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WEIGHTED INEQUALITIES FOR HARMONIC MEANS

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Abstract. We characterize the weighted weak and strong type (p,q) inequalities for the harmonic averaging operator

$$Tf(x) = \frac{x}{\int_0^x \frac{1}{f}}$$

in the cases $0 < p \leqslant q < \infty$ and $0 < q < p < \infty$.

1. Introduction

The heart of the theory of Hardy's inequalities is the characterization of the pairs of weights (w,v) such that

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f\right)^q w(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty f^p v\right)^{\frac{1}{p}} \tag{1.1}$$

holds for all $f \geqslant 0$.

This problem was solved by Bradley ([3]) in the case $1 \leqslant p \leqslant q < \infty$, by Maz'ja ([14]) in the case $1 \leqslant q < p < \infty$ and by Sinnamon ([20]) in the case $0 < q < 1, p > 1$.

Inequality (1.1) is the weighted strong type (p,q) inequality for the (arithmetic) averaging operator $Pf(x) = \frac{1}{x}\int_0^x f$.

The theory of Hardy's inequalities has developed following many directions (see [8] and [16]). Among all of them, we will fix our attention to the study of weighted inequalities for other kind of means different from the arithmetic one. In this sense, Heinig, Kerman and Krbec ([7]), Opic and Gurka ([15]) and Pick and Opic ([17]) have characterized the weighted strong type (p,q) inequalities for the geometric averaging operator $Gf(x) = \exp\left(\frac{1}{x}\int_0^x \log f\right)$. More recently, Jain, Persson and Wedestig ([9]) have studied weighted inequalities for power means $P_\alpha f(x) = \left(\frac{1}{x}\int_0^x f^\alpha\right)^{\frac{1}{\alpha}}$ with positive α.

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This paper deals with harmonic means. If f is a positive function defined on $(0,A)$ $(A \leqslant \infty)$, the harmonic mean of f in the interval $(0,x)$ is the number

$$Hf(x) = \frac{x}{\int_0^x \frac{1}{f}}.$$

Extensive information about harmonic means and their relationships with arithmetic and geometric means can be found in the books [4] and [5].

We will characterize the pairs of weights (w,v) such that

$$\left(\int_0^A Hf(x)^q w(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^A f^p v\right)^{\frac{1}{p}} \tag{1.2}$$

holds for all positive functions f in the cases $0 < p \leqslant q < \infty$ and $0 < q < p < \infty$.

We will also work with weighted weak type (p,q) inequalities, i.e., inequalities of the form

$$(w(\{x \in (0,A) : Hf(x) > \lambda\}))^{\frac{1}{q}} \leqslant \frac{C}{\lambda}\left(\int_0^A f^p v\right)^{\frac{1}{p}}, \tag{1.3}$$

which were studied for Hardy-type operators in [1], [11] and [12].

The techniques we are going to apply have worked successfully in the study of Hardy type inequalities ([11], [12]) and in weighted inequalities for other one-sided operators ([10], [13]). We will also use some ideas from Cruz-Uribe's, Neugebauer's and Olesen's paper about the one-sided minimal operator ([6]). In fact, this minimal operator is very closely related to the maximal operator of harmonic means.

The paper is organized as follows. Section 2 is devoted to the weak type inequalities. In section 3, we prove the theorems about the strong type inequalities. A final remark about power means P_α with negative exponents is also included.

All along the paper, the letter C will design a positive constant, not necessarily the same at each occurrence, and w, v will design positive measurable functions on $(0,A)$. Finally, we will assume the conventions $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$.

2. Weak type inequalities

The weighted weak type inequality (1.3) is equivalent to

$$\left(w(\{x \in (0,A) : \frac{1}{x}\int_0^x f < \frac{1}{\lambda}\})\right)^{\frac{1}{q}} \leqslant \frac{C}{\lambda}\left(\int_0^A \frac{v}{f^p}\right)^{\frac{1}{p}}.$$

In fact, we will work in a more general context. We will consider a positive function g and characterize the inequality

$$\left(w(\{x \in (0,A) : g(x)\int_0^x f < \frac{1}{\lambda}\})\right)^{\frac{1}{q}} \leqslant \frac{C}{\lambda}\left(\int_0^A \frac{v}{f^p}\right)^{\frac{1}{p}}. \tag{2.1}$$

In the statements of the theorems, we refer to the spaces $L^{s,\infty}(w)$ and their norms. By $L^{s,\infty}(w)$ we denote the set of the functions f such that $\|f\|_{s,\infty;w} < \infty$, where

$$\|f\|_{s,\infty;w} = \sup_{\lambda>0} \lambda \left(\int_{\{x\in(0,A):|f(x)|>\lambda\}} w \right)^{\frac{1}{s}}.$$

Our first theorem deals with the case $0 < p \leqslant q < \infty$.

