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GENERALIZATION AND SHARPNESS OF FINSLER–HADWIGER'S INEQUALITY AND ITS APPLICATIONS

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Abstract. In this paper, the following generalization and sharpness of Finsler-Hadwiger's inequality is established

$$\sum a^{\lambda} \geqslant 2^{\lambda} 3^{1-\frac{\lambda}{4}} F^{\frac{\lambda}{2}} + \sum |a-b|^{\lambda} + \sum_{n=1}^{m} \sum |a_n - b_n|^{\lambda}.$$

As consequence, an exponential Finsler-Hadwiger type inequality for tetrahedron is derived.

1. Introduction and main results

In what follows, for a given triangle ABC, we denote by a, b, c the lengths of its sides, F denotes the area of triangle ABC. We will customarily use the symbol of cyclic sum such as $\sum f(a) = f(a) + f(b) + f(c)$, $\sum f(a,b) = f(a,b) + f(b,c) + f(c,a)$.

In 1937, Finsler and Hadwiger presented an important geometric inequality related to the sides and the area of a triangle (see [1]), as follows

$$\sum a^2 \geqslant 4\sqrt{3}F + \sum (a-b)^2. \tag{1}$$

Inequality (1) is known in the literature as Finsler-Hadwiger's inequality. This celebrated inequality has motivated a large number of research papers involving its new proofs, various generalizations, analogues and applications etc.(see [2–4] and references therein). The purpose of this paper is to establish a new generalization and sharpness of Finsler-Hadwiger's inequality. In Section 2, the obtained result will be used to establish an exponential Finsler-Hadwiger type inequality for tetrahedron.

Our main result is stated in the following Theorem:

THEOREM 1. *Let m be a natural number and let λ be a real number, and $\lambda \geqslant 2$. Then for any triangle ABC we have*

$$\sum a^{\lambda} \geqslant 2^{\lambda} 3^{1-\frac{\lambda}{4}} F^{\frac{\lambda}{2}} + \sum |a-b|^{\lambda} + \sum_{n=1}^{m} \sum |a_n - b_n|^{\lambda}, \tag{2}$$

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where $a_n,\ b_n,\ c_n$ *are defined by* $a_0 = a,\ b_0 = b,\ c_0 = c$, *satisfying*

$$a_n = \sqrt{a_{n-1}(b_{n-1}+c_{n-1}-a_{n-1})}, \quad b_n = \sqrt{b_{n-1}(c_{n-1}+a_{n-1}-b_{n-1})},$$
$$c_n = \sqrt{c_{n-1}(a_{n-1}+b_{n-1}-c_{n-1})}, \quad n = 1,\ 2,\ 3, \cdots, m.$$

The equality holds in (2) *if and only if the triangle is equilateral.*

REMARK. The Lemma 3 below will reveals that $a_n,\ b_n,\ c_n\ (n = 1,\ 2,\ 3, \cdots, m)$ are exactly the lengths of sides of a triangle.

LEMMA 1. (Power means inequality [5,pp. 26]) *Let* $x_1, x_2, \cdots, x_n$ *be nonnegative numbers, and* $p \geqslant 1$. *Then*

$$\sum_{i=1}^{n} x_i^p \geqslant n^{1-p}\Big(\sum_{i=1}^{n} x_i\Big)^p, \tag{3}$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$ *or* $p = 1$.

$$(x_1 + x_2)^p \geqslant x_1^p + x_2^p, \tag{4}$$

with equality holding if and only if $x_1 = 0$ *or* $x_2 = 0$ *or* $p = 1$.

LEMMA 2. *Let* λ *be real numbers, and* $\lambda \geqslant 2$. *Then for any triangle* ABC *we have*

$$\sum a^\lambda \geqslant \sum |a-b|^\lambda + \sum \big[\sqrt{a(b+c-a)}\,\big]^\lambda, \tag{5}$$

with equality holding if and only if the triangle is equilateral, or $\lambda = 2$.

