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ON THE REAL LINEAR POLARIZATION CONSTANT PROBLEM

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Abstract. The present paper deals with lower bounds for the norm of products of linear forms. It has been proved by J. Arias-de-Reyna [2], that the so-called n^{th} linear polarization constant $c_n(\mathbb{C}^n)$ is $n^{n/2}$, for arbitrary $n \in \mathbb{N}$. The same value for $c_n(\mathbb{R}^n)$ is only conjectured. In a recent work A. Pappas and S. Révész prove that $c_n(\mathbb{R}^n) = n^{n/2}$ for $n \leqslant 5$. Moreover, they show that if the linear forms are given as $f_j(x) = \langle x, a_j \rangle$, for some unit vectors a_j $(1 \leqslant j \leqslant n)$, then the product of the f_j's attains at least the value $n^{-n/2}$ at the normalized signed sum of the vectors $\{a_j\}_{j=1}^n$ having maximal length. Thus they asked whether this phenomenon remains true for arbitrary $n \in \mathbb{N}$. We show that for vector systems $\{a_j\}_{j=1}^n$ close to an orthonormal system, the Pappas-Révész estimate does hold true. Furthermore, among these vector systems the only system giving $n^{-n/2}$ as the norm of the product is the orthonormal system. On the other hand, for arbitrary vector systems we answer the question of A. Pappas and S. Révész in the negative when $n \in \mathbb{N}$ is large enough. We also discuss various further examples and counterexamples that may be instructive for further research towards the determination of $c_n(\mathbb{R}^n)$.

1. Introduction and notation

For convenience we recall the basic definitions needed to discuss polynomials on a Banach space. If $\mathbb{K}$ is the real or complex field and X is a Banach space over $\mathbb{K}$, then by B_X and S_X we denote the closed unit ball and the unit sphere of X respectively. A map $P : X \to \mathbb{K}$ is a (continuous) n*-homogeneous polynomial* if there is a (continuous) symmetric n-linear mapping $L : X^n \to \mathbb{K}$ for which $P(x) = L(x, \dots, x)$ for all $x \in X$. In this case it is convenient to write $P = \widehat{L}$. We let $\mathcal{P}(^nX)$ denote the space of scalar valued continuous n-homogeneous polynomials on X. We define the norm of a (continuous) homogeneous polynomial $P : X \to \mathbb{K}$ by

$$\|P\| = \sup\{|P(x)| : \ x \in B_X\}.$$

For general background on polynomials, we refer to [7].

If $P_k \in \mathcal{P}(^{n_k}X)$ $(1 \leqslant k \leqslant m)$ then the pointwise product of the P_k's given by $(P_1 \cdots P_m)(x) := P_1(x) \cdots P_m(x)$ for every $x \in X$ is also a homogeneous polynomial, in fact if $n = n_1 + \ldots + n_m$ we have $P_1 \cdot P_2 \cdots P_m \in \mathcal{P}(^nX)$. Clearly $\|P_1 \cdots P_m\| \leqslant$

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$\|P_1\|\cdots\|P_m\|$. An estimate in the other direction is more difficult to establish, but still possible. Indeed, there is an absolute constant $C_{n_1,\ldots,n_m}$ such that

$$\|P_1\|\cdot\|P_2\|\cdots\|P_m\| \leqslant C_{n_1,\ldots,n_m}\|P_1\cdot P_2\cdots P_m\|. \tag{1}$$

Products of polynomials and estimates like (1) have been studied by several authors. For a general account on this problem we recommend [6] and the references therein.

