the functional equation*

$$\mathcal{M}_{-1,u}(\mathcal{M}_{-1,a},\mathcal{M}_{-1,b}) = \mathcal{M}_{-1,v}$$

*are*

$$(a=b=v) \text{ and } (u,v) = \left(\sqrt{2}, \sqrt{\frac{a^2+b^2}{2}}\right).$$

*Consequently,* $\mathcal{M}_{-1,a}\otimes\mathcal{M}_{-1,b} = \mathcal{M}_{-1,v}$ *if and only if* $v=\sqrt{2}$ *and* $a^2+b^2=4$.

*Proof.* Let $A=a^2, B=b^2, U=u^2, V=v^2$. If

$$Q(x,y) = (\mathcal{M}_{-1,u}(\mathcal{M}_{-1,a}(x,y),\mathcal{M}_{-1,b}(x,y)))^2 - (\mathcal{M}_{-1,v}(x,y))^2,$$

then direct calculations show that $16Q(x,y) = Q_1 - Q_2$, where

$$Q_1 = (AU+BU-4V)(x^2+y^2) - (2AU+2BU-8U-8V+16)xy$$
$$Q_2 = 2(U-2)\sqrt{(A(x-y)^2+4xy)(B(x-y)^2+4xy)}$$

and that

$$Q_1^2 - Q_2^2 = (x-y)^2(c_2(y^2+x^2)+c_1xy)$$

where

$$c_2 = U^2(A-B)^2 - 8U(AV+BV-2AB) + 16(V^2-AB)$$
$$c_1 = 32(U-2)(A+B-2V) - 2c_2$$

If $Q=0$, then $c_1=c_2=0$ and either $U=2$ or $V=(A+B)/2$. Substituting $U=2$ in $c_2$ results in $4(A+B-2V)^2$, and substituting $V=(A+B)/2$ in $c_2$ results in $(U-2)^2(A-B)^2$. Thus discarding the trivial case $A=B$, we see that $Q=0$ if and only if $U=2$ and $V=(A+B)/2$, as claimed. The last statement follows from solving $U=V$. □

Note that the special case $(a,b,v)=(0,2,\sqrt{2})$ of $\mathcal{M}_{-1,a}\otimes\mathcal{M}_{-1,b}=\mathcal{M}_{-1,v}$ is nothing but [5, Exercise 4(d), pages 253-254].

We now turn to compounding within $\mathcal{G}$. It follows from Theorem 4 above that the compound of the arithmetic and harmonic means is the geometric mean, i.e.,

$$\frac{x+y}{2}\otimes\frac{2xy}{x+y} = \sqrt{xy} \tag{20}$$

This simple fact is an elegant illustration of [5, Exercise 6, page 278] which we adapt as Theorem 8 for ease of reference.

THEOREM 8. *Let* $\mu$ *and* $\nu$ *be homogeneous symmetric rational means. If* $\mu\otimes\nu$ *is algebraic, then* $(\mu\otimes\nu)^j$ *is a rational function for some integer* $j$.

As an illustration to Theorem 8, the following two theorems explore the conditions on $\mu, \nu \in \mathcal{G}$ under which $\mu \otimes \nu \in \mathcal{G} \cup \mathcal{M}$.

THEOREM 9. *The only solutions of the functional equation $\mathcal{G}_a(\mathcal{G}_b, \mathcal{G}_c) = \mathcal{G}_t$ are*

$$(b = c = t) \text{ and } \left(b + c = 2t \text{ and } a = \frac{1}{2}\right).$$

*Consequently, $\mathcal{G}_b \otimes \mathcal{G}_c = \mathcal{G}_a$ if and only if $b + c = 2a = 1$.*

*Proof.* If $Q(x,y) = \mathcal{G}_a(\mathcal{G}_b(x,y), \mathcal{G}_c(x,y)) - \mathcal{G}_t(x,y)$, then direct calculations show that

$$\frac{(x+y)((b+c)(x-y)^2 + 4xy)Q(x,y)}{(x-y)^2} = c_1x^2 + c_2xy + c_1y^2$$

where

$$c_1 = a(b-c)^2 + 2bc - t(b+c), \quad c_2 = 2(b+c-2t) - 2c_1.$$

Therefore $c_1 = c_2 = 0$ if and only if $b + c = 2t$ and $(2a-1)(b-c)^2 = 0$. Thus either ($b = c = t$) or ($a = 1/2$ and $t = (b+c)/2$), as claimed. The last statement follows from solving $a = t$. $\square$

Note that the special case $(b,c,a) = (1/4, 3/4, 1/2)$ of $\mathcal{G}_b \otimes \mathcal{G}_c = \mathcal{G}_a$ is nothing but [5, Exercise 4(e), pages 253-254].

