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FUNDAMENTALS OF EQUILIBRIUM PROBLEMS

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Abstract. Equilibrium problems theory provides us with a unified, natural, innovative and general framework for studying a wide class of linear and nonlinear problems arising in finance, economics, image reconstructions, medical imaging, ecology, network analysis, transportation. elasticity, operations research and optimization. In this work, we consider some new classes of equilibrium problems in the framework of convexity, invexity, g-convexity and prox-regular convexity. We also study a class of equilibrium problems involving the nondifferentability Lipschitz continuous functions, which is known as the hemiequilibrium problems. The auxiliary principle technique is used to suggest and analyze several iterative schemes for solving these classes of equilibrium problems. We consider the convergence analysis of these iterative algorithms under some mild conditions. We also introduce the concept of well-posedness for the equilibrium problems and obtain some interesting results. As special cases, we obtain several known and new results for variational inequalities and related optimization problems. Results obtained in this paper can be viewed as a nice and novel applications of the auxiliary principle technique in this fast growing and fascinating field.

1. Introduction

Equilibrium problems theory has had a great impact and influence in the development of several branches of pure and applied sciences. This theory has witnessed an explosive growth in theoretic advances, algorithmic aspects and applications across all discipline of mathematical and engineering sciences. In recent years, it has been shown that the equilibrium problems theory provides a novel and unified treatment of a wide class of linear and nonlinear problems arising in economics, finance, image reconstruction, medical imaging, ecology, transportation, network analysis, structural analysis, elasticity and nonlinear optimization. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative. It has been shown that the equilibrium problems include variational inequalities, variational-like inequalities, hemicariational inequalities, complmentarity problems, fixed-point, Nash equilibrium and game theory as special cases. Hence collectively equilibrium problems cover a vast range of applications. In the present form, the equilibrium problems were first introduced by Blum and Oettli [1] and Noor and Oettli [47] in 1994. A classical assumption in this theory and in the algorithms for equilibrium problems is

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the convexity of the set and objective function. We call such type of the equilibrium problems as the classical equilibrium problems. Almost all the results obtained in this theory are in the setting of convexity. It has been noted that these results may not hold for the nonconvexity setting. The concept of the convexity has been generalized and extended in many directions, which has potential and important applications in various fields. Some of these generalizations are the introduction of preinvex (invex) functions, g-convex functions, g-convex sets, invex sets as well as the prox-regular sets. It is well known that the preinvex (invex) and g-convex functions may not be convex functions. At the same time, the invex sets, g-convex sets and prox-regular sets are also nonconvex sets. Motivated by these concepts, Noor [33, 37] has introduced and considered some new classes of the equilibrium problems, which are known as invex equilibrium problems, nonconvex (g-convex) equilibrium problems, regular equilibrium problems and hemiequilibrium problems. These new classes are quite different classes of the equilibrium problems and have no inter relationship with each other. However all these classes of the equilibrium problems include the classical equilibrium problems and variational inequalities as special cases. We also consider another class of equilibrium problems, which is called the hemiequilibrium problems related to the concept of the hemivariational inequalities involving the nonlinear nondifferentiable Lipschitz continuous functions. Hemivariational inequalities were introduced by Panagiotopoulos [48] in early 1980's.

In recent years, several numerical methods including projection technique, Wiener-Hopf (resolvent) equations, auxiliary principle technique have been developed for variational inequalities and related optimization problems. Unfortunately, the projection method and its variant forms including the Wiener-Hopf equations can not be extended for solving equilibrium problems, since it is not possible to find the projection of the bifunction from the whole space onto the convex set. To overcome this drawback, one usually uses the auxiliary principle technique, the origin of which can be traced back to Lions and Stampacchia [14]. The main and basic idea in this technique is to consider an auxiliary problem related to the original problem. This way one defines a mapping connecting the solutions of these problems. In this case, one has to show that the mapping connecting the solutions is a contraction mapping and consequently it has a fixed point, which is the solution of the original problem. Glowinski, Lions and Tremolieres [10] has used this technique to study the existence of a solution of mixed variational inequalities, whereas Noor [26-43] has used this technique to suggest and analyze a number of iterative methods for solving various classes of variational inequalities and equilibrium problems. We again use the auxiliary principle technique to suggest and analyze some classes of iterative methods for solving these classes of equilibrium problems. We have studied the convergence criteria of these methods under some mild conditions. As a consequence of this approach, we construct the gap (merit) function for equilibrium problems, which can be used to develop descent-type methods for solving equilibrium problems. We also introduce the concept of well-posedness for equilibrium problems and obtain some results. The interested reader is urged to explore these problems further and discover some new, novel and innovative applications of the reqularized equilibrium problems in the setting of different normed space. Our results can be viewed as significant extension and generalization of the previously known results for solving classical variational inequalities and equilibrium problems.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. Let $T : H \longrightarrow H$ be a nonlinear operator. For a given nonlinear function $F(.,.,.) : K \times K \times K \longrightarrow R$, consider the problem of finding $u \in K$ such that

$$F(u, Tu, v) \geqslant 0, \quad \forall v \in K, \tag{2.1}$$

which is called the *equilibrium problem with trifunction* considered and investigated by Noor and Oettli [47] in 1994.

If $F(u, Tu, v) \equiv F(u, v)$, then problem (2.1) is equivalent to finding $u \in K$ such that

$$F(u, v) \geqslant 0, \quad \forall v \in K. \tag{2.2}$$

Problem (2.2) is known as the classical equilibrium problem introduced and studied by Blum and Oettli [1] and Noor and Oettli [47]. It has been shown a wide class of problems including fixed-point, Nash equilibrium, transportaion and variational inequalities can be obtained as special cases of problems (2.1) and (2.2), see the references.

If $F(u, v) = \langle Tu, v - u \rangle$, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geqslant 0, \quad \forall v \in K, \tag{2.3}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [53] in 1964. It is well-known that a wide class of obstacle, unilateral, contact, free, moving and equilibrium problems arising in mathematical, engineering, economics and finance can be studied in the unified and general framework of the variational inequalities of type (2.3). For the physical and mathematical formulation of problems (2.1) and (2.2), see [1-56] and the references therein.

We also need the following concepts and results.

LEMMA 2.1. $\forall u, v \in H$,

$$2\langle u, v \rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2. \tag{2.4}$$

LEMMA 2.2. *For a given* $z \in H$, $u \in K$ *satisfies the inequality*

$$\langle u - z, v - u \rangle \geqslant 0, \quad \forall v \in K,$$

if and only if

$$u = P_K z,$$

where P_K *is the projection of* H *onto the convex closed set* K.