In this paper we will restrict ourselves to the case where the P_k's are continuous linear functionals $f_1,f_2,\ldots,f_n$ on X. Then the product $(f_1f_2\cdots f_n)(x) := f_1(x)f_2(x)\cdots f_n(x)$ is a continuous n-homogeneous polynomial on X and from (1) there exists $C_n > 0$ such that

$$\|f_1\|\|f_2\|\cdots\|f_n\| \leqslant C_n\|f_1f_2\cdots f_n\|\,. \tag{2}$$

Estimate (2) was also studied by R. A. Ryan and B. Turett in [18]. In [6] it was proved, in the case of *complex* Banach spaces, that $C_n \leqslant n^n$ and the constant n^n is best possible in general. However n^n can be improved for specific spaces. The best fitting constant in (2) was defined by C. Benítez, Y. Sarantopulos and A. Tonge in [6] as

$$c_n(X) := \inf\{M>0 : \|f_1\|\cdots\|f_n\| \leqslant M\cdot\|f_1\cdots f_n\|, \forall f_1,\ldots,f_n \in X^*\},$$

and in the literature it is referred to as the n^{th} linear polarization constant of X.

Let us represent the Hilbert space of the n-tuples of elements of $\mathbb{K}$ endowed with the Euclidean norm $\|\cdot\|_2$ by $\mathbb{K}^n$. Then, it has been proved by S. Révész and Y. Sarantopoulos [17] that

$$c_n(X) \geqslant c_n(\mathbb{K}^n), \quad \forall n\in\mathbb{N},$$

for any infinite dimensional Banach space X. This inequality shows the importance of $c_n(\mathbb{K}^n)$ when estimating $c_n(X)$, at least for infinite dimensional Banach spaces.

J. Arias-de-Reyna proved in [2] that $c_n(\mathbb{C}^n) = n^{\frac{n}{2}}$, however his proof does not adapt to the real case. It has been conjectured in [6] that $c_n(\mathbb{R}^n) = n^{\frac{n}{2}}$ also holds, but no proof has been given yet.

Observe that in order to determine $c_n(\mathbb{R}^n)$ one only needs to consider norm-one functionals $f_1,f_2,\ldots,f_n$ in (2). Therefore by the Riesz Representation Theorem a polynomial $P_n(x) := f_1(x)\cdots f_n(x)$, where $f_k \in S_{(\mathbb{R}^n)^*}$, $1\leqslant k\leqslant n$, can be written in the form

$$P_n(x) = \langle x,a_1\rangle\cdot\langle x,a_2\rangle\cdot\ldots\langle x,a_n\rangle\ , \tag{3}$$

where $a_j \in S_{(\mathbb{R}^n)^*} = S_{\mathbb{R}^n}$.

If $\mathcal{B}_n = \{e_j : 1\leqslant j\leqslant n\}$ is the canonical basis of $\mathbb{R}^n$ and we put $a_j = e_j$ for $1\leqslant j\leqslant n$ in (3), then for $x\in B_{\mathbb{R}^n}$ with coordinates $(x_1,\ldots,x_n)$, we have (using the fact that the geometric mean is smaller than the quadratic mean):

$$|P_n(x)| = |x_1\cdots x_n| \leqslant n^{-\frac{n}{2}}(x_1^2+\ldots+x_n^2)^{\frac{n}{2}} = n^{-\frac{n}{2}}\|x\|_2^n \leqslant n^{-\frac{n}{2}},$$

from which $\|P_n\|\leqslant n^{-\frac{n}{2}}$. In addition to this

$$|P_n(\frac{1}{\sqrt{n}},\ldots,\frac{1}{\sqrt{n}})| = n^{-\frac{n}{2}},$$

and therefore $\|P_n\| = n^{-\frac{n}{2}}$. This lets us state that $c_n(\mathbb{R}^n) \geqslant n^{\frac{n}{2}}$. In order to prove the reverse inequality one could try and find for every P_n as in (3) a norm one vector x in $\mathbb{R}^n$ satisfying $|P_n(x)| \geqslant n^{-\frac{n}{2}}$. This has been done in the real case for $n \leqslant 5$ by A. Pappas and S. Révész in [16], taking x as the normalization of the signed combination of the functional vectors $\{a_j\}_{j=1}^n$ with maximal length (see next section).

A. Pappas and S. Révész also asked whether their argument can be generalized for any dimension. We show in Section 4 that their argument fails in spaces of large enough dimension.