Proof. Using Lemma 1, it follows that

$$\begin{aligned}
\sum a^\lambda &= \sum \big[(b-c)^2 + (c+a-b)(a+b-c)\big]^{\frac{\lambda}{2}} \\
&\geqslant \sum \big[|b-c|^\lambda + (c+a-b)^{\frac{\lambda}{2}}(a+b-c)^{\frac{\lambda}{2}}\big] \\
&= \sum |b-c|^\lambda + \sum (c+a-b)^{\frac{\lambda}{2}}(a+b-c)^{\frac{\lambda}{2}} \\
&= \sum |b-c|^\lambda + \frac{1}{2}\sum (b+c-a)^{\frac{\lambda}{2}}\big[(c+a-b)^{\frac{\lambda}{2}} + (a+b-c)^{\frac{\lambda}{2}}\big] \\
&\geqslant \sum |b-c|^\lambda + 2^{-\frac{\lambda}{2}}\sum (b+c-a)^{\frac{\lambda}{2}}(c+a-b+a+b-c)^{\frac{\lambda}{2}} \\
&= \sum |a-b|^\lambda + \sum \big[\sqrt{a(b+c-a)}\,\big]^\lambda.
\end{aligned}$$

Thus

$$\sum a^\lambda \geqslant \sum |a-b|^\lambda + \sum \big[\sqrt{a(b+c-a)}\,\big]^\lambda.$$

The condition of the equality in (5) follows immediately from inequalities (3) and (4). The Lemma 2 is proved.

LEMMA 3. *Let* $a,\ b,\ c$ *be the lengths of sides of triangle* ABC, *and let*

$$a_1 = \sqrt{a(b+c-a)}, \quad b_1 = \sqrt{b(c+a-b)}, \quad c_1 = \sqrt{c(a+b-c)}. \tag{6}$$

Then there exists a triangle $A_1B_1C_1$ *with the lengths of sides* $a_1,\ b_1,\ c_1$, *and its area* $F_1 = F$.

Proof. From the assumption in (6), we have

$$\begin{aligned}(a_1+b_1)^2 &= a(b+c-a)+b(c+a-b)+2\sqrt{ab(b+c-a)(c+a-b)}\\ &> a(b+c-a)+b(c+a-b)\\ &= c(a+b-c)+(b+c-a)(c+a-b)\\ &> c(a+b-c)=c_1^2\,.\end{aligned}$$

This yields $a_1+b_1>c_1$. Similarly to the above, we can obtain $b_1+c_1>a_1$ and $c_1+a_1>b_1$. Consequently, there exists a triangle $A_1B_1C_1$ with the sides $a_1,\ b_1,\ c_1$. Now we calculate the area of the triangle $A_1B_1C_1$.

Utilizing the Heron's formula for area of triangle, we get that

$$\begin{aligned}16F_1^2 = 2\sum a_1^2b_1^2-\sum a_1^4 &= 2\sum ab(b+c-a)(c+a-b)-\sum[a(b+c-a)]^2\\ &= 2\sum a^2b^2-\sum a^4=16F^2,\end{aligned}$$

which leads to $F_1=F$. The proof of Lemma 3 is complete.

Proof of Theorem 1. Define a sequence $\{a_n,\ b_n,\ c_n\}$ by a, b, c, as follows

$$\begin{gathered}a_0=a,\ \ b_0=b,\ \ c_0=c,\\ a_n=\sqrt{a_{n-1}(b_{n-1}+c_{n-1}-a_{n-1})},\quad b_n=\sqrt{b_{n-1}(c_{n-1}+a_{n-1}-b_{n-1})},\\ c_n=\sqrt{c_{n-1}(a_{n-1}+b_{n-1}-c_{n-1})},\quad n=1,\ 2,\ 3,\cdots,m+1.\end{gathered}$$

By Lemma 3, we find that there exists a sequence of triangle $\{\triangle A_nB_nC_n\}$ $(n=1,\ 2,\ 3,\cdots,m+1)$ with the lengths of sides $a_n,\ b_n,\ c_n$ $(n=1,\ 2,\ 3,\cdots,m+1)$, and their areas satisfy that $F_n=F$ for $n=1,\ 2,\ 3,\cdots,m+1$.