THEOREM 1. *Let* $0 < p \leqslant q < \infty$ *and let* g *be a positive function. Then there exists* $C > 0$ *such that* (2.1) *holds for all positive functions* f *on* $(0,A)$ *if and only if*

$$D = \sup_{0<b<A} \|\frac{1}{g}\chi_{(0,b)}\|_{q,\infty;w} \left(\int_0^b v^{\frac{1}{p+1}} \right)^{-\frac{p+1}{p}} < \infty. \tag{2.2}$$

Proof. Suppose that condition (2.2) holds. Let f be a positive function, $\lambda > 0$ and $O_\lambda = \{x \in (0,A) : g(x)\int_0^x f < \frac{1}{\lambda}\}$. If $\int_0^A \frac{v}{f^p} = \infty$, there is nothing to prove. Suppose, therefore, that $\int_0^A \frac{v}{f^p} < \infty$. Suppose also that $\int_0^A v^{\frac{1}{p+1}} < \infty$. Let us consider the sequence $\{x_i\}$ defined by $x_0 = A$ and $\int_0^{x_{i+1}} v^{\frac{1}{p+1}} = \int_{x_{i+1}}^{x_i} v^{\frac{1}{p+1}}$. Let $E_i = (x_{i+1}, x_i) \cap O_\lambda$. Then, if $x \in E_i$ we have

$$\frac{1}{\lambda} > g(x)\int_0^x f > g(x)\int_0^{x_{i+1}} f > g(x)\int_{x_{i+2}}^{x_{i+1}} f.$$

Therefore, $\frac{1}{\lambda} \geqslant (\sup_{z\in E_i} g(z))\int_{x_{i+2}}^{x_{i+1}} f$. Now, Hölder inequality with exponents $p+1$ and $\frac{p+1}{p}$ gives

$$\frac{1}{\lambda} \geqslant (\sup_{z\in E_i} g(z)) \left(\int_{x_{i+2}}^{x_{i+1}} v^{\frac{1}{p+1}} \right)^{\frac{p+1}{p}} \left(\int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p} \right)^{\frac{-1}{p}},$$

or equivalently,

$$1 \leqslant \frac{1}{\lambda}(\inf_{z\in E_i} \frac{1}{g(z)}) \left(\int_{x_{i+2}}^{x_{i+1}} v^{\frac{1}{p+1}} \right)^{-\frac{p+1}{p}} \left(\int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p} \right)^{\frac{1}{p}}.$$

Multiplying both sides of the above inequality by $\left(\int_{E_i} w\right)^{\frac{1}{q}}$ and taking into account the definition of the sequence $\{x_i\}$, we obtain

$$\begin{aligned}
\left(\int_{E_i} w\right)^{\frac{1}{q}} &\leqslant \frac{1}{\lambda}(\inf_{z\in E_i} \frac{1}{g(z)}) \left(\int_{E_i} w\right)^{\frac{1}{q}} \left(\int_{x_{i+2}}^{x_{i+1}} v^{\frac{1}{p+1}} \right)^{-\frac{p+1}{p}} \left(\int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p} \right)^{\frac{1}{p}} \\
&\leqslant \frac{1}{\lambda} \|\frac{1}{g}\chi_{(x_{i+1},x_i)}\|_{q,\infty;w} \left(\int_{x_{i+2}}^{x_{i+1}} v^{\frac{1}{p+1}} \right)^{-\frac{p+1}{p}} \left(\int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p} \right)^{\frac{1}{p}} \\
&\leqslant \frac{C}{\lambda} \|\frac{1}{g}\chi_{(0,x_i)}\|_{q,\infty;w} \left(\int_0^{x_i} v^{\frac{1}{p+1}} \right)^{-\frac{p+1}{p}} \left(\int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p} \right)^{\frac{1}{p}}.
\end{aligned}$$

Applying condition (2.2) and raising to the q-th power, we have

$$\begin{aligned}\int_{E_i} w &\leqslant \frac{C}{\lambda^q}\|\frac{1}{g}\chi_{(0,x_i)}\|^q_{q,\infty;w}\left(\int_0^{x_i} v^{\frac{1}{p+1}}\right)^{-\frac{(p+1)q}{p}}\left(\int_{x_{i+2}}^{x_{i+1}}\frac{v}{f^p}\right)^{\frac{q}{p}}\\ &\leqslant \frac{C}{\lambda^q}\left(\int_{x_{i+2}}^{x_{i+1}}\frac{v}{f^p}\right)^{\frac{q}{p}}.\end{aligned}$$

Summing up in i and taking into account that $p\leqslant q$, we obtain the weak type inequality.

Let us suppose now that $\int_0^A v^{\frac{1}{p+1}} = \infty$. If $\int_0^x v^{\frac{1}{p+1}} = \infty$ for all $x\in(0,A)$, then $\int_0^x f = \infty$ for all x, since $\int_0^x v^{\frac{1}{p+1}} \leqslant \left(\int_0^x \frac{v}{f^p}\right)^{\frac{1}{p+1}}\left(\int_0^x f\right)^{\frac{p}{p+1}}$. This implies that $O_\lambda = \emptyset$ for all $\lambda>0$ and the weak type inequality is trivial.

If $\int_0^{x_0} v^{\frac{1}{p+1}} < \infty$ for some $x_0\in(0,A)$, then there exists $A_0\in(0,A]$ such that $\int_0^x v^{\frac{1}{p+1}} < \infty$ for all $x\in(0,A_0)$ and $\int_0^x v^{\frac{1}{p+1}} = \infty$ for all $x>A_0$. Then $\int_0^x f = \infty$ if $x>A_0$ and

$$\{x\in(0,A): g(x)\int_0^x f < \frac{1}{\lambda}\} = \{x\in(0,A_0): g(x)\int_0^x f < \frac{1}{\lambda}\}.$$

Let $\{a_n\}$ be a strictly increasing sequence with limit A_0. Since condition (2.2) holds in $(0,a_n)$ for all n and $\int_0^{a_n} v^{\frac{1}{p+1}} < \infty$, we have

$$\left(w(\{x\in(0,a_n): g(x)\int_0^x f < \frac{1}{\lambda}\})\right)^{\frac{1}{q}} \leqslant \frac{C}{\lambda}\left(\int_0^{a_n}\frac{v}{f^p}\right)^{\frac{1}{p}}$$

for all n. Then the monotone convergence theorem gives us the weak type inequality (2.1).