Another approach to the problem of estimating $c_n(\mathbb{R}^n)$ is using complexification arguments together with the result of J. Arias-de-Reyna in $\mathbb{C}^n$. In fact, this idea has been used by S. Révész and Y. Sarantopoulos in [17] to prove

$$c_n(\mathbb{R}^n) \leqslant 2^{\frac{n}{2}-1} n^{\frac{n}{2}}, \tag{4}$$

which is the best known estimate on $c_n(\mathbb{R}^n)$ (see also [15] for a complete account on results on complexifications of polynomials).

With the notation $\|P_n\|_{S_{\mathbb{K}^n}} := \sup\{|P_n(\xi)| : \xi \in S_{\mathbb{K}^n}\}$ one could aim to prove that $\|P_n\|_{S_{\mathbb{R}^n}} = \|P_n\|_{S_{\mathbb{C}^n}}$ for any $\{a_j\}_{j=1}^n \subset S_{\mathbb{R}^n}$. Then it would follow that

$$\|P_n\|_{S_{\mathbb{R}^n}} = \|P_n\|_{S_{\mathbb{C}^n}} \leqslant n^{-\frac{n}{2}},$$

from which $c_n(\mathbb{R}^n) = n^{\frac{n}{2}}$. However, it is shown in Section 5 that $\|P_n\|_{S_{\mathbb{R}^n}}$ does not necessarily coincide with $\|P_n\|_{S_{\mathbb{C}^n}}$. Moreover, we prove that using complexification arguments it is not possible to improve the estimate (4).

2. Mean vectors of maximal length

In the following we will refer to a real choice of signs ϵ as an n-tuple, $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, with $\epsilon_j = \pm 1$ $(1 \leqslant j \leqslant n)$. If $a_1, \ldots, a_n \in S_{\mathbb{R}^n}$ are n vectors in $\mathbb{R}^n$ we define

$$a_\epsilon := \sum_{i=1}^n \epsilon_i a_i. \tag{5}$$

If we select ϵ to maximize $\|a_\epsilon\|_2$ it can be easily shown that $\langle a_\epsilon, \epsilon_j a_j \rangle \geqslant 1$ for $1 \leqslant j \leqslant n$. Indeed, if we fix j $(1 \leqslant j \leqslant n)$ and ϵ' is the choice of signs given by $\epsilon'_k = \epsilon_k$ if $k \neq j$ and $\epsilon'_j = -\epsilon_j$ then

$$\|a_\epsilon\|_2^2 \geqslant \|a_{\epsilon'}\|_2^2 = \|a_\epsilon\|^2 + 4(1 - \langle a_\epsilon, \epsilon_j a_j \rangle),$$

from which $\langle a_\epsilon, \epsilon_j a_j \rangle \geqslant 1$ follows immediately. Note that if we replace the a_j's in (3) by $\epsilon_j a_j$, the norm of P_n does not change. Therefore, we can assume without loss of generality that the choice $\epsilon = (1, 1, \ldots, 1)$ gives the maximal length, and then, by the argument above we have

$$\begin{aligned}
y_1 &:= \langle a_1, a_1 \rangle + \langle a_1, a_2 \rangle + \ldots + \langle a_1, a_n \rangle \geqslant 1 \\
y_2 &:= \langle a_2, a_1 \rangle + \langle a_2, a_2 \rangle + \ldots + \langle a_2, a_n \rangle \geqslant 1 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
y_n &:= \langle a_n, a_1 \rangle + \langle a_n, a_2 \rangle + \ldots + \langle a_n, a_n \rangle \geqslant 1.
\end{aligned} \tag{6}$$

A. Pappas and S. Révész used in [16] this type of signed combinations. In particular they considered the normalized mean vector

$$x := \frac{a_\epsilon}{\|a_\epsilon\|_2} = \frac{a_1 + \cdots + a_n}{\|a_1 + \cdots + a_n\|_2}, \tag{7}$$

(with a_ϵ having maximal length), and they obtained the following:

THEOREM 1. (A. Pappas and S. Révész [16]) *Let* $n \leqslant 5$. *If* P_n *is as in* (3) *and* x *as in* (7) *then*

$$|P_n(x)| = \prod_{j=1}^{n} |\langle x , a_j \rangle| \geqslant n^{-n/2}. \tag{8}$$

REMARK 1. Let us note that (6) is a special case of a more general result, known as Bang's Lemma, see [5]. The argument for this special case occurs at several places, see eg. [10, Lemma 2.4.1 (i)], [12] and [16].