Lemma 2.2 is known as the *Projection Lemma* and plays fundamental role in the studies of the variational inequalities and related optimization problems. It is well known that the projection operator P_K is nonexpansive and monotone, that is,

$$\|P_K u - P_K v\| \leqslant \|u - v\|, \quad \forall u, v \in K$$
$$\langle P_K u - P_K v, u - v \rangle \geqslant 0, \quad \forall u, v \in K.$$

DEFINITION 2.1. The trifunction $F(.,.,.)$ and the operator T are said to be:
(i) *strongly jointly monotone,* if there exists a constant $\gamma > 0$ such that

$$F(u, Tu, v) + F(v, Tv, u) \leqslant -\gamma\|u - v\|^2, \quad \forall u, v \in K.$$

(ii) *jointly pseudomonotone,* if

$$F(u, Tu, v) \geqslant 0 \quad \Longrightarrow -F(v, Tv, u) \geqslant 0, \quad \forall u, v \in K.$$

(iii) *partially relaxed strongly jointly monotone,* if there exists a constant $\alpha > 0$ such that

$$F(u, Tu, v) + F(v, Tv, z) \leqslant \alpha\|z - u\|^2, \quad \forall u, v, z \in K.$$

(iv) *jointly hemicontinuous,* if $\forall u, v \in K,$ the mapping $t \in [0, 1]$ implies that $F(u + t(v - u), T(u + t(v - u)), v)$ is continuous.

Note that for $z = u,$ partially relaxed strongly jointly monotonicity reduces to

$$F(u, Tu, v) + F(v, Tv, u) \leqslant 0, \quad \forall u, v \in K,$$

which is known as jointly monotonicity. It is known that jointly monotonicity implies jointly pseudomonotonicity, but the converse is not true.

DEFINITION 2.2. A function f is said to be strongly convex function on the convex set K with modulus $\mu,$ if,

$$f(u + t(v - u)) \leqslant (1 - t)f(u) + tf(v) - t(1 - t)\mu\|v - u\|^2, \quad \forall u, v \in K, \quad t \in [0, 1].$$

Clearly the differentiable strongly convex function f is equivalent to

$$f(v) - f(u) \geqslant \langle f'(u), v - u) \rangle + 2\mu\|v - u\|^2, \quad \forall u, v \in K.$$

3. Equilibrium problems

Blum and Oettli [1] and Noor and Oettli [47] have shown that the variational inequalities and mathematical programming problems can be viewed as special realization of the abstract equilibrium problems. Equilibrium problems have numerous applications, including but not limited to problems in economics, game theory, finance, traffic analysis, circuit network analysis and mechanics, see the references. We suggest and analyze some proximal methods for solving the equilibrium problems (2.1) using the auxiliary principle technique.

For a given $u \in K,$ consider the auxiliary problem of finding $w \in K$ such that

$$\rho F(w, Tw, v) + \langle w - u + \gamma(u - u), v - w\rangle \geqslant 0, \quad \forall v \in K, \tag{3.1}$$

where $\rho > 0$ and $\gamma > 0$ are constants. We note that, if $w = u,$ then clearly w is solution of the equilibrium problem (2.1). This observation enables us to suggest and analyze the following iterative method for solving (2.1).

Algorithm 3.1. For a given $u_0 \in H,$ compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, Tu_{n+1}, v) + \langle u_{n+1} - u_n + \gamma_n(u_n - u_{n-1}), v - u_{n+1} \rangle \geqslant 0, \quad \forall v \in K,$$

which is known as the inertial proximal method for solving equilibrium problem (2.1). Such type of inertial proximal methods have been considered by Noor [27, 29, 42] for solving variational inequalities and equilibrium problems.

If $F(u, Tu, v) = F(u, v),$ then Algorithm 3.1 collapses to:

Algorithm 3.2. For a given $u_0 \in H,$ compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, v) + \langle u_{n+1} - u_n + \gamma_n(u_n - u_{n-1}), v - u_{n+1} \rangle \geqslant 0, \quad \forall v \in K,$$

which is known as the inertial proximal method for solving equilibrium problem (2.2).

For $\gamma_n = 0,$ Algorithm 3.1 collapses to:

Algorithm 3.3. For a given $u_0 \in H,$ compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, Tu_{n+1}, v) + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geqslant 0, \quad \forall v \in K, \tag{3.2}$$

which is called the proximal method for solving problem (2.1). This shows that the inertial proximal methods include the classical proximal methods as a special case.

If $F(u, Tu, v) = \langle Tu, v - u \rangle$, then Algorithm 3.1 reduces to the following iterative method for solving the variational inequalities (2.3).

Algorithm 3.4. For a given $u_0 \in K,$ compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Tu_{n+1} + u_{n+1} - u_n + \gamma_n(u_n - u_{n-1}), v - u_{n+1} \rangle \geqslant 0, \quad \forall v \in K,$$

which can be written as

$$u_{n+1} = P_K[u_n - \rho Tu_{n+1} + \gamma_n(u_n - u_{n-1})], \quad n = 1, 2, \ldots,$$

where P_K is the projection of H onto the convex set $K.$ Algorithm 3.4 is known as the inertial proximal point algorithm for solving variational inequalities (2.3) and has been studied by Noor [27, 29, 42]. In a similar way, one can obtain several iterative methods for variational inequalities (2.3) and their special cases.

We now study the convergence analysis of Algorithm 3.3. The convergence analysis of Algorithms 3.1, 3.2 and 3.4 can be studied in a similar way.

THEOREM 3.1. *Let $\bar{u} \in K$ be a solution of* (2.1) *and u_{n+1} be the approximate solution obtained from Algorithm* 3.3. *If the trifunction $F(.,.,.)$ and T are jointly pseudomonotone, then*

$$\|u_{n+1} - \bar{u}\|^2 \leqslant \|u_n - \bar{u}\|^2 - \|u_{n+1} - u_n\|^2. \tag{3.3}$$

Proof. Let $\bar{u} \in K$ be a solution of (2.1). Then

$$F(\bar{u}, T\bar{u}, v) \geqslant 0, \quad \forall v \in K,$$

which implies that

$$-F(v, Tv, \bar{u}) \geqslant 0, \quad \forall v \in K, \tag{3.4}$$

since $F(.,.,.)$ and T are jointly pseudomonotone.