Conversely, suppose that the weak type inequality (2.1) holds. Let b with $0<b<A$, $f=\frac{v^{\frac{1}{p+1}}}{\chi_{(0,b)}}$ and $\alpha>0$. If $z\in(0,b)$ and $\frac{1}{g(z)}>\alpha$, then

$$g(z)\int_0^z f \leqslant g(z)\int_0^b f = g(z)\int_0^b v^{\frac{1}{p+1}} < \frac{1}{\alpha}\int_0^b v^{\frac{1}{p+1}}.$$

This shows that

$$\{z\in(0,b): \frac{1}{g(z)}>\alpha\} \subset \{z\in(0,A): g(z)\int_0^z f < \frac{1}{\alpha}\int_0^b v^{\frac{1}{p+1}}\}.$$

Then, by the weak type inequality,

$$\begin{aligned}\left(\int_{\{z\in(0,b):\frac{1}{g(z)}>\alpha\}} w\right)^{\frac{1}{q}} &\leqslant \left(\int_{\{z\in(0,A):g(z)\int_0^z f<\frac{1}{\alpha}\int_0^b v^{\frac{1}{p+1}}\}} w\right)^{\frac{1}{q}}\\ &\leqslant \frac{C}{\alpha}\left(\int_0^b v^{\frac{1}{p+1}}\right)^{\frac{p+1}{p}}.\end{aligned}$$

Since this inequality holds for all positive α, we have

$$\|\frac{1}{g}\chi_{(0,b)}\|_{q,\infty;w} \leqslant C\left(\int_0^b v^{\frac{1}{p+1}}\right)^{\frac{p+1}{p}},$$

and we are done.

The next theorem characterizes (2.1) in the case $0 < q < p < \infty$. In this case we will suppose that g is monotone.

THEOREM 2. *Let* $0 < q < p < \infty$ *and* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. *Let* g *be a positive monotone function. Then, there exists* $C > 0$ *such that inequality* (2.1) *holds for all positive functions* f *on* $(0,A)$ *if and only if the function* Φ *defined on* $(0,A)$ *by*

$$\Phi(x) = \sup_{0<a<x<b<A} \frac{1}{\sup\limits_{y\in(a,b)} g(y)} \left(\int_a^b w\right)^{\frac{1}{p}} \left(\int_0^b v^{\frac{1}{p+1}}\right)^{-\frac{p+1}{p}}$$

belongs to $L^{r,\infty}(w)$.

Proof. First let suppose that inequality (2.1) holds. We have to show that

$$\sup_{\lambda>0} \lambda \left(\int_{\{x\in(0,A):\Phi(x)>\lambda\}} w\right)^{\frac{1}{r}} < \infty.$$

Let $\lambda > 0$ and $S_\lambda = \{x \in (0,A) : \Phi(x) > \lambda\}$. If $z \in S_\lambda$, then there exist a_z, b_z with $0 < a_z < z < b_z < A$ such that

$$\frac{1}{\sup\limits_{y\in(a_z,b_z)} g(y)} \left(\int_{a_z}^{b_z} w\right)^{\frac{1}{p}} \left(\int_0^{b_z} v^{\frac{1}{p+1}}\right)^{-\frac{p+1}{p}} > \lambda. \tag{2.3}$$

Let K be a compact subset of S_λ. Then there exist (a_{z_1}, b_{z_1}), (a_{z_2}, b_{z_2}),..., (a_{z_n}, b_{z_n}) which cover K. We may suppose $\sum \chi_{(a_{z_j},b_{z_j})} \leqslant 2\chi_{\cup(a_{z_j},b_{z_j})}$. Let f be the function defined by

$$f(x) = \left(\sum_{j=1}^n \chi_{(0,b_{z_j})}(x)(\sup_{y\in(a_{z_j},b_{z_j})} g(y))^p \left(\int_0^{b_{z_j}} v^{\frac{1}{p+1}}\right)^p v^{-\frac{p}{p+1}}(x)\right)^{\frac{-1}{p}}.$$

If $z \in (a_{z_j}, b_{z_j})$, then

$$g(z)\int_0^z f \leqslant g(z)\int_0^{b_{z_j}} f \leqslant g(z)(\sup_{y\in(a_{z_j},b_{z_j})} g(y))^{-1} \left(\int_0^{b_{z_j}} v^{\frac{1}{p+1}}\right)^{-1} \int_0^{b_{z_j}} v^{\frac{1}{p+1}} \leqslant 1.$$

This shows that $\cup(a_{z_j}, b_{z_j}) \subset \{x \in (0,A) : g(x)\int_0^x f \leqslant 1\}$. Then, (2.1) gives

$$\int_{\cup_{j=1}^n (a_{z_j}, b_{z_j})} w \leqslant C\left(\int_0^A \frac{v}{f^p}\right)^{\frac{q}{p}} = C\left(\sum_{j=1}^n \big(\sup_{y\in(a_{z_j},b_{z_j})} g(y)\big)^p \left(\int_0^{b_{z_j}} v^{\frac{1}{p+1}}\right)^{p+1}\right)^{\frac{q}{p}} \tag{2.4}$$

Now, since (2.3) holds, (2.4) is less than

$$C\left(\sum_{j=1}^n \frac{1}{\lambda^p}\int_{a_{z_j}}^{b_{z_j}} w\right)^{\frac{q}{p}} = \frac{C}{\lambda^q}\left(\sum_{j=1}^n \int_{a_{z_j}}^{b_{z_j}} w\right)^{\frac{q}{p}} \leqslant \frac{C}{\lambda^q}\left(\int_{\cup_{j=1}^n (a_{z_j}, b_{z_j})} w\right)^{\frac{q}{p}}.$$