REMARK 2. We observe that signed combinations of vectors such as (5) have been used in many other constructions (see for instance [8] and [10]). In all cases the usual approach is to choose the combination of maximal length.

3. A Question of A. Pappas and S. Révész and its complex version

Estimate (8) entails the conjectured value $n^{n/2}$ of the polarization constant for dimensions $n \leqslant 5$. That is why the next question was posed by A. Pappas and S. Révész [16].

Question. (A. Pappas and S. Révész [16]). Is it true, that for any n and unit vectors $a_j \in \mathbb{R}^n$ $(1 \leqslant j \leqslant n)$, with signs of the unit vectors a_j $(1 \leqslant j \leqslant n)$ chosen to maximize the length of the signed sum (and thus satisfying (6), too), the mean value vector (7) satisfies $|P(x)| \geqslant n^{-n/2}$?

This question has a natural analogue in the complex case. Instead of ± 1 we consider complex choices of sign, which we define by n-tuples $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, where $\epsilon_j = e^{i\varphi_j}$ for some $\varphi_j \in [0, 2\pi)$ $(1 \leqslant j \leqslant n)$. In this setting we consider complex vectors $a_j \in S_{\mathbb{C}^n}$ $(j = 1, \ldots, n)$ and in this case a_ϵ is defined as

$$a_\epsilon := \sum_{j=1}^{n} \epsilon_j a_j = \sum_{j=1}^{n} e^{i\varphi_j} a_j.$$

If we choose $e^{i\varphi_j}$ $(1 \leqslant j \leqslant n)$ so as to maximize the norm of a_ϵ, as in the real case it can be easily shown that

$$\langle a_\epsilon, \epsilon_j a_j \rangle = \langle \sum_{k=1}^{n} e^{i\varphi_k} a_j, e^{i\varphi_j} a_j \rangle \geqslant 1,$$

for $1 \leqslant j \leqslant n$. Indeed, taking $1 \leqslant j \leqslant n$, $u := \sum_{k \neq j} e^{i\varphi_k} a_k$, $v := e^{i\varphi_j} a_j$ and $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\langle u, \lambda v \rangle = |\langle u, v \rangle|$, we have

$$\|u+v\|_2^2 \geqslant \|u+\lambda v\|_2^2 \Leftrightarrow \|u\|_2^2 + \|v\|_2^2 + 2\mathrm{Re}\,(\langle u, v \rangle) \geqslant \|u\|_2^2 + \|v\|_2^2 + 2\mathrm{Re}\,(\langle u, \lambda v \rangle)$$
$$\Leftrightarrow \mathrm{Re}\,(\langle u, v \rangle) \geqslant \mathrm{Re}\,(\langle u, \lambda v \rangle) = |\langle u, v \rangle|.$$

Therefore $\operatorname{Im}(\langle u,v\rangle)=0$ and $\langle u,v\rangle = |\langle u,v\rangle| \geqslant 0$. This implies the desired result:

$$\langle a_\epsilon, \epsilon_j a_j\rangle = \langle \sum_{k=1}^{n} e^{i\varphi_k} a_j, e^{i\varphi_j} a_j\rangle = \langle u+v, v\rangle = \langle u,v\rangle + 1 \geqslant 1.$$

Now, since

$$|P_n(z)| = |\langle a_1, z\rangle \ldots \langle a_n, z\rangle| = |\langle e^{i\varphi_1} a_1, z\rangle \ldots \langle e^{i\varphi_n} a_n, z\rangle|,$$

we can replace the a_j's in (3) by $e^{i\varphi_j} a_j$ without loss of generality. In other words, writing $y_j := \langle \sum_{k=1}^{n} a_k, a_j\rangle$, we find $y_j \geqslant 1$, $j=1,\ldots,n$, as in (6). From here, the argument proving Theorem 1 coincides with the real case (cf. [16]), considering the normalized mean vector z:

$$z := \frac{a_1 + \ldots + a_n}{\|a_1 + \ldots + a_n\|_2}.$$

The arguments of [16] (applied verbatim) then show that z always satisfies $|P_n(z)| \geqslant n^{-n/2}$, for $n \leqslant 5$.