Applying Lemma 2 together with the above definition, we have

$$\begin{aligned}\sum a^\lambda &\geqslant \sum|a-b|^\lambda+\sum a_1^\lambda\\ &\geqslant \sum|a-b|^\lambda+\sum|a_1-b_1|^\lambda+\sum a_2^\lambda\\ &\geqslant\cdots\geqslant\sum|a-b|^\lambda+\sum|a_1-b_1|^\lambda+\cdots+\sum|a_m-b_m|^\lambda+\sum a_{m+1}^\lambda.\end{aligned}$$

From Lemma 1 and the above inequality we obtain

$$\sum a^\lambda\geqslant\sum|a-b|^\lambda+\sum_{n=1}^{m}\sum|a_n-b_n|^\lambda+3^{1-\frac{\lambda}{2}}\left(\sum a_{m+1}^2\right)^{\frac{\lambda}{2}}.\tag{7}$$

On the other hand, in view of $a_{m+1},\ b_{m+1},\ c_{m+1}$ are the lengths of sides of triangle $A_{m+1}B_{m+1}C_{m+1}$, it is known that the following Weitzenböeck's inequality holds[1]:

$$\sum a_{m+1}^2\geqslant 4\sqrt{3}F_{m+1},\tag{8}$$

with the equality if and only if the triangle is equilateral.

Combining inequalities (7), (8) and the relation $F_{m+1}=F$, we deduce that

$$\sum a^\lambda\geqslant\sum|a-b|^\lambda+\sum_{n=1}^{m}\sum|a_n-b_n|^\lambda+2^\lambda 3^{1-\frac{\lambda}{4}}F^{\frac{\lambda}{2}}.\tag{9}$$

The condition of equality in (2) follows immediately from (8) and Lemma 2. This completes the proof of Theorem 1.

Putting $m = 1$ in Theorem 1, we obtain the following fresh inequality:

COROLLARY 1. *Let λ be real numbers, and $\lambda \geqslant 2$. Then for any triangle ABC we have*

$$\sum a^{\lambda} \geqslant 2^{\lambda} 3^{1-\frac{\lambda}{4}} F^{\frac{\lambda}{2}} + \sum |a-b|^{\lambda} + \sum \left| \sqrt{a(b+c-a)} - \sqrt{b(c+a-b)} \right|^{\lambda}. \tag{10}$$

In (10), the special case $\lambda = 2$ yields a remarkable sharpness of Finsler-Hadwiger's inequality, as follows

COROLLARY 2. *For any triangle ABC, the following inequality holds*

$$\sum a^2 \geqslant 4\sqrt{3}F + \sum (a-b)^2 + \sum [\sqrt{a(b+c-a)} - \sqrt{b(c+a-b)}\,]^2. \tag{11}$$

2. Application to the tetrahedron inequality

In this section we will establish an exponential Finsler-Hadwiger type inequality for tetrahedron by using the above results.

For a tetrahedron $A_1A_2A_3A_4$, we state that $A_1,\ A_2,\ A_3,\ A_4$ denote the vertexes of tetrahedron, a_1, a_2, a_3, a_4, a_5, a_6 denote the edge lengths of tetrahedron, V is the volume of tetrahedron.