We have proved

$$\int_{\cup_{j=1}^n (a_{z_j}, b_{z_j})} w \leqslant \frac{C}{\lambda^q}\left(\int_{\cup_{j=1}^n (a_{z_j}, b_{z_j})} w\right)^{\frac{q}{p}},$$

which is equivalent to

$$\lambda\left(\int_{\cup_{j=1}^n (a_{z_j}, b_{z_j})} w\right)^{\frac{1}{r}} \leqslant C.$$

Since $K \subset \cup_{j=1}^n (a_{z_j}, b_{z_j})$, we have

$$\lambda\left(\int_K w\right)^{\frac{1}{r}} \leqslant C.$$

This inequality holds for all compact subsets K of S_λ and all $\lambda > 0$. Then

$$\sup_{\lambda>0} \lambda\left(\int_{S_\lambda} w\right)^{\frac{1}{r}} \leqslant C,$$

which means that $\Phi \in L^{r,\infty}(w)$.

Conversely, let us suppose that $\Phi \in L^{r,\infty}(w)$. Let $f > 0$ such that $\int_0^A \frac{v}{f^p} = 1$, let $\lambda > 0$ and $O_\lambda = \{x \in (0,A) : g(x)\int_0^x f < \frac{1}{\lambda}\}$. Then

$$\int_{O_\lambda} w = \int_{O_\lambda \cap \{x\in(0,A):\Phi(x)>\lambda^{\frac{q}{r}}\}} w + \int_{O_\lambda \cap \{x\in(0,A):\Phi(x)\leqslant\lambda^{\frac{q}{r}}\}} w. \tag{2.5}$$

The first summand in (2.5) is, by the definition of $\|.\|_{r,\infty;w}$, less than $\frac{\|\Phi\|^r_{r,\infty;w}}{\lambda^q}$. In order to estimate the second summand in (2.5), we may suppose that $\int_0^A v^{\frac{1}{p+1}} < \infty$. Let us consider the sequence $\{x_i\}$ defined by $x_0 = A$ and $\int_0^{x_{i+1}} v^{\frac{1}{p+1}} = \int_{x_{i+1}}^{x_i} v^{\frac{1}{p+1}}$. Let $E_i = (x_{i+1}, x_i) \cap O_\lambda \cap \{x \in (0,A) : \Phi(x) \leqslant \lambda^{\frac{q}{r}}\}$, $\alpha_i = \inf E_i$ and $\beta_i = \sup E_i$. Suppose that $\int_{E_i} w > 0$. Then, if $x \in E_i$ we have

$$\frac{1}{\lambda} > g(x)\int_0^x f > g(x)\int_0^{x_{i+1}} f > g(x)\int_{x_{i+2}}^{x_{i+1}} f.$$

Therefore, $\frac{1}{\lambda} \geqslant (\sup_{z\in E_i} g(z)) \int_{x_{i+2}}^{x_{i+1}} f$, and this inequality implies, by the monotonicity of g , that

$$\frac{1}{\lambda} \geqslant (\sup_{z\in(\alpha_i,\beta_i)} g(z)) \int_{x_{i+2}}^{x_{i+1}} f. \tag{2.6}$$

Then, (2.6), Hölder inequality with exponents $p+1$ and $\frac{p+1}{p}$ and the choice of the sequence $\{x_i\}$ yield

$$\begin{aligned}\frac{1}{\lambda} &\geqslant (\sup_{z\in(\alpha_i,\beta_i)} g(z)) \left(\int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p}\right)^{\frac{-1}{p}} \left(\int_{x_{i+2}}^{x_{i+1}} v^{\frac{1}{p+1}}\right)^{\frac{p+1}{p}} \\ &= \frac{1}{4^{\frac{p+1}{p}}}(\sup_{z\in(\alpha_i,\beta_i)} g(z)) \left(\int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p}\right)^{\frac{-1}{p}} \left(\int_0^{x_i} v^{\frac{1}{p+1}}\right)^{\frac{p+1}{p}},\end{aligned}$$

which is equivalent to

$$1 \leqslant \frac{C}{\lambda} \frac{1}{\sup_{z\in(\alpha_i,\beta_i)} g(z)} \left(\int_0^{x_i} v^{\frac{1}{p+1}}\right)^{-\frac{p+1}{p}} \left(\int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p}\right)^{\frac{1}{p}}.$$

Multiplying both sides of the above inequality by $\left(\int_{\alpha_i}^{\beta_i} w\right)^{\frac{1}{p}}$ and taking into account that $\beta_i \leqslant x_i$, we obtain

$$\left(\int_{\alpha_i}^{\beta_i} w\right)^{\frac{1}{p}} \leqslant \frac{C}{\lambda \sup_{z\in(\alpha_i,\beta_i)} g(z)} \frac{\left(\int_{\alpha_i}^{\beta_i} w\right)^{\frac{1}{p}}}{\left(\int_0^{\beta_i} v^{\frac{1}{p+1}}\right)^{\frac{p+1}{p}}} \left(\int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p}\right)^{\frac{1}{p}}. \tag{2.7}$$

Now, we observe that

$$\frac{1}{\sup_{z\in(\alpha_i,\beta_i)} g(z)} \left(\int_{\alpha_i}^{\beta_i} w\right)^{\frac{1}{p}} \left(\int_0^{\beta_i} v^{\frac{1}{p+1}}\right)^{-\frac{p+1}{p}} \leqslant \Phi(y)$$

for almost every $y \in E_i$, and since $\Phi(y) \leqslant \lambda^{\frac{q}{r}}$ for all $y \in E_i$, we have

$$\frac{1}{\sup_{z\in(\alpha_i,\beta_i)} g(z)} \left(\int_{\alpha_i}^{\beta_i} w\right)^{\frac{1}{p}} \left(\int_0^{\beta_i} v^{\frac{1}{p+1}}\right)^{-\frac{p+1}{p}} \leqslant \lambda^{\frac{q}{r}}.$$