The advantage of this result (in these low dimensional cases) compared to [2] and [3] is that it gives a *construction* for the vector z as opposed to proving existence only.

The analogue to the question of A. Pappas and S. Révész in the complex case is to ask whether this construction of z works in *all* dimensions.

4. Mean vectors of maximal length for systems close to orthonormal vectors

It is plausible to expect that the only system of vectors for which $\|P_n\| \leqslant n^{-n/2}$ is the orthonormal system (which satisfies $\|P_n\| = n^{-n/2}$ as shown in Section 1). Even in the complex case, this statement does not follow from the considerations in [2] and [3], and remains an open question. In this section we prove the *local* uniqueness of the orthonormal system in both the real and the complex case. In Theorem 2 below we show that for vectors close to the orthonormal system we always have $\|P_n\| \geqslant n^{-n/2}$ and the Pappas-Révész choice (7) of mean vector belonging to maximal length always provides strict inequality (unless the system is orthonormal, in which case strict inequality is not possible). With the help of some specific examples, however, we will show later that the Pappas-Révész choice (7) of mean vector x does not satisfy $|P_n(x)| \geqslant n^{-n/2}$ in general.

THEOREM 2. *Let H be any real or complex Hilbert space and $\{a_j\}_{j=1}^{n} \subset S_H$. We assume (without loss of generality) that the choice of signs $\epsilon = (1,1,\ldots,1)$ gives maximal length among the mean vectros a_ϵ. (This also means that the system $\{a_j\}_{j=1}^{n}$ satisfies condition (6).) Assume also, with the notation used in (6), that $y_j \leqslant 3.5$, for $j=1,\ldots,n$. Then $\|P_n\| \geqslant n^{-n/2}$ holds with equality only when $y_j = 1$ for $j=1,\ldots,n$, that is, only for the orthonormal vector system.*

Proof. Before proceeding with the proof we note that the condition $y_j \leqslant 3.5$ is certainly satisfied in a small neighbourhood of the orthonormal system. (Due to the dimension being finite, all definitions of "neighbourhood" are equivalent. Vector systems in a small neighbourhood are obtained by small perturbations of the vectors of an orthonormal system.)

Let x be the mean vector defined as in (7). The assertion $|P_n(x)| \geqslant n^{-n/2}$ is equivalent to state that the inequality

$$y_1^2 y_2^2 \cdots y_n^2 \geqslant \left(\frac{y_1 + y_2 + \cdots + y_n}{n}\right)^n \tag{9}$$

holds true (see [16]). With a slight reformulation, we put $t_i := y_i - 1 \quad (i = 1, \dots, n)$ and write

$$(1+t_1)^2(1+t_2)^2 \cdots (1+t_n)^2 \geqslant \left(1 + \frac{t_1 + t_2 + \cdots + t_n}{n}\right)^n . \tag{10}$$

Note that here $0 \leqslant t_i \leqslant n-1$ for all $i = 1, \dots, n$. Now it suffices to show (10) for real quantities $0 \leqslant t_j \leqslant 2.5$ ($j = 1, \dots, n$). To start with, observe that

$$(1+t)^2 \geqslant e^t. \tag{11}$$

is satisfied for $0 \leqslant t \leqslant t_0$, where t_0 is the (unique) root of $f(t) := (1+t)^2 - e^t$ in $(1, \infty)$. Moreover, since $t_0 > 2.5$, (11) follows with strict inequality if $0 < t \leqslant 2.5$.