Taking $v = u_{n+1}$ in (3.4), we have

$$-F(u_{n+1}, Tu_{n+1}, \bar{u}) \geqslant 0. \tag{3.5}$$

Now taking $v = \bar{u}$ in (3.2), we obtain

$$\rho F(u_{n+1}, Tu_{n+1}, \bar{u}) + \langle u_{n+1} - u_n, \bar{u} - u_{n+1} \rangle \geqslant 0. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\langle u_{n+1} - u_n, \bar{u} - u_{n+1} \rangle \geqslant -\rho F(u_{n+1}, Tu_{n+1}, \bar{u}) \geqslant 0. \tag{3.7}$$

Setting $u = \bar{u} - u_{n+1}$ and $v = u_{n+1} - u_n$ in (2.4), we obtain

$$2\langle u_{n+1} - u_n, \bar{u} - u_{n+1} \rangle = \|\bar{u} - u_n\|^2 - \|\bar{u} - u_{n+1}\|^2 - \|u_n - u_{n+1}\|^2. \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$\|u_{n+1} - \bar{u}\|^2 \leqslant \|u_n - \bar{u}\|^2 - \|u_{n+1} - u_n\|^2,$$

the required result. □

THEOREM 3.2. *Let H be a finite dimensional space. If u_{n+1} is the approximate solution obtained from Algorithm* 3.2 *and $\bar{u} \in K$ is a solution of* (2.1)*, then* $\lim_{n \longrightarrow \infty} u_n = \bar{u}$.

Proof. Let $\bar{u} \in K$ be a solution of (2.1). From (3.3), it follows that the sequence $\{\|\bar{u} - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (3.3), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leqslant \|u_0 - \bar{u}\|^2,$$

which implies that

$$\lim_{n \longrightarrow \infty} \|u_{n+1} - u_n\| = 0. \tag{3.9}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (3.2) and taking the limit $n_j \longrightarrow \infty$ and using (3.9), we have

$$F(\hat{u}, T\hat{u}, v) \geqslant 0, \quad \forall v \in K,$$

which implies that $\hat{u}$ solves the equilibrium problem (2.1) and

$$\|u_{n+1} - u_n\|^2 \leqslant \|u_n - \bar{u}\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\longrightarrow\infty} u_n = \hat{u}$. $\square$

It is known that in order to implement the inertial proximal and proximal algorithms, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we suggest another iterative method for solving the equilibrium problem (2.1).

For a given $u \in K$, consider the auxiliary problem of finding $w \in K$ such that

$$\rho F(u, Tu, v) + \langle w - u, v - w\rangle \geqslant 0, \quad \forall v \in K, \tag{3.10}$$

where $\rho > 0$ is a constant.

We note that if $w = u$, then clearly w is solution of the equilibrium problem (2.1). Note that problems (3.1) and (3.10) are quite different. In fact, problem (3.10) is equivalent to an optimization problem. This observation enables us to suggest and analyze the following iterative method for solving the equilibrium problem (2.1).

Algorithm 3.5. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_n, Tu_n, v) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geqslant 0, \quad \forall v \in K. \tag{3.11}$$

If $F(u, Tu, v) \equiv \langle Tu, v-u\rangle$, then Algorithm 3.5 is equivalent to the following iterative method for solving variational inequalities (2.3).

Algorithm 3.6. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Tu_n + u_{n+1} - u_n, v - u_{n+1}\rangle \geqslant 0, \quad \forall v \in K,$$

or equivalently

$$u_{n+1} = P_K[u_n - \rho Tu_n], \quad n = 0, 1, 2, \ldots$$

where P_K is the projection operator. Algorithm 3.6 has been studied extensively. For suitable and appropriate choice of the function $F(.,.,.)$ and the space H, one can obtain several iterative schemes for solving the problems (2.1)-(2.2) and related optimization problems.

We now study the convergence analysis of Algorithm 3.4.

THEOREM 3.3. *Let $\bar{u} \in K$ be a solution of* (2.1) *and u_{n+1} be the approximate solution obtained from Algorithm* 3.4*. If $F(.,.,.)$ and T are partially relaxed strongly jointly monotone with constant $\alpha > 0$, then*

$$\|u_{n+1} - \bar{u}\|^2 \leqslant \|u_n - \bar{u}\|^2 - (1 - 2\alpha\rho)\|u_{n+1} - u_n\|^2. \tag{3.12}$$

Proof. Let $\bar{u} \in K$ be a solution of (2.1). Then

$$F(\bar{u}, T\bar{u}, v) \geqslant 0, \quad \forall v \in K. \tag{3.13}$$

Taking $v = u_{n+1}$ in (3.13) and $v = u$ in (3.11), we have

$$F(\bar{u}, T\bar{u}, u_{n+1}) \geqslant 0. \tag{3.14}$$

and

$$\rho F(u_n, Tu_n, \bar{u}) + \langle u_{n+1} - u_n, \bar{u} - u_{n+1} \rangle \geqslant 0. \tag{3.15}$$

From (3.14) and (3.15), we have

$$\begin{aligned} \langle u_{n+1} - u_n, \bar{u} - u_{n+1} \rangle &\geqslant -\rho\{F(u_n, Tu_n, \bar{u}) + F(\bar{u}, T\bar{u}, u_{n+1})\} \\ &\geqslant -\alpha\rho\|u_n - u_{n+1}\|^2, \end{aligned} \tag{3.16}$$

since $F(.,.,.)$ and T are partially relaxed strongly jointly monotone with a constant $\alpha > 0$.

Combining (3.8) and (3.16), we have

$$\|u_{n+1} - \bar{u}\|^2 \leqslant \|u_n - \bar{u}\|^2 - (1 - 2\rho\alpha)\|u_{n+1} - u_n\|^2.$$

THEOREM 3.4. *Let* H *be a finite dimensional space and let* $0 < \rho < \frac{1}{2\alpha}$. *If* u_{n+1} *is the approximate solution obtained from Algorithm* 3.4 *and* $\bar{u} \in H$ *is a solution of* (2.1)*, then* $\lim_{n\longrightarrow\infty} u_n = \bar{u}$.

Proof. Its proof is similar to Theorem 3.2.

We now again use the auxiliary principle technique in conjunction with Bregman function to suggest another class of iterative methods for solving equilibrium problems (2.1). Convergence analysis of this class of iterative methods is distinctly different from the previous analysis.

For a given $u \in K$, consider the problem of finding $w \in K$ such that

$$\rho F(w, Tw, v) + \langle E'(w) - E'(u), v - w \rangle \geqslant 0, \quad \forall v \in K, \tag{3.17}$$

which is known as the auxiliary equilibrium problem. Here $E'(u)$ is the differential of a strongly convex function $E(u)$ at the point $u \in K$. Problem (3.17) has a unique solution, since the functions E is a strongly convex function.