Therefore, from (2.7) we obtain that

$$\int_{\alpha_i}^{\beta_i} w \leqslant \frac{C}{\lambda^q} \int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p}.$$

Since $E_i \subset (\alpha_i, \beta_i)$ and $\int_0^A \frac{v}{f^p} = 1$, we have

$$\int_{E_i} w \leqslant \frac{C}{\lambda^q}.$$

It is clear that this inequality also holds if $\int_{E_i} w = 0$. Then, summing up in i we obtain

$$\int_{O_\lambda \cap \{x\in(0,A):\Psi(x)\leqslant \lambda^{\frac{q}{r}}\}} w \leqslant \frac{C}{\lambda^q}$$

and we are done.

3. Strong type inequalities

The strong type inequality (1.2) is equivalent to

$$\left(\int_0^A \left(\frac{x}{\int_0^x f}\right)^q w(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^A \frac{v}{f^p}\right)^{\frac{1}{p}}. \tag{3.1}$$

The following theorem characterizes (3.1) in the case $0 < p \leqslant q < \infty$.

THEOREM 3. *Let* $0 < p \leqslant q < \infty$. *Then there exists* $C > 0$ *such that* (3.1) *holds for all positive functions* f *on* $(0,A)$ *if and only if*

$$B = \sup_{0<b<A} \left(\int_0^b x^q w(x)dx\right)^{\frac{1}{q}} \left(\int_0^b v^{\frac{1}{p+1}}\right)^{-\frac{p+1}{p}} < \infty. \tag{3.2}$$

Proof. Let us suppose that condition (3.2) holds. Let f be a positive function. As we have seen in theorems 1 and 2, we may suppose that $\int_0^A v^{\frac{1}{p+1}} < \infty$. Let $\{x_i\}$ be the decreasing sequence defined by $x_0 = A$ and $\int_0^{x_{i+1}} v^{\frac{1}{p+1}} = \int_{x_{i+1}}^{x_i} v^{\frac{1}{p+1}}$. Then, by Hölder inequality with exponents $p+1$ and $\frac{p+1}{p}$, the choice of the sequence $\{x_i\}$ and condition (3.2), we have

$$\begin{aligned}
\int_0^A \left(\frac{x}{\int_0^x f}\right)^q w(x)dx &= \sum_{i=0}^{\infty} \int_{x_{i+1}}^{x_i} \left(\frac{x}{\int_0^x f}\right)^q w(x)dx \\
&\leqslant \sum_{i=0}^{\infty} \int_{x_{i+1}}^{x_i} \left(\frac{x}{\int_0^{x_{i+1}} f}\right)^q w(x)dx \leqslant \sum_{i=0}^{\infty} \left(\int_0^{x_i} x^q w(x)dx\right)\left(\frac{1}{\int_{x_{i+2}}^{x_{i+1}} f}\right)^q \\
&\leqslant \sum_{i=0}^{\infty} \left(\int_0^{x_i} x^q w(x)dx\right)\left(\int_{x_{i+2}}^{x_{i+1}} v^{\frac{1}{p+1}}\right)^{-\frac{(p+1)q}{p}} \left(\int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p}\right)^{\frac{q}{p}} \\
&= 4^{\frac{(p+1)q}{p}} \sum_{i=0}^{\infty} \left(\int_0^{x_i} x^q w(x)dx\right) \left(\int_0^{x_i} v^{\frac{1}{p+1}}\right)^{-\frac{(p+1)q}{p}} \left(\int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p}\right)^{\frac{q}{p}} \\
&\leqslant CB^q \sum_{i=0}^{\infty} \left(\int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p}\right)^{\frac{q}{p}} \leqslant C\left(\int_0^A \frac{v}{f^p}\right)^{\frac{q}{p}}.
\end{aligned}$$

Conversely, let us suppose that inequality (3.1) holds. Let b be a positive number with $0<b<A$ and let $f=\frac{v^{\frac{1}{p+1}}}{\chi_{(0,b)}}$. On one hand we have

$$\begin{aligned}\left(\int_0^A\left(\frac{x}{\int_0^x f}\right)^q w(x)dx\right)^{\frac{1}{q}} &\geqslant \left(\int_0^b\left(\frac{x}{\int_0^x f}\right)^q w(x)dx\right)^{\frac{1}{q}}\\ &\geqslant \left(\int_0^b\left(\frac{x}{\int_0^b v^{\frac{1}{p+1}}}\right)^q w(x)dx\right)^{\frac{1}{q}} \qquad (3.3)\\ &= \frac{\left(\int_0^b x^q w(x)dx\right)^{\frac{1}{q}}}{\int_0^b v^{\frac{1}{p+1}}}.\end{aligned}$$

On the other hand,

$$\left(\int_0^A \frac{v}{f^p}\right)^{\frac{1}{p}} = \left(\int_0^b v^{\frac{1}{p+1}}\right)^{\frac{1}{p}}. \qquad (3.4)$$

Finally, inequality (3.1) together with (3.3) and (3.4) give

$$\left(\int_0^b x^q w(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^b v^{\frac{1}{p+1}}\right)^{\frac{p+1}{p}},$$

which is condition (3.2).