THEOREM 10. *The only solutions of the functional equation*

$$\mathcal{M}_{-1,u}(\mathcal{G}_b, \mathcal{G}_c) = \mathcal{M}_{-1,v}$$

*are*

$$u^2 = \frac{-(2b-1)(2c-1)}{(c-b)^2} \text{ and } v^2 = 2b + 2c - 1.$$

*Consequently, $\mathcal{G}_b \otimes \mathcal{G}_c = \mathcal{M}_{-1,u}$ if and only if*

$$2b^2 + 2c^2 + b + c - 4bc = 1 \text{ and } u^2 = 2b + 2c - 1.$$

*Proof.* If $Q(x,y) = (\mathcal{M}_{-1,u}(\mathcal{G}_b, \mathcal{G}_c))^2 - (\mathcal{M}_{-1,v})^2$, then direct calculations show that

$$Q(x,y) = \frac{(x-y)^2}{4(x+y)^2}(c_2y^2 + 2c_1xy + c_2x^2),$$

where

$$c_2 = u^2(b-c)^2 + 4bc - v^2, \quad c_1 = 4b + 4c - 2 - 2v^2 - c_2.$$

Therefore $c_1 = c_2 = 0$ if and only if $u$ and $v$ are as given above. The last statement follows from solving $u = v$. $\square$

The last statements of Theorems 10 and 9 can be viewed as sources of examples that illustrate Theorem 8. We restate them as follows. Let $\mathcal{P}$ be the part of the parabola defined by $2b^2 + 2c^2 + b + c - 4bc = 1$ (or equivalently by $(b+c) + 2(b-c)^2 = 1$) that lies in the first quadrant of the $(b,c)$-plane. This parabola has its vertex at $(1/2, 1/2)$, its axis on the line $b=c$ and it intercepts the $b-$ and $c-$ axes at $(1/2,0)$ and $(0,1/2)$. Let $\mathcal{L}$ be the part of its tangent line at the vertex $(1/2,1/2)$ that lies in the first quadrant; i.e., $\mathcal{L}$ is given by $a+b=1,\ 0\leqslant a,b\leqslant 1$. Then $\mathcal{G}_b\otimes\mathcal{G}_c=\mathcal{M}_{-1,\sqrt{2b+2c-1}}$ if $(b,c)\in\mathcal{P}$ and is the arithmetic mean $\mathcal{G}_{1/2}$ if $(b,c)\in\mathcal{L}$. In detail, we have

$$\frac{b(x-y)^2+2xy}{x+y}\otimes\frac{c(x-y)^2+2xy}{x+y} = \begin{cases}\sqrt{(2b+2c-1)((x-y)/2)^2+xy} & \text{if}\quad (b,c)\in\mathcal{P},\\ (x+y)/2 & \text{if}\quad (b,c)\in\mathcal{L},\end{cases} \tag{21}$$

of which (20) is the special case $(b,c)=(1/2,0)$. One wonders whether (21) (together with the trivial case $b=c$) covers all the algebraic means that are compounds of elements in $\mathcal{G}$.

We will not discuss compoundability in $\mathcal{N}$, as this may unduly increase the length of this article and as we expect this issue to be computationally more involved.

## 5. Behavior under equal increments of the variables

Given a mean $\mu$, the family $\{\mu_t : t\geqslant 0\}$ of means defined by

$$\mu_t(x,y)=\mu(x+t,y+t)-t$$

was introduced in [21] and was further investigated in [8], [2] and [3] where it was shown that

$$\lim_{t\longrightarrow\infty}\mu_t(x,y) = \frac{x+y}{2} \tag{22}$$

under very mild conditions on $\mu$. Our family $\{\mathcal{M}_{0,b} : -1\leqslant b\leqslant 1\}$ of root-quadratic means supplies examples of means that do not have the pleasant property (22). In fact, if $\mu=\mathcal{M}_{0,b}$, then it is obvious that

$$\mu_t(x,y)=\frac{x+y}{2}+\frac{b\mid x-y\mid}{2}\quad\forall\, t \tag{23}$$

On the other hand, if $\mu=\mathcal{M}_{a,b}$ and $a\neq 0$, then $\mu(x,y)$ is differentiable at $(x,y)=(1,1)$ and it follows from [2, Proposition 4] or [3, Proposition 2] that (22) holds. In this case, it is direct to see that $\mu_t$ increases or decreases, as a function of $t$, according as $a+b$ is less or greater than 0.

If $\mu=\mathcal{G}_b,\ b\in[0,1]$, then it is again easy to see that (22) holds and that $\mu_t$ increases or decreases according as $b$ is less or greater than 0.

If $\mu=\mathcal{N}_{b,c}$, then although (22) again holds, the convergence in (22) is not necessarily monotone. In fact, direct calculations show that

$$(\mu_t-\mu_s)(1,y)=\frac{-b(t-s)(y-1)^2}{((y-1)^2+2c(1+t)(y+t))((y-1)^2+2c(1+s)(y+s))}H$$

where

$$H=(c-1)y^2+(2+ct+cs)y+(c+ct+cs+2cst-1).$$

If $c \geqslant 1$, then $H \geqslant 0$ for all $y \geqslant 0$ and $\mu_t$ increases or decreases with $t$ according as $b$ is less or greater than 0. However, if $c < 1$, then the sum of the roots of $H$ is positive and therefore at least one of the roots is positive. Also, $H$ cannot have a multiple root since its discriminant is 0 if and only if

$$c = \frac{-8(t+1)(s+1)}{(t-s)^2 - 4(t+1)(s+1)}.$$

The assumption $c > 0$ implies that $(t-s)^2 - 4(t+1)(s+1) < 0$ and the assumption that $c < 1$ would then lead to the contradiction $(t-s)^2 + 4(t+1)(s+1) < 0$. Thus $H$ has two distinct roots of which at least one is positive. Therefore $(\mu_t - \mu_s)(1,y)$ changes sign as $y$ crosses that positive root, and $\mu_s$ and $\mu_t$ are not comparable.