REMARK 3.1. The function $B(z,u) = E(z) - E(u) - \langle E'(u), z - u \rangle$ associated with the convex function $E(u)$ is called the Bregman function. For the applications of the Bregman function in solving variational inequalities and complementarity problems, see the references.

We remark that if $w = u$, then w is a solution of (2.1). On the basis of this observation, we suggest and analyze the following iterative algorithm for solving (2.1) as long as (3.17) is easier to solve than (2.1).

Algorithm 3.7. For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, Tu_{n+1}, v) + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1} \rangle \geqslant 0, \quad \forall v \in K, \tag{3.18}$$

Algorithm 3.7 is called the proximal point method for solving the equilibrium problems (2.1). Note that for $F(u, Tu, v) = \langle Tu, v - u \rangle$, Algorithm 3.7 reduces to the following method for solving classical variational inequalities (2.3).

Algorithm 3.8. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_{n+1} + E'(u_{n+1}) - E'(u_n), v - u_{n+1} \rangle \geqslant 0, \quad \forall v \in K,$$

which is called the proximal method and has been studied extensively in recent years.

We now consider the convergence analysis of Algorithm 3.7 and this is the main motivation of our next result.

THEOREM 3.5. *Let the function $F(.,.,.)$ and T be jointly pseudomonotone. If E is differentiable strongly convex function with modulus $\beta > 0$, then the approximate solution u_{n+1} obtained from Algorithm 3.7 converges to a solution $u \in K$ satisfying (2.1).*

Proof. Let $u \in K$ be a solution of (2.1). Then

$$-F(v, Tv, u) \geqslant 0, \quad \forall v \in K \tag{3.19}$$

since $F(.,.,.)$ and T are jointly pseudomonotone.

Taking $v = u_{n+1}$ in (3.19) and $v = u$ in (3.18), we have

$$-F(u_{n+1}, Tu_{n+1}, u) \geqslant 0, \tag{3.20}$$

and

$$\rho F(u_{n+1}, Tu_{n+1}, u) + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1} \rangle \geqslant 0. \tag{3.21}$$

We consider the function,

$$\begin{aligned} B(u,z) &= E(u) - E(z) - \langle E'(z), u - z \rangle \\ &\geqslant \beta \|u - z\|^2, \end{aligned} \tag{3.22}$$

where we have used the fact that the function E is a strongly convex function with modulus $\beta > 0$.

Combining (3.20), (3.21) and (3.22), we have

$$\begin{aligned} B(u,u_n) - B(u,u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), u - u_n \rangle + \langle E'(u_{n+1}), u - u_{n+1} \rangle \\ &= E(u_{n+1}) - E(u_n) - \langle E'(u_n) - E'(u_{n+1}), u - u_{n+1} \rangle \\ &\quad - \langle E'(u_n), u_{n+1} - u_n \rangle \\ &\geqslant \beta \|u_{n+1} - u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1} \rangle \\ &\geqslant \beta \|u_{n+1} - u_n\|^2 - \rho F(u_{n+1}, Tu_{n+1}, u) \\ &\geqslant \beta \|u_{n+1} - u_n\|^2. \end{aligned}$$

If $u_{n+1} = u_n$, then clearly u_n is a solution of the equilibrium problem (2.1). Otherwise, the sequence $B(u,u_n) - B(u,u_{n+1})$ is nonnegative and we must have $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$. Now by using the technique of Zhu and Marcotte [56], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\overline{u}$ satisfying the equilibrium problem (2.1). □

We again use the auxiliary principle technique to suggest and analyze another iterative method for solving the equilibrium problem (2.1).

For a given $u \in K,$ consider the problem of finding $w \in K$ such that

$$\rho F(u, Tu, v) + \langle E'(w) - E'(u), v - w \rangle \geqslant 0, \quad \forall v \in K, \tag{3.23}$$

which is called the auxiliary equilibrium problem. From the strongly convexity of the differentiable function $E,$ it follows that problem (3.23) has a unique solution. Note that problems (3.23) and (3.17) are quite different. It is clear that if $w = u,$ then w is a solution of the equilibrium problem (2.1). This observation enables to suggest and analyze the following iterative method for solving (2.1).

Algorithm 3.9. For a given $u_0 \in K,$ calculate the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_n, Tu_n, v) + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1} \rangle \geqslant 0, \quad \forall v \in K. \tag{3.24}$$

Note that if $F(u, Tu, v) = \langle Tu, v - u \rangle$, then Algorithm 3.9 reduces to the following iterative scheme for variational inequalities (2.3).

Algorithm 3.10. For a given $u_0 \in K,$ find the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Tu_n + E'(u_{n+1}) - E'(u_n), v - u_{n+1} \rangle \geqslant 0, \quad \forall v \in K,$$

which can be written as

Algorithm 3.11. For a given $u_0 \in K,$ compute the approximate solution

$$E'(u_{n+1}) = P_K[E'(u_n) - \rho Tu_n], \quad n = 0, 1, 2, \ldots.$$

Note that for $K = H,$ Algorithm 3.11 reduces to:

Algorithm 3.12. For a given $u_0 \in H,$ compute the approximate solution

$$E'(u_{n+1}) = E'(u_n) - \rho Tu_n, \quad n = 0, 1, 2, \ldots.$$

which is similar to the interior point method.

One can study the convergence analysis of Algorithm 3.9 using essentially the technique of Theorem 3.5. However, we give its proof for the sake of completeness and to convey an idea.

THEOREM 3.6. *Let the function $F(.,.,.)$ and T be partially relaxed strongly jointly monotone with constant $\alpha > 0$ and let $E(u)$ be strongly convex function with modulus $\beta > 0.$ If $0 < \rho < \frac{\beta}{\alpha},$ then approximate solution u_{n+1} obtained from Algorithm* 3.9 *converges to a solution $u \in K$ of the equilibrium problem* (2.1).

Proof. Let $u \in K$ be a solution of (2.1). Then taking $v = u_{n+1}$ in (2.1) and $v = u$ in (3.24), we have

$$F(u, Tu, u_{n+1}) \geqslant 0. \tag{3.25}$$

and

$$\rho F(u_n, Tu_n, u) + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1} \rangle \geqslant 0. \tag{3.26}$$

Now combining (3.22), (3.25) and (3.26), we have

$$\begin{aligned} B(u,u_n) - B(u,u_{n+1}) &\geqslant \beta\|u_{n+1}-u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \\ &\geqslant \beta\|u_{n+1}-u_n\|^2 - \rho\{F(u,u_n) + F(u_{n+1},u)\} \\ &\geqslant \{\beta - \alpha\rho\}\|u_{n+1}-u_n\|^2, \end{aligned}$$

where we have used the fact that the bifunction $F(.,.,.)$ and T are partially relaxed strongly jointly monotone with constant $\alpha > 0$.