Now we will deal with the case $0<q<p<\infty$. We will give three conditions, each one equivalent to (3.1). The first one is a Sawyer type condition ([19]), the second one is a Maz'ja type condition ([14]) and the third one is a new type of condition which has been recently studied by A. L. Bernardis, F. J. Martin-Reyes and P. Ortega [2] in relation to the Hardy operator with weights. We get the connection between some of the conditions by mean of an argument inspired in the paper [18] by Qinsheng Lai. The result is the following one:

THEOREM 4. *Let $p,\ q,\ r\in R$ with $0<q<p<\infty$ and $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$. The following statements are equivalent:*

(i) *There exists $C>0$ such that (3.1) holds for all positive functions f on $(0,A)$.*

(ii) *There exists a positive constant C such that*

$$\sum_{i=0}^{\infty}\left(\int_{x_{i+1}}^{x_i} x^q w(x)dx\right)^{\frac{r}{q}}\left(\int_0^{x_i} v^{\frac{1}{p+1}}\right)^{-\frac{(p+1)r}{p}} \leqslant C$$

for all decreasing sequences $\{x_i\}$ such that $x_0=A$ and $\lim_{i\to\infty} x_i=0$.

(iii) *The function Θ defined on $(0,A)$ by*

$$\Theta(x)=\left(\int_0^x t^q w(t)dt\right)^{\frac{1}{p}}\left(\int_0^x v^{\frac{1}{p+1}}\right)^{-\frac{p+1}{p}}$$

belongs to $L^r(x^q w(x))$.

(iv) *The function* Ψ *defined on* $(0,A)$ *by*

$$\Psi(x) = \sup_{0<x<b\leqslant A} \left(\int_0^b t^q w(t)dt\right)^{\frac{1}{p}} \left(\int_0^b v^{\frac{1}{p+1}}\right)^{-\frac{p+1}{p}}$$

belongs to $L^r(x^q w(x))$.

Proof.

$(i) \Rightarrow (ii)$

Let us suppose that inequality (3.1) holds. Let $\{x_i\}$ be a decreasing sequence with $x_0 = A$ and $\lim_{i\to\infty} x_i = 0$ and let $\{a_i\}$ be a sequence of positive numbers. Let f be the function defined by

$$f(x) = (\sum_{i=0}^{\infty} \chi_{(0,x_i)} a_i^p v^{-\frac{p}{p+1}})^{\frac{-1}{p}}.$$

Then, by (3.1),

$$\begin{aligned}
\left(\sum_{i=0}^{\infty} a_i^p \int_0^{x_i} v^{\frac{1}{p+1}}\right)^{\frac{1}{p}} &= \left(\int_0^A \frac{v}{f^p}\right)^{\frac{1}{p}} \\
&\geqslant C \left(\int_0^A \left(\frac{x}{\int_0^x f}\right)^q w(x)dx\right)^{\frac{1}{q}} \\
&= C \left(\sum_{i=0}^{\infty} \int_{x_{i+1}}^{x_i} \left(\frac{x}{\int_0^x f}\right)^q w(x)dx\right)^{\frac{1}{q}} \\
&\geqslant C \left(\sum_{i=0}^{\infty} \left(\int_{x_{i+1}}^{x_i} x^q w(x)dx\right) \frac{1}{\left(\int_0^{x_i} f\right)^q}\right)^{\frac{1}{q}} \\
&\geqslant C \left(\sum_{i=0}^{\infty} a_i^q \left(\int_{x_{i+1}}^{x_i} x^q w(x)dx\right) \left(\int_0^{x_i} v^{\frac{1}{p+1}}\right)^{-q}\right)^{\frac{1}{q}} \\
&= C \left(\sum_{i=0}^{\infty} \left[a_i^q \left(\int_0^{x_i} v^{\frac{1}{p+1}}\right)^{\frac{q}{p}}\right] \left(\int_{x_{i+1}}^{x_i} x^q w(x)dx\right) \left(\int_0^{x_i} v^{\frac{1}{p+1}}\right)^{\frac{-(p+1)q}{p}}\right)^{\frac{1}{q}}.
\end{aligned}$$

Since this inequality holds for all sequences of positive numbers $\{a_i\}$, we have that the sequence

$$\left\{\left(\int_{x_{i+1}}^{x_i} x^q w(x)dx\right) \left(\int_0^{x_i} v^{\frac{1}{p+1}}\right)^{\frac{-(p+1)q}{p}}\right\}$$

belongs to $l^{\frac{r}{q}}$ with norm bounded by C, the constant of the strong type inequality. This proves (ii).

$(ii) \Rightarrow (iii)$

We may suppose, without loss of generality, that $\int_0^A x^q w(x)dx < \infty$. Let $\{x_i\}$ be the sequence defined by $x_0 = A$ and

$$\int_0^{x_{i+1}} t^q w(t)dt = \int_{x_{i+1}}^{x_i} t^q w(t)dt.$$

It is clear that $\{x_i\}$ decreases and $\lim_{i\to\infty} x_i = 0$. Then, the definition of the sequence $\{x_i\}$ and condition (ii) give us