THEOREM 2. *Let λ be real numbers, and $\lambda \geqslant 2$, Then for any tetrahedron $A_1A_2A_3A_4$ we have*

$$\sum_{i=1}^{6} a_i^{\lambda} \geqslant 2^{1+\frac{\lambda}{2}} 3^{1+\frac{\lambda}{3}} V^{\frac{\lambda}{3}} + (2+6^{\frac{\lambda}{2}-1})^{-1} \sum_{1 \leqslant i<j \leqslant 6} |a_i - a_j|^{\lambda}, \tag{12}$$

with equality holding if and only if the tetrahedron is regular.

Proof. In tetrahedron $A_1A_2A_3A_4$, without loss of generality we may assume that $a_1 = A_1A_2$, $a_2 = A_1A_3$, $a_3 = A_1A_4$, $a_4 = A_3A_4$, $a_5 = A_2A_4$, $a_6 = A_2A_3$, and F_1, F_2, F_3, F_4 denote respectively the areas of the faces $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$.

Applying Corollary 1 to triangle $A_2A_3A_4$, triangle $A_1A_3A_4$, triangle $A_1A_2A_4$ and triangle $A_1A_2A_3$ respectively, we obtain

$$a_4^{\lambda} + a_5^{\lambda} + a_6^{\lambda} \geqslant 2^{\lambda} 3^{1-\frac{\lambda}{4}} F_1^{\frac{\lambda}{2}} + |a_4 - a_5|^{\lambda} + |a_4 - a_6|^{\lambda} + |a_5 - a_6|^{\lambda},$$
$$a_2^{\lambda} + a_3^{\lambda} + a_4^{\lambda} \geqslant 2^{\lambda} 3^{1-\frac{\lambda}{4}} F_2^{\frac{\lambda}{2}} + |a_2 - a_3|^{\lambda} + |a_2 - a_4|^{\lambda} + |a_3 - a_4|^{\lambda},$$
$$a_1^{\lambda} + a_3^{\lambda} + a_5^{\lambda} \geqslant 2^{\lambda} 3^{1-\frac{\lambda}{4}} F_3^{\frac{\lambda}{2}} + |a_1 - a_3|^{\lambda} + |a_1 - a_5|^{\lambda} + |a_3 - a_5|^{\lambda},$$
$$a_1^{\lambda} + a_2^{\lambda} + a_6^{\lambda} \geqslant 2^{\lambda} 3^{1-\frac{\lambda}{4}} F_4^{\frac{\lambda}{2}} + |a_1 - a_2|^{\lambda} + |a_1 - a_6|^{\lambda} + |a_2 - a_6|^{\lambda}.$$

Summing both sides of the above inequalities respectively with a simple calculation, we get

$$\sum_{i=1}^{6} a_i^{\lambda} \geqslant 2^{\lambda-1} 3^{1-\frac{\lambda}{4}} \sum_{i=1}^{4} F_i^{\frac{\lambda}{2}} + \frac{1}{2}\left(-|a_1-a_4|^{\lambda} - |a_2-a_5|^{\lambda} - |a_3-a_6|^{\lambda} + \sum_{1\leqslant i<j\leqslant 6} |a_i-a_j|^{\lambda}\right). \tag{13}$$