If $u_{n+1} = u_n$, then clearly u_n is a solution of (2.1). Otherwise, for $0 < \rho < \frac{\beta}{\alpha}$, the sequence $B(u,u_n) - B(u,u_{n+1})$ is nonnegative and we must have $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$. Now by using the technique of Zhu and Marcotte [56], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\overline{u}$ satisfying the equilibrium problem (2.1). □

It is obvious that the auxiliary equilibrium problem (3.10) is equivalent to finding the minimum of the functional $I[w]$ over the convex set K, where

$$I[w] = (1/2)\langle w-u, w-u\rangle - F(u,Tu,w), \tag{3.27}$$

which is known as the auxiliary energy (virtual work, potential) function associated with the problem (3.10). Using this functional $I[w]$, one can reformulate the equilibrium problem (2.1) as an equivalent optimization problem:

$$\Psi_\alpha(u) = \max_{w\in K}\{-F(u,Tu,w) - (\alpha/2)\|u-w\|^2\}, \tag{3.28}$$

where $\alpha > 0$ is a constant. Function of the type $\Psi(u)$ defined by (3.28) is called the regular gap function for the equilibrium problem. Note that for $\alpha = 0$, and $F(u,v) \equiv \langle Tu, v-u\rangle$, we obtain the original gap function for the variational inequality (2.2), which is due to Fukushima [7]. From the above discussion and observation, it is clear that can obtain the gap (merit) function for the equilibrium problems (2.1) by using the auxiliary principle technique. In passing, we remark this is observation is due to Noor [27, 29], where it has been shown that the auxiliary principle technique can be used to construct gap functions for several variational inequalities. This equivalent optimization formulation of the equilibrium problems can be used to develop some descent-type algorithms for solving equilibrium problems under suitable conditions on the function $F(.,.)$ by using the technique of Fukushima [7]. See also Masteroeni [18].

4. Invex equilibrium problems

In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson [13]. Hanson's initial result inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. Weir and Mond [54] and Noor [34-36] have studied the basic properties of the preinvex functions and their role in optimization and variational-like inequalities. It is well-known that the preinvex functions and invex sets may not be convex functions and convex sets. Noor [36] has proved that the minimum of the differentiable preinvex (invex) functions on the invex sets in normed spaces can be characterized by a class of variational inequalities, known as variational-like (pre-variational) inequalities. Thus it is clear that the concept of invexity plays exactly the same role in variational-like inequalities as the classical convexity plays in variational inequalities. This shows that the variational-like inequalities are well-defined only in the setting of invexity. Ironically, we note that all the results for variational-like inequalities are being obtained in the setting of classical convexity. No attempt has been made to utilize the concept of invexity. Since the the preinvex and invex functions are not convex functions, so all these results for variational-like inequalities are wrong and meaningless, since these results have been obtained using the KKM mappings and diagonally convexity. It is still an open problem to prove that the subdifferential of a differentiable preinvex function is maximal monotone operator. This implies one cannot define the resolvent operator associated with the proper, preinvex and lower-semicontinuous functions as some authors have defined. In brief, we would like to emphasize the fact that the variational-like inequalities must be studied in the setting of invexity. There is a very delicate and subtle difference between the concepts of invexity and convexity, which should be taken into account while considering variational-like inequalities and related optimization problems.

Related to the variational-like inequalities, we consider a new class of equilibrium problems. This class of equilibrium problems is called the invex equilibrium problems, which was introduced by Noor [25, 33, 37]. He used the auxiliary principle technique to suggest and analyze some iterative schemes for solving invex equilibrium problems. It has been shown that the invex equilibrium problems include variational-like inequalities, equilibrium problems and variational inequalities as special cases.

First of all, we recall the following well know results and concepts.

Let $f : K \to H$ and $\eta(.,.) : K \times K \to H$ be continuous functions.

DEFINITION 4.1. Let $u \in K$. Then the set K is said to be invex at u with respect to $\eta(.,.)$, if ,

$$u + t\eta(v,u) \in K, \quad \forall u, v \in K, \quad t \in [0,1].$$

K is said to be an invex set with respect to η, if K is invex at each $u \in K.$ The invex set K is also called η-connected set.

From now onward K is a nonempty closed invex set in H with respect to $\eta(.,.)$, unless otherwise specified.

DEFINITION 4.2. The function $f : K \to H$ is said to be preinvex with respect to η, if,

$$f(u + t\eta(v,u)) \leqslant (1-t)f(u) + tf(v), \quad \forall u, v \in K, \quad t \in [0,1].$$

The function $f : K \to H$ is said to be preconcave if and only if $-f$ is preinvex. Also note that for $t = 1,$ preinvex function reduces to:

$$f(u + \eta(v,u)) \leqslant f(v), \quad \forall u, v \in K.$$

DEFINITION 4.3. The differentiable function $f : K \to H$ is said to be an invex function with respect to $\eta(.,.)$, if,

$$f(v)-f(u) \geqslant \langle f'(u), \eta(v,u)\rangle, \quad \forall u, v \in K,$$

where $f'(u)$ is the differential of f at $u \in K$. The concepts of the invex and preinvex functions have played very important role in the development of convex programming. From definitions 4.2 and 4.3, it is clear that the differentiable preinvex function are invex functions and the converse is also true under certain conditions, see [44]. From definitions 4.2 and 4.3, it follows that the minimum of the differentiable preinvex(invex) function on the invex set K in H can be characterized by the inequality of the type:

$$\langle f'(u), \eta(v,u)\rangle \geqslant 0, \quad \forall \quad v \in K,$$

which is known as the variational-like inequality, see Noor [34-36]. From this formulation, it is clear that the set K involved in the variational-like inequality problem is an invex set, otherwise the variational-like inequality problem is not well-defined.

DEFINITION 4.4. A function f is said to be strongly preinvex function on K with respect to the function $\eta(.,.)$ with modulus μ, if,

$$f(u+t\eta(v,u)) \leqslant (1-t)f(u)+tf(v)-t(1-t)\mu\|\eta(v,u)\|^2, \quad \forall u, v \in K, \quad t \in [0,1].$$

Clearly the differentiable strongly preinvex function F is a strongly invex functions with module constant μ, that is,

$$f(v)-f(u) \geqslant \langle f'(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2,$$

and the converse is also true under certain conditions, see Noor [44]. Clearly for $\mu = 0$, definition 5.4 includes definition 5.2 as a special case. Also strongly invexity implies invexity.