Multiplying together this inequality for all t_i with $i = 1, \dots, n$, we obtain

$$(1+t_1)^2(1+t_2)^2 \cdots (1+t_n)^2 \geqslant e^{t_1+t_2+\cdots+t_n} . \tag{12}$$

Since $e^x \geqslant (1 + x/n)^n$ for all $x \geqslant 0$ and $n \in \mathbb{N}$, we conclude (10), and hence the assertion.

We have equality in (11) only for $t = 0$, hence for equality in (12) we must have $t_1 = \cdots = t_n = 0$. That is, $\{a_j\}_{j=1}^n$ must be an orthonormal system of vectors.

The proof of the theorem above heavily relied on the assumption $y_j \leqslant 3.5$. A thorough analysis of the proof above leads us to give a negative answer to the question of A. Pappas and S. Révész in high dimensions.

THEOREM 3. *If n is large enough, then there exist vectors $\{a_j\}_{j=1}^n \subset S_{\mathbb{R}^n}$ so that taking the mean vector (7) of maximal length, we have*

$$|P_n(x)| = \prod_{j=1}^{n} |\langle x, a_j\rangle| < n^{-n/2} . \tag{13}$$

The analogous results holds for $a_j \in S_{\mathbb{C}^n}$ ($j = 1, \dots, n$) and complex signs $e^{i\varphi_j}$.

Proof. The proof is a specific example obtained by analyzing the proof of Theorem 2 above.

Take $n \geqslant 34$ and let $a_1, \dots, a_n$ be the n unit vectors in H defined as follows:

$$a_j := b := \left(\frac{1}{\sqrt{6}}, \overset{(6)}{\dots}, \frac{1}{\sqrt{6}}, 0 \dots, 0\right) \quad \text{for } 1 \leqslant j \leqslant 6$$
$$a_j := e_j \quad \text{for } 6 < j \leqslant n,$$

where $\{e_j : 1 \leqslant j \leqslant n\}$ is the canonical basis of $\mathbb{K}^n$.

Then it is obvious that a choice of signs maximizing the length of the mean vector (both in the real and complex case) is $z = 6b + \sum_{j=7}^{n} e_j$, and for this vector we have

$$|P_n(z/\|z\|)| = \frac{6^6}{(n+30)^{\frac{n}{2}}} < \frac{1}{n^{\frac{n}{2}}} \quad \text{for}\, n \geqslant 34.$$

REMARK 3. In the example above there exists a combination of signs such that the corresponding mean vector x satisfies $|P_n(x)| \geqslant n^{-n/2}$. But this choice of signs *does not correspond* to the sum vector of maximal length. Indeed, the choice $x = a_1 + a_2 + a_3 + a_4 - a_5 - a_6 + \sum_{j=7}^{n} a_j = 2b + \sum_{j=7}^{n} e_j$ is such a choice.

REMARK 4. The example used in the previous proof can be easily generalized by considering natural numbers $n > d$ and the system of vectors $\{a_j\}_{j=1}^{n}$ given by

$$\begin{aligned} a_j &:= b := \left(\frac{1}{\sqrt{d}}, \overset{(d)}{\ldots}, \frac{1}{\sqrt{d}}, 0 \ldots, 0\right) \quad \text{for}\, 1 \leqslant j \leqslant d \\ a_j &:= e_j \quad \text{for}\, d < j \leqslant n. \end{aligned}$$

In this case $z = db + \sum_{j=d+1}^{n} e_j$ and

$$|P(z/\|z\|)| = \frac{d^d}{(n+d^2-d)^{\frac{n}{2}}}.$$

The reader can check easily that for every $d \in \mathbb{N}$ we can find $n_0 \in \mathbb{N}$ with

$$\frac{d^d}{(n+d^2-d)^{\frac{n}{2}}} < \frac{1}{n^{\frac{n}{2}}} \quad \text{for}\, n \geqslant n_0.$$

It is also a simple exercise to see that the smallest possible value for n_0 is 34 and it is achieved when $d = 6$.