On the other hand, from Lemma 1 we have

$$\begin{aligned}
&|a_4-a_5|^{\lambda} + |a_4-a_6|^{\lambda} + |a_5-a_6|^{\lambda} + |a_2-a_3|^{\lambda} + |a_2-a_4|^{\lambda} + |a_3-a_4|^{\lambda} \\
&+ |a_1-a_3|^{\lambda} + |a_1-a_5|^{\lambda} + |a_3-a_5|^{\lambda} + |a_1-a_2|^{\lambda} + |a_1-a_6|^{\lambda} + |a_2-a_6|^{\lambda} \\
\geqslant\ & 12^{1-\frac{\lambda}{2}} [(a_4-a_5)^2 + (a_4-a_6)^2 + (a_5-a_6)^2 + (a_2-a_3)^2 + (a_2-a_4)^2 + (a_3-a_4)^2 \\
&+ (a_1-a_3)^2 + (a_1-a_5)^2 + (a_3-a_5)^2 + (a_1-a_2)^2 + (a_1-a_6)^2 + (a_2-a_6)^2]^{\frac{\lambda}{2}} \\
=\ & 12^{1-\frac{\lambda}{2}} [2(a_1-a_4)^2 + 2(a_2-a_5)^2 + 2(a_3-a_6)^2 \\
&+ (a_1+a_4-a_2-a_5)^2 + (a_1+a_4-a_3-a_6)^2 + (a_2+a_5-a_3-a_6)^2]^{\frac{\lambda}{2}} \\
\geqslant\ & 12^{1-\frac{\lambda}{2}} 2^{\frac{\lambda}{2}} [(a_1-a_4)^2 + (a_2-a_5)^2 + (a_3-a_6)^2]^{\frac{\lambda}{2}} \\
\geqslant\ & 12^{1-\frac{\lambda}{2}} 2^{\frac{\lambda}{2}} \left(|a_1-a_4|^{\lambda} + |a_2-a_5|^{\lambda} + |a_3-a_6|^{\lambda}\right).
\end{aligned}$$

Further, the above inequality can be rewritten as

$$\sum_{1\leqslant i<j\leqslant 6} |a_i-a_j|^{\lambda} \geqslant \left(1 + 12^{1-\frac{\lambda}{2}} 2^{\frac{\lambda}{2}}\right)\left(|a_1-a_4|^{\lambda} + |a_2-a_5|^{\lambda} + |a_3-a_6|^{\lambda}\right),$$

that is

$$|a_1-a_4|^{\lambda} + |a_2-a_5|^{\lambda} + |a_3-a_6|^{\lambda} \leqslant \left(1 + 12^{1-\frac{\lambda}{2}} 2^{\frac{\lambda}{2}}\right)^{-1} \sum_{1\leqslant i<j\leqslant 6} |a_i-a_j|^{\lambda}. \tag{14}$$

We deduce from inequalities (13) and (14) that

$$\sum_{i=1}^{6} a_i^{\lambda} \geqslant 2^{\lambda-1} 3^{1-\frac{\lambda}{4}} \sum_{i=1}^{4} F_i^{\frac{\lambda}{2}} + \left(2 + 6^{\frac{\lambda}{2}-1}\right)^{-1} \sum_{1\leqslant i<j\leqslant 6} |a_i-a_j|^{\lambda}. \tag{15}$$

In addition, by the arithmetic-geometric mean inequality and the known inequality for tetrahedron [6]:

$$\prod_{i=1}^{4} F_i \geqslant \frac{81\sqrt[3]{9}}{16} V^{\frac{8}{3}}, \tag{16}$$

with the equality if and only if the tetrahedron is regular.

We obtain that

$$\sum_{i=1}^{4} F_i^{\frac{\lambda}{2}} \geqslant 4\left(\prod_{i=1}^{4} F_i\right)^{\frac{\lambda}{8}} \geqslant 2^{2-\frac{\lambda}{2}} 3^{\frac{7\lambda}{12}} V^{\frac{\lambda}{3}}. \tag{17}$$

Combining inequalities (15) and (17) leads to inequality (12) immediately. Moreover, Corollary 1 and inequality (16) show that the equality holds in (12) if and only if the tetrahedron is regular. The proof of Theorem 2 is complete.

In particular, putting $\lambda = 2$ in Theorem 2, an interesting Finsler-Hadwiger type inequality for tetrahedron is derived as follows

COROLLARY 3. *For any tetrahedron $A_1A_2A_3A_4$ we have*

$$\sum_{i=1}^{6} a_i^2 \geqslant 12\sqrt[3]{9}V^{\frac{2}{3}} + \frac{1}{3}\sum_{1\leqslant i<j\leqslant 6}(a_i - a_j)^2. \tag{18}$$

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