For given continuous bifunction function $F(.,.,.)$ and the operator $T4$, consider the problem of finding $u \in K$ such that

$$F(u, Tu, v) \geqslant 0, \quad \forall v \in K, \tag{4.1}$$

which is called an *invex equilibrium (or equilibrium-like) problem* introduced and studied by Noor [33] recently. Here the set K is an invex set in H.

If $F(u, Tu, v) \equiv \langle Tu, \eta(v,u)\rangle$, then problem (4.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, \eta(v,u)\rangle \geqslant 0, \quad \forall v \in K, \tag{4.2}$$

which is known as the variational-like inequality problem. Problem (4.2) and its variant forms have been studied extensively by many authors in the setting of convexity using the KKM mappings and fixed-point theory. It is worth mentioning the concept of variational-like inequalities in the convexity setting is not well-defined and consequently all the results so far obtained in the convexity (scalar and vector) are misleading and wrong. Noor [34-36] and Yang and Chen [55] have shown that the minimum of the differentiable preinvex (invex) functions $f(u)$ on the invex sets in the normed spaces can be characterized by a class of variational-like inequalities (4.2) with $Tu = f'(u)$, where $f'(u)$ is the differential of the preinvex function $f(u)$. This shows that the concept of variational-like inequalities is closely related to the concept of invexity.

If $\eta(v,u) = v-u$, then the invex set K becomes the convex set and problem (4.1) is finding $u \in K$ such that

$$F(u,Tu,v) \geqslant 0, \quad \forall v \in K, \tag{4.3}$$

which is exactly the equilibrium problems (2.1) considered in Section 3.

Also the variational-like inequality (4.2) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geqslant 0, \quad \forall v \in K, \tag{4.4}$$

which is known as the classical variational inequality (2.3) introduced by Stampacchia [53].

DEFINITION 4.5. The function $F(.,.,.)$ and the operator T are said to be:

(i) *jointly pseudomonotone,* if

$$F(u,Tu,v) \geqslant 0 \quad \Longrightarrow -F(v,Tv,u) \geqslant 0, \quad \forall u,v \in K.$$

(ii) *partially relaxed strongly jointly η-monotone,* if there exists a constant $\alpha > 0$ such that

$$F(u,Tu,v) + F(v,Tv,z) \leqslant \alpha\|\eta(z,u)\|^2, \quad \forall u,v,z \in K.$$

(iii) *jointly hemicontinuous,* if $\forall u,v \in K$ and $t \in [0,1]$, the mapping $F(u+t\eta(v,u), T(u+t\eta(v,u)), v)$ is continuous.

Note that for $z = u$, partially relaxed strongly jointly monotonicity reduce to

$$F(u,Tu,v) + F(v,Tv,u) \leqslant 0, \quad \forall u,v \in K,$$

which is known as the jointly monotonicity.

We also need the following condition for the function $\eta(.,.)$ which is due to Mohan and Neogy [19].

Condition C. We assume that the function $\eta(.,.)$ satisfies the following:

$$\eta(u, u+t\eta(v,u)) = -t\eta(v,u)$$
$$\eta(v, u+t\eta(v,u)) = (1-t)\eta(v,u), \quad \forall u,v \in K, \quad t \in [0,1]$$

Clearly for $t=0$, we have $\eta(u,v) = 0$, if and only if $u = v, \forall u,v \in K$. Furthermore, it can be shown that $\eta(u+t\eta(v,u),u) = t\eta(v,u), \quad \forall u,v, \in K$.

LEMMA 4.1. *Let the function $F(.,..,)$ and T be jointly pseudomonotone and jointly hemicontinuous. If the function $F(.,.,.)$ is preinvex in the third argument, then problem (4.1) is equivalent to finding $u \in K$ such that*

$$F(v,Tv,u) \leqslant 0, \quad \forall v \in K. \tag{4.5}$$

Proof. Let $u \in K$ be a solution of invex equilibrium problem (4.1). Then

$$F(u, Tu, v) \geqslant 0, \quad \forall v \in K,$$

implies

$$F(v, Tv, u) \leqslant 0, \quad \forall v \in K, \tag{4.6}$$

since $F(.,.,.)$ and T are jointly pseudomonotone.

Since K is an invex set, $\forall u, v \in K, t \in [0,1]$, there exits an operator $\eta(.,.)$ such that $v_t = u + t\eta(v,u) \in K$. Taking $v = v_t$, in (4.6), we have

$$F(v_t, Tv_t, u) \leqslant 0, \quad \forall v_t \in K. \tag{4.7}$$

Now, using (4.7), we have

$$0 \leqslant F(v_t, Tv_t, v_t) \leqslant tF(v_t, Tv_t, v) + (1-t)F(v_t, Tv_t, u) \leqslant tF(v_t, Tv_t, v). \tag{4.8}$$

Dividing the inequality (4.8) by t and taking the limit as $t \longrightarrow 0$, since $F(.,.,.)$ and T are jointly hemicontinuous, we have

$$F(u, Tu, v) \geqslant 0, \quad \forall v \in K,$$

which shows that $u \in K$ is a solution of (2.1), the required result. □

For $F(u, Tu, v) = \langle Tu, \eta(v,u) \rangle$ Lemma 4.1 collapses to:

LEMMA 4.2. *Let T be η-pseudomonotone and η-hemicontinuous. If Condition C holds, then problem (4.2) is equivalent to finding $u \in K$ such that*

$$\langle Tv, \eta(u,v) \rangle \leqslant 0, \quad \forall v \in K. \tag{4.9}$$

Proof. Let $u \in K$ be solution of (4.2). Then (4.2) implies (4.9), since T is η-pseudomonotone. Since K is an invex set, $\forall u, v \in K, \quad t \in [0,1], \quad v_t = u + t\eta(v,u) \in K$. Taking $v = v_t$ in (4.9) and using Condition C, we have

$$0 \leqslant -\langle Tv_t, \eta(u, v_t) \rangle \leqslant t\langle Tv_t, \eta(v,u) \rangle .$$

Using the η-hemicontinuity of T and taking the limit as $t \longrightarrow 0$ in the above inequality, we have

$$\langle Tu, \eta(v,u) \geqslant 0, \quad \forall v \in K,$$

the required (4.2). □

Lemma 4.2 and Lemma 4.1 can be viewed as an extension and generalization of Minty's Lemma for variational-like inequalities and invex equilibrium problems, see Noor[27, 29] and Kinderlehrer and Stampacchia [12]. Problems (4.5) and (4.9) are also called the *dual invex equilibrium problems and dual variational-like inequalities.*

We use the auxiliary principle technique to suggest and analyze some iterative algorithms for solving invex equilibrium problem (4.1). For a given $u \in K$, consider the problem of finding $w \in K$ such that

$$\rho F(w, Tw, v) + \langle E'(w) - E'(u), \eta(v,w) \rangle \geqslant 0, \quad \forall v \in K, \tag{4.10}$$

which is known as the auxiliary invex equilibrium problem. Here $E'(u)$ is the differential of a strong preinvex function $E(u)$ at the point $u \in K.$ Problem (4.10) has a unique solution, since the functions E is a strongly preinvex function.