5. Norms over $\mathbb{R}^n$ and over $\mathbb{C}^n$

Complexifications provide another natural approach towards the determination of real polarization constants. In this context, it is natural to ask whether the norm of a real polynomial of the form (3) remains the same when considered over the complex unit ball $S_{\mathbb{C}^n}$ instead of $S_{\mathbb{R}^n}$. In view of the result of Arias-de-Reyna [2], this would imply the conjectured value $c_n(\mathbb{R}^n) = n^{n/2}$. However, in this section we show that if $n \geqslant 3$ then the norm of the complex polynomial can be strictly larger than that of the real one.

The main result in this section is based on the following well-known estimate by G. A. Muñoz, Y. Sarantopoulos and A. Tonge [15]:

THEOREM 4. *If $P : \mathbb{R}^n \to \mathbb{R}$ $(n \geqslant 2)$ is an n-homogeneous polynomial then*

$$\|P\|_{S_{\mathbb{C}^n}} \leqslant 2^{\frac{n-2}{2}} \|P\|_{S_{\mathbb{R}^n}}. \tag{14}$$

Moveover, the constant $2^{\frac{n-2}{2}}$ *is sharp and equality in* (14) *is achieved for the polynomial* $R_n : \mathbb{R}^n \to \mathbb{R}$ *defined by*

$$R_n(x_1, x_2, \ldots, x_n) := \mathrm{Re}\,(x_1 + ix_2)^n.$$

Regarding the real linear polarization problem, one could hope to achieve a better estimate than (14) by restricting attention to products of functionals instead of *all* n-homogeneous polynomials. However, we shall prove in this section that the polynomials R_n $(n \in \mathbb{N})$ can be written as a product of functionals as in (3). This fact leads us to conclude that the inequality (14) cannot be improved even if we consider only polynomials as described in (3).

REMARK 5. The polynomials R_n $(n \geqslant 2)$ were originally defined in [15] on $\mathbb{R}^2$. For simplicity we shall consider the restriction of R_n to $\mathbb{R}^2$, in other words

$$R_n(x, y) := \mathrm{Re}\,(x + iy)^n. \tag{15}$$

The following elementary result shall be required in order to prove that R_n $(n \in \mathbb{N})$ can be factored as the product of n linear forms:

LEMMA 5. *If* $P \in \mathcal{P}(^n\mathbb{R}^2)$ *satisfies* $P(x_0, y_0) = 0$ *for some* $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(0,0)\}$, *then there exists* $Q \in \mathcal{P}(^{n-1}\mathbb{R}^2)$ *such that*

$$P(x, y) = (-y_0 x + x_0 y) Q(x, y),$$

for every $(x, y) \in \mathbb{R}^2$.

Proof. Suppose that $x_0 \neq 0$ and define $p(t) := P(1, t)$ for all $t \in \mathbb{R}$. Then p is a real polynomial of degree at most n such that

$$p\left(\frac{y_0}{x_0}\right) = P\left(1, \frac{y_0}{x_0}\right) = \frac{1}{x_0^n} P(x_0, y_0) = 0.$$

Therefore $p(t) = (t - \frac{y_0}{x_0}) q(t)$ for some real polynomial $q(t) := a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$ of degree at most $n - 1$. Hence, if we take $(x, y) \in \mathbb{R}^2$ with $x \neq 0$ we have

$$P(x, y) = x^n p\left(\frac{y}{x}\right) = \left(\frac{y}{x} - \frac{y_0}{x_0}\right) x^n q\left(\frac{y}{x}\right) = (-y_0 x + x_0 y) \frac{x^{n-1}}{x_0} q\left(\frac{y}{x}\right).$$

On the other hand the mapping $Q : \mathbb{R}^2 \to \mathbb{R}$ given by

$$Q(x, y) = \begin{cases} \frac{x^{n-1}}{x_0} q(\frac{y}{x}) & \text{if } x \neq 0 \\ \frac{a_{n-1}}{x_0} y^{n-1} & \text{if } x = 0 \end{cases},$$

is clearly an $(n-1)$-homogeneous polynomial. By continuity it follows immediately that $P(x, y) = (-y_0 x + x_0 y) Q(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

THEOREM 6. *If* R_n $(n \in \mathbb{N})$ *is as in* (15) *then there exist* $\{a_j\} \subset S_{\mathbb{R}^2}$ *and a constant* $K \neq 0$ *so that*

$$R_n(v) = K\langle v, a_1\rangle \cdots \langle v, a_n\rangle,$$

for all $v \in \mathbb{R}^2$.