$$\begin{aligned}
\|\Theta\|^r_{r;x^q w(x)} &= \int_0^A \left(\int_0^x t^q w(t)dt\right)^{\frac{r}{p}} \left(\int_0^x v^{\frac{1}{p+1}}\right)^{-\frac{(p+1)r}{p}} x^q w(x)dx \\
&= \sum_{i=0}^{\infty} \int_{x_{i+1}}^{x_i} \left(\int_0^x t^q w(t)dt\right)^{\frac{r}{p}} \left(\int_0^x v^{\frac{1}{p+1}}\right)^{-\frac{(p+1)r}{p}} x^q w(x)dx \\
&\leqslant \sum_{i=0}^{\infty} \left(\int_0^{x_{i+1}} v^{\frac{1}{p+1}}\right)^{-\frac{(p+1)r}{p}} \int_{x_{i+1}}^{x_i} \left(\int_0^x t^q w(t)dt\right)^{\frac{r}{p}} x^q w(x)dx \\
&\leqslant \sum_{i=0}^{\infty} \left(\int_0^{x_{i+1}} v^{\frac{1}{p+1}}\right)^{-\frac{(p+1)r}{p}} \left(\int_0^{x_i} t^q w(t)dt\right)^{\frac{r}{q}} \\
&= 4^{\frac{r}{q}} \sum_{i=0}^{\infty} \left(\int_0^{x_{i+1}} v^{\frac{1}{p+1}}\right)^{-\frac{(p+1)r}{p}} \left(\int_{x_{i+2}}^{x_{i+1}} t^q w(t)dt\right)^{\frac{r}{q}} \leqslant C.
\end{aligned}$$

$(iii) \Rightarrow (iv)$

It is clear that $\Psi(x) \leqslant \Theta(x) + \tilde{\Psi}(x)$, where

$$\tilde{\Psi}(x) = \sup_{0<x<b\leqslant A} \left(\int_x^b t^q w(t)dt\right)^{\frac{1}{p}} \left(\int_0^b v^{\frac{1}{p+1}}\right)^{-\frac{p+1}{p}}.$$

Then, by (iii), it will suffice to show that $\tilde{\Psi} \in L^r(x^q w(x))$. In order to do this, we will see that

$$\int_{\{y\in(0,A):\tilde{\Psi}(y)>\lambda\}} x^q w(x)dx \leqslant 2\int_{\{y\in(0,A):\Theta(y)>\lambda\}} x^q w(x)dx$$

for all positive λ.

Since

$$\{y : \tilde{\Psi}(y) > \lambda\} \subset \{y : \Theta(y) > \lambda\} \cup \{y : \tilde{\Psi}(y) > \lambda \geqslant \Theta(y)\},$$

we have

$$\int_{\{y:\tilde{\Psi}(y)>\lambda\}} x^q w(x)dx \leqslant \int_{\{y:\Theta(y)>\lambda\}} x^q w(x)dx + \int_{\{y:\tilde{\Psi}(y)>\lambda\geqslant\Theta(y)\}} x^q w(x)dx.$$

Therefore, again by (iii), it suffices to show that if $E = \{y : \tilde{\Psi}(y) > \lambda \geqslant \Theta(y)\}$, then

$$\int_E x^q w(x)dx \leqslant \int_{\{y:\Theta(y)>\lambda\}} x^q w(x)dx. \tag{3.5}$$

Let $z \in E$. There exists $b \in (z,A]$ such that

$$\frac{\left(\int_0^z x^q w(x)dx\right)^{\frac{1}{p}}}{\left(\int_0^z v^{\frac{1}{p+1}}\right)^{\frac{p+1}{p}}} \leqslant \lambda < \frac{\left(\int_z^b x^q w(x)dx\right)^{\frac{1}{p}}}{\left(\int_0^b v^{\frac{1}{p+1}}\right)^{\frac{p+1}{p}}}.$$

These inequalities imply

$$\int_0^z x^q w(x)dx \leqslant \int_z^b x^q w(x)dx.$$

If $(z,b) \subset \{y : \Theta(y) > \lambda\}$, then

$$\int_0^z x^q w(x)dx \leqslant \int_{\{y:\Theta(y)>\lambda\}} x^q w(x)dx.$$

If $F = \{y \in (z,b) : \Theta(y) \leqslant \lambda\}$ is nonempty and $y \in F$, we have

$$\frac{\left(\int_0^y x^q w(x)dx\right)^{\frac{1}{p}}}{\left(\int_0^y v^{\frac{1}{p+1}}\right)^{\frac{p+1}{p}}} \leqslant \lambda < \frac{\left(\int_z^b x^q w(x)dx\right)^{\frac{1}{p}}}{\left(\int_0^b v^{\frac{1}{p+1}}\right)^{\frac{p+1}{p}}},$$

which implies

$$\int_0^y x^q w(x)dx \leqslant \int_z^b x^q w(x)dx.$$

It follows that

$$\int_0^z x^q w(x)dx \leqslant \int_y^b x^q w(x)dx.$$

Since the above inequality holds for all $y \in F$, calling $\beta = \sup F$, we obtain

$$\int_0^z x^q w(x)dx \leqslant \int_\beta^b x^q w(x)dx \leqslant \int_{\{y:\Theta(y)>\lambda\}} x^q w(x)dx.$$

Anyway, we have shown

$$\int_0^z x^q w(x)dx \leqslant \int_{\{y:\Theta(y)>\lambda\}} x^q w(x)dx.$$

Since this inequality holds for all $z \in E$, we get (3.5).