REMARK 4.1. The function $B(z,u) = E(z) - E(u) - \langle E'(u), \eta(z,u)\rangle$ associated with the preinvex function $E(u)$ is called the generalized Bregman function. We note that if $\eta(z,u) = z - u,$ then $B(z,u) = E(z) - E(u) - \langle E'(u), z-u\rangle$ is the well known Bregman function.

We remark that if $w = u,$ then w is a solution of (4.1). On the basis of this observation, we suggest and analyze the following iterative algorithm for solving (4.1) as long as (4.10) is easier to solve than (4.1).

Algorithm 4.1. For a given $u_0 \in H,$ calculate the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, Tu_{n+1}, v) + \langle E'(u_{n+1}) - E'(u_n), \eta(v, u_{n+1})\rangle \geqslant 0, \quad \forall v \in K, \tag{4.11}$$

Algorithm 4.1 is called the proximal point method for solving invex equilibrium problems (4.1). Note that if $\eta(v,u) = v - u,$ then Algorithm 4.1 reduces to Algorithm 3.7 for solving classical equilibrium problems (2.1).

If $F(u,Tu,v) = \langle Tu, \eta(v,u)\rangle,$ then Algorithm 4.1 collapse to the following method for solving variational-like inequalities (4.2).

Algorithm 4.2. For a given $u_0 \in H,$ calculate the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Tu_{n+1} + E'(u_{n+1}) - E'(u_n), \eta(v, u_{n+1})\rangle \geqslant 0 \quad \forall v \in K.$$

Here $E'(u)$ is the differential of a differentiable strongly preinvex function $E(u)$ at a point $u \in K,$ an invex set in $H.$ Algorithm 4.2 can be considered as a correct algorithm for solving variational-like inequalities (4.2). Note all the algorithms and their analysis which have been proposed and investigated in the setting of convexity are wrong. As we have pointed out earlier that the variational-like inequalities are only well-defined in the setting of invexity. In view of these facts and comments, results obtained must be modified and studied in the setting of invexity. In similar way, one can obtain the proximal point method for solving classical variational inequalities (2.3).

For the convergence analysis of Algorithms, we also need the following condition.

Assumptiom 4.1. The function $\eta(.,.)$ satisfies the following

$$\eta(u,v) = \eta(u,z) + \eta(z,v), \quad \forall u,v,z \in H. \tag{4.12}$$

Assumption 4.1 has been used to study the existence of a solution of variational-like inequalities by many authors. Note that $\eta(u,v) = 0$ if and only if $u = v \quad \forall u,v \in H.$

THEOREM 4.1. *Let the function $F(.,.,.)$ and T be jointly pseudomonotone. If E is differentiable strongly preinvex function with modulus $\beta > 0$ and (4.12) holds, then the approximate solution u_{n+1} obtained from Algorithm 4.1 converges to a solution $u \in K$ satisfying the invex equilibrium problems (4.1).*

Proof. Let $u \in K$ be a solution of (4.1). Then

$$-F(v, Tv, u) \geqslant 0, \quad \forall v \in K \tag{4.13}$$

since $F(.,.,.)$ and T are jointly pseudomonotone.

Taking $v = u_{n+1}$ in (4.13) and $v = u$ in (4.11), we have

$$-F(u_{n+1}, Tu_{n+1}, u) \geqslant 0, \tag{4.14}$$

and

$$\rho F(u_{n+1}, Tu_{n+1}, u) + \langle E'(u_{n+1}) - E'(u_n), \eta(u, u_{n+1})\rangle \geqslant 0. \tag{4.15}$$

Consider the function,

$$B(u,z) = E(u) - E(z) - \langle E'(z), \eta(u,z)\rangle \geqslant \beta\|\eta(u,z)\|^2, \tag{4.16}$$

using the strongly invexity of E.

Combining (4.12), (4.14), (4.15) and (4.16), we have

$$\begin{aligned} B(u,u_n)-B(u,u_{n+1}) &= E(u_{n+1})-E(u_n)-\langle E'(u_n), \eta(u,u_n)\rangle + \langle E'(u_{n+1}), \eta(u,u_{n+1})\rangle \\ &= E(u_{n+1}) - E(u_n) - \langle E'(u_n) - E'(u_{n+1}), \eta(u,u_{n+1})\rangle \\ &\quad - \langle E'(u_n), \eta(u_{n+1}, u_n)\rangle \\ &\geqslant \beta\|\eta(u_{n+1},u_n)\|^2 + \langle E'(u_{n+1}) - E'(u_n), \eta(u,u_{n+1})\rangle \\ &\geqslant \beta\|\eta(u_{n+1},u_n)\|^2 - F(u_{n+1}, Tu_{n+1}, u) \\ &\geqslant \beta\|\eta(u_{n+1},u_n)\|^2. \end{aligned}$$

If $u_{n+1} = u_n$, then clearly u_n is a solution of the invex equilibrium problem (4.1). Otherwise, the sequence $B(u,u_n) - B(u,u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} \|\eta(u_{n+1},u_n)\| = 0.$$

Now by using the technique of Zhu and Marcotte [56], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\overline{u}$ satisfying the invex equilibrium problem (4.1). □

For a given $u \in K$, consider the problem of finding $w \in K$ such that

$$\rho F(u, Tu, v) + \langle E'(w) - E'(u), \eta(v,w)\rangle \geqslant 0, \quad \forall v \in K, \tag{4.17}$$

which is called the auxiliary invex equilibrium problem. From the strongly preinvexity of the differentiable function E, it follows that problem (4.17) has a unique solution. Note that problems (4.17) and (4.10) are quite different. It is clear that if $w = u$, then w is a solution of invex equilibrium problem (2.1). This observation enables to suggest and analyze the following iterative method for solving (4.1).