Proof. Assume first that n is odd and define (identifying $\mathbb{R}^2$ with $\mathbb{C}$)

$$(x_j, y_j) := \left(\sin\frac{j\pi}{n}, -\cos\frac{j\pi}{n}\right) = e^{i(\frac{j\pi}{n}-\frac{\pi}{2})} \quad \text{for} j = 0, 1, \ldots, n-1.$$

Then, since $2j-n$ is always an odd integer for all odd $n \in \mathbb{N}$ and $0 \leqslant j \leqslant n-1$, it follows that

$$R_n(x_j, y_j) := \operatorname{Re} e^{in(\frac{j\pi}{n}-\frac{\pi}{2})} = \operatorname{Re} e^{i(2j-n)\frac{\pi}{2}} = 0,$$

and hence by Lemma 5 for every $v \in \mathbb{R}^2$

$$R_n(v) = K\langle v, a_1\rangle \cdots \langle v, a_n\rangle,$$

where $a_j = (\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n})$ for $0 \leqslant j \leqslant n-1$ and

$$K = \frac{1}{\prod_{j=0}^{n-1}\cos\frac{j\pi}{n}} = (-1)^{\frac{n-1}{2}} 2^{n-1}.$$

Now suppose that n is even and define

$$(x_j, y_j) := \left(\sin\frac{(2j+1)\pi}{2n}, -\cos\frac{(2j+1)\pi}{2n}\right) = e^{i(\frac{(2j+1)\pi}{2n}-\frac{\pi}{2})} \quad \text{for} j = 0, 1, \ldots, n-1.$$

Then since $2j+1-n$ is always an odd integer for all even $n \in \mathbb{N}$ and $1 \leqslant j \leqslant n-1$ we have

$$R_n(x_j, y_j) = \operatorname{Re} e^{in(\frac{(2j+1)\pi}{2n}-\frac{\pi}{2})} = \operatorname{Re} e^{i(2j+1-n)\frac{\pi}{2}} = 0.$$

and hence by Lemma 5 for every $v \in \mathbb{R}^2$

$$R_n(v) = K\langle v, a_1\rangle \cdots \langle v, a_n\rangle,$$

where $a_j = (\cos\frac{(2j+1)\pi}{2n}, \sin\frac{(2j+1)\pi}{2n})$ for $0 \leqslant j \leqslant n-1$ and

$$K = \frac{1}{\prod_{j=0}^{n-1}\cos\frac{(2j+1)\pi}{2n}} = (-1)^{\frac{n}{2}} 2^{n-1}.$$

REMARK 6. If $a_j = (a_j^1, a_j^2)$ $(1 \leqslant j \leqslant n)$ are the unit vectors obtained in Theorem 6 and we regard the a_j's as vectors in $\mathbb{R}^n$ $(n \geqslant 2)$ by setting $a_j := (a_j^1, a_j^2, 0\ldots, 0) \in \mathbb{R}^n$ $(1 \leqslant j \leqslant n)$, then the n-homogeneous polynomial $P_n : \mathbb{R}^n \to \mathbb{R}$ defined by

$$P_n(x) := \langle x, a_1\rangle \cdots \langle x, a_n\rangle,$$

is of the type (3) and satisfies

$$\|P_n\|_{S_{\mathbb{C}^n}} = 2^{\frac{n-2}{2}}\|P_n\|_{S_{\mathbb{R}^n}}.$$

This shows that the complexification argument used to prove estimate (4) cannot be improved.

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