We summarize all of this in the following theorem, in which the subscripts of $\mathcal{G}$, $\mathcal{M}$ and $\mathcal{N}$ are understood to take values in the respective domains.

THEOREM 11.

*(i) If* $\mu = \mathcal{G}_b$, *then* (22) *holds and the convergence is monotone increasing or decreasing according as* $b < 0$ *or* $b > 0$.

*(ii) If* $\mu = \mathcal{M}_{0,b}$, *then* (22) *does not hold, and* (23) *holds instead. If* $\mu = \mathcal{M}_{a,b}$, $a \neq 0$, *then* (22) *holds and the convergence is monotone increasing or decreasing according as* $a+b < 0$ *or* $a+b > 0$.

*(iii) If* $\mu = \mathcal{N}_{b,c}$, *then* (22) *holds. If* $c \geqslant 1$, *then the convergence in* (22) *is monotone increasing or decreasing according as* $b < 0$ *or* $b > 0$. *If* $c < 1$ *and* $b \neq 0$, *then no two elements in the family* $\{\mu_t : t \geqslant 0\}$ *are comparable.*

## 6. Concluding remarks

It is worth remarking that for root-quadratic and rational-quadratic means, internality at the two test pairs $(1,1)$ and $(1,0)$ was sufficient to ensure internality at all $(x,y)$ with $x,y \geqslant 0$. This is not so for rational-cubic means where it is easy to see that if $C(x,y)$ and $Q(x,y)$ are symmetric forms of degrees 3 and 2, then $\mu(x,y) := C(x,y)/Q(x,y)$ is internal at the test pairs $(1,1)$ and $(1,0)$ if and only if $\mu(x,y)$ has the form

$$\frac{x+y}{2}\,\frac{(1+b)(x-y)^2 + 2cxy}{(x-y)^2 + 2cxy}$$

with $-1 \leqslant b \leqslant 1$ and $c \geqslant 0$. However, Theorem 1 says that the extra condition that $(b,c)$ do not lie below the circle $b^2 + (c-1)^2 = 1$ is needed to guarantee internality of $\mu(x,y)$ for all $x,y \geqslant 0$. It would be interesting to develop methods by which a minimal number of test pairs can be generated for each given family. In this regard, we mention a similar situation in the theory of positive semi-definite symmetric forms, where a cubic such form $C(x_1,\cdots,x_n)$ is known to take non-negative values for all $x_i \geqslant 0$ if and only if it does so on the test tuples whose coordinates consist of 0's and 1's [10], while the same does not hold for quartics even when the number of variables is 3; see [25] for a symmetric quartic in three variables that is positive at each of the points $(1,1,1)$, $(1,1,0)$, $(1,0,0)$ and $(1,3,5)$ but negative at $(1,1,3)$.

We also remark that according to Theorem 8, the compound $\mu \otimes \nu$ of two rational means (if it exists) is either transcendental or else a $j$-th root of a rational function for some $j$. Although Theorems 9 and 10 (as exhibited in (21)) provide an illustration of this fact, (21) does not claim to give all algebraic means that arise as compounds of rational-quadratic means. For this to be achieved, it seems natural to first find an upper bound on the number $j$ for which $(\mu \otimes \nu)^j$ is rational in terms of the degrees of $\mu$ and $\nu$.

Regarding (22), the means discussed in the literature all seem to have the property that convergence in (22) is monotone; see [2], [3], [8], [18] and [21]. On the other hand, the means $\mu = \mathcal{N}_{b,c}$, $c < 1$, provide opposite extreme examples where no two elements of $\{\mu_t : t \geqslant 0\}$ are comparable. It would be interesting to explore necessary and sufficient conditions on a given mean $\mu$ under which $\mu_t$ is monotone (with $t$), and conditions under which no two elements of the family $\{\mu_t : t \geqslant 0\}$ are comparable.

Finally, we mention that the questions raised in this article with regards to means of two positive numbers have their natural analogues for multi-dimensional means. Answering these questions for several variables involves more technicalities and gives rise to subtle challenges that are currently being taken up by the authors. In particular, a generalization of Theorem 1 $(i)$ to all dimensions has already appeared in [1].

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*Raghib M. Abu-Saris*
*Department of Basic Sciences*
*University of Sharjah*
*P. O. Box 27272, Sharjah*
*United Arab Emirates*
*e-mail:* rabusaris@sharjah.ac.ae

*Mowaffaq Hajja*
*Department of Mathematics*
*Yarmouk University*
*Irbid*
*Jordan*
*e-mail:* mhajja@yu.edu.jo