$(iv) \Rightarrow (i)$

Let f be a positive function and let $\{x_i\}$ be the sequence defined as in the previous theorems. Then, working as above

$$\int_0^A \left(\frac{x}{\int_0^x f}\right)^q w(x)dx = \sum_{i=0}^{\infty} \int_{x_{i+1}}^{x_i} \left(\frac{x}{\int_0^x f}\right)^q w(x)dx$$

$$= 4^{\frac{(p+1)q}{p}} \sum_{i=0}^{\infty} \left(\int_{x_{i+1}}^{x_i} x^q w(x)dx\right) \left(\int_0^{x_i} v^{\frac{1}{p+1}}\right)^{-\frac{(p+1)q}{p}} \left(\int_{x_{i+2}}^{x_{i+1}} \frac{v}{f^p}\right)^{\frac{q}{p}}.$$

If we apply now Hölder inequality for sums with exponents
$\frac{r}{q}$ and $\frac{p}{q}$ and then condition (iv), the last term is less than

$$
\begin{aligned}
& C\left(\sum_{i=0}^{\infty}\left(\int_{x_{i+1}}^{x_i} x^q w(x)dx\right)^{\frac{r}{q}}\left(\int_0^{x_i} v^{\frac{1}{p+1}}\right)^{-\frac{(p+1)r}{p}}\right)^{\frac{q}{r}}\left(\sum_{i=0}^{\infty}\int_{x_{i+2}}^{x_{i+1}}\frac{v}{f^p}\right)^{\frac{q}{p}} \\
& \quad = C\left(\sum_{i=0}^{\infty}\int_{x_{i+1}}^{x_i}\left(\int_0^{x_i} t^q w(t)dt\right)^{\frac{r}{p}}\left(\int_0^{x_i} v^{\frac{1}{p+1}}\right)^{-\frac{(p+1)r}{p}} x^q w(x)dx\right)^{\frac{q}{r}}\left(\int_0^A \frac{v}{f^p}\right)^{\frac{q}{p}} \\
& \quad \leqslant C\left(\sum_{i=0}^{\infty}\int_{x_{i+1}}^{x_i}\Psi(x)^r x^q w(x)dx\right)^{\frac{q}{r}}\left(\int_0^A \frac{v}{f^p}\right)^{\frac{q}{p}} \\
& \quad = C\|\Psi\|_{r;x^q w(x)}^q\left(\int_0^A \frac{v}{f^p}\right)^{\frac{q}{p}}.
\end{aligned}
$$

FINAL REMARK. It is clear that if $\alpha < 0$ and $P_\alpha f(x) = \left(\frac{1}{x}\int_0^x f^{\alpha}\right)^{\frac{1}{\alpha}}$, then $P_\alpha f(x) = (H(f^{|\alpha|})(x))^{\frac{1}{|\alpha|}}$, and, therefore, the characterizations of the weighted weak and strong type inequalities for P_α reduce easily to those of H.

REFERENCES

[1] K. F. ANDERSEN AND B. MUCKENHOUPT, *Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions*, Studia Math. **72** (1982), 9–26.

[2] A. L. BERNARDIS, F. J. MARTÍN-REYES AND P. ORTEGA SALVADOR, *A new proof of the characterization of the weighted Hardy inequality*, preprint (2004).

[3] J. BRADLEY, *Hardy inequalities with mixed norms*, Canad. Math. Bull. **21** (1978), 405–408.

[4] P. S. BULLEN, *Handbook of means and their inequalities*, Kluwer Academic Pub., 2003.

[5] P. S. BULLEN , D. S. MITRINOVIC AND P. M. VASIC (EDS .), *Means and Their inequalities*, D. Reidel Publishing Company, 1988.

[6] D. CRUZ-URIBE, S.F.O., C. J. NEUGEBAUER AND V. OLESEN, *The one-sided minimal operator and the one-sided reverse Hölder inequality*, Studia Math. **116** no. 3 (1995), 255–270.

[7] H. P. HEINIG, R. KERMAN AND M. KRBEC, *Weighted exponential inequalities*, Georgian Math. J. **8** no. 1 (2001), 69–86.

[8] A. KUFNER AND L. E. PERSSON, *Weighted inequalities of Hardy type*, World Scientific, 2003.

[9] P. JAIN, L.E. PERSSON AND A. WEDESTIG, *Carlemann-Knopp type inequalities via Hardy inequalities*, Mathematical Inequalities and Applications **4** no. 3 (2001), 343–356.

[10] F. J. MARTÍN-REYES, *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc. **117** (1993), 691–698.

[11] F. J. MARTÍN-REYES AND P. ORTEGA SALVADOR, *On weighted weak type inequalities for modified Hardy operators*, Proc. Amer. Math. Soc. **126** no. 6 (1998), 1739–1746.

[12] F. J. MARTÍN-REYES, P. ORTEGA SALVADOR AND M. D. SARRIÓN GAVILÁN, *Boundedness of operators of Hardy type in $\Lambda^{p,q}$ spaces and weighted mixed inequalities for singular integral operators*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), 157–170.

[13] F. J. MARTÍN-REYES AND A. DE LA TORRE, *Some weighted inequalities for general one-sided maximal operators*, Studia Math. **122** (1997), 1–14.

[14] V. G. MAZ'JA, *Sobolev Spaces*, Springer-Verlag Berlin Heidelberg New York Tokyo, 1985.

[15] B. OPIC AND P. GURKA, *Weighted inequalities for geometric means*, Proc. Amer. Math. Soc. **120** no. 3 (1994), 771–779.

[16] B. OPIC AND A. KUFNER, *Hardy-type inequalities*, Longman, 1990.
[17] L. PICK AND B. OPIC, *On the geometric mean operator*, J. Math. Anal. Appl. **183** (1994), 652–662.
[18] QINSHENG LAI, *Weighted modular inequalities for Hardy type operators*, Proc. London Math. Soc. **79** no. 3 (1999), 649–672.
[19] E. SAWYER, *Weighted Lebesgue and Lorentz norm inequaliti es for the Hardy operator*, Trans. Amer. Math. Soc. **281** (1984), 329–337.
[20] G. SINNAMON, *Operators on Lebesgue spaces with general measures*, Doctoral Thesis, McMaster University, 1987.

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