Algorithm 4.3. For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_n, Tu_n, v) + \langle E'(u_{n+1}) - E'(u_n), \eta(v, u_{n+1})\rangle \geqslant 0, \quad \forall v \in K. \tag{4.18}$$

Note that, if $F(u,Tu,v) = \langle Tu, \eta(v,u)\rangle$, then Algorithm 4.3 reduces to the following iterative scheme for variational-like inequalities (4.2).

Algorithm 4.4. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_n + E'(u_{n+1}) - E'(u_n), \eta(v,u_{n+1})\rangle \geqslant 0, \quad \forall v \in K,$$

For $\eta(v,u) = v-u$, the invex set K becomes the convex set K, and consequently Algorithms 4.2 and 4.4 are exactly the iterative methods for solving convex equilibrium problems (2.1) and variational inequalities (2.3), which have been considered and analyzed in Section 3.

One can study the convergence analysis of Algorithm 4.3 using essentially the technique of Theorem 4.1. However, we give its proof for the sake of completeness and to convey an idea.

THEOREM 4.2. *Let the function $F(.,.,.)$ and T be partially relaxed strongly jointly η-monotone with constant $\alpha > 0$ and let $E(u)$ be strongly preinvex function with modulus $\beta > 0$. If $0 < \rho < \frac{\beta}{\alpha}$ and (4.12) holds, then approximate solution u_{n+1} obtained from Algorithm 4.3 converges to a solution $u \in K$ of the invex equilibrium problem (4.1).*

Proof. Let $u \in K$ be a solution of (4.1). Then taking $v = u_{n+1}$ in (4.1) and $v = u$ in (4.18), we have

$$F(u,Tu,u_{n+1}) \geqslant 0. \tag{4.19}$$

and

$$\rho F(u_n,Tu_n,u) + \langle E'(u_{n+1}) - E'(u_n), \eta(u,u_{n+1})\rangle \geqslant 0. \tag{4.20}$$

Now combining (4.16), (4.19) and (4.20), we have

$$\begin{aligned} B(u,u_n) - B(u,u_{n+1}) &\geqslant \beta\|\eta(u_{n+1},u_n)\|^2 + \langle E'(u_{n+1}) - E'(u_n), \eta(u,u_{n+1})\rangle \\ &\geqslant \beta\|\eta(u_{n+1},u_n)\|^2 - \rho\{F(u,Tu,u_n) + F(u_{n+1},Tu_{n+1},u)\} \\ &\geqslant \{\beta - \alpha\rho\}\|\eta(u_{n+1},u_n)\|^2, \end{aligned}$$

where we have used the fact that the bifunction $F(.,.,.)$ and T are partially relaxed strongly jointly monotone with constant $\alpha > 0$.

If $u_{n+1} = u_n$, then clearly u_n is a solution of (4.1). Otherwise, for $0 < \rho < \frac{\beta}{\alpha}$, the sequence $B(u,u_n) - B(u,u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} \|\eta(u_{n+1},u_n)\| = 0.$$

Now by using the technique of Zhu and Marcotte [56], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\overline{u}$ satisfying the invex equilibrium problem (4.1).

5. Regularized equiblibriun problems

Recently, the concept of convex set has been generalized in many directions, which has potential and important applications in various fields. A significant generalization of the convex sets is the introduction of uniformly prox-regular (smooth) sets, see [3, 51]. It is known that uniformly prox-regular sets are nonconvex sets and include convex sets as special case. In this Section, we introduce and consider a new class of equilibrium problems, known as regularized equilibrium problems. These regularized equilibrium problems are more general and include classical equilibrium problems, variational inequalities and related optimization problems as special cases. Since the underlying set is a nonconvex set, it is not possible to extend the usual projection and resolvent techniques for solving regularized mixed quasi equilibrium problems. Fortunately, these difficulties can be overcome by using the auxiliary principle, which has been used to develop some iterative schemes for solving various classes of equilibrium problems and variational inequalities in Sections 3 and 4. We point out that this technique does not involve the projection or resolvent of the operator and is flexible. Here we show that the auxiliary principle technique can be used to suggest and analyze a class of iterative methods for solving regularized (nonconvex) mixed quasi equilibrium problems. We also prove that the convergence of these new methods either require pseudomonotonicity or partially relaxed strongly monotonicity. As special cases, one can obtain several known and new results for variational inequalities and related optimization problems.

First of all, we recall the following well-known concepts from nonlinear convex analysis, see [3, 51].

DEFINITION 5.1. The proximal normal cone of K at u is given by

$$N^P(K;u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset $K,$ that is

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N^P(K;u)$ has the following characterization.

Let K be a closed subset in H. Then $\zeta \in N^P(K;u)$ if and only if there exists a constant $\alpha > 0$ such that

$$\langle \zeta, v - u\rangle \leqslant \alpha\|v - u\|^2, \quad \forall v \in K.$$

DEFINITION 5.2. The Clarke normal cone, denoted by $N^C(K;u)$, is defined as

$$N^C(K;u) = \overline{co}[N^P(K;u)],$$

where $\overline{co}$ means the closure of the convex hull.

Poliquin et al [51] and Clarke et al [3] have introduced and studied a new class of nonconvex sets, which are also called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. In particular, we have:

DEFINITION 5.3. For a given $r \in (0,\infty]$, a subset K is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K can be realized by an r-ball, that is, $\forall u \in K$ and $0 \neq \xi \in N^P(K;u)$ with $\|\xi\| = 1$, one has

$$\langle \xi, v-u\rangle \leqslant (1/2r)\|v-u\|^2, \quad \forall v \in K.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [3, 51]. It is clear that if $r = \infty$, then uniform r-prox-regularity of K is equivalent to the convexity of K. This fact plays an important part in the analysis of regularized equilibrium problems.

It is known that if K is a uniformly r-prox-regular set, then the proximal normal cone $N^P(K;u)$ is closed as a set-valued mapping. Thus, we have $N^C(K;u) = N^P(K;u)$. For sake of simplicity, we denote $N(K;u) = N^C(K;u) = N^P(K;u)$ and take $\gamma = \frac{1}{2r}$. Clealry $\gamma = 0$ if and only if $r = \infty$.

From now onward, the set K is uniformly r-prox-regular set, unless otherwise specified.

For given nonlinear continuous trifunction $F(.,.,.)$, we consider the problem of finding $u \in K$ such that

$$F(u,Tu,v) + \gamma\|v-u\|^2 \geqslant 0, \quad \forall v \in K, \tag{5.1}$$

which is called the *regularized equilibrium problem* introduced and investigated by Noor and Noor [45].

Note that if $\gamma = 0$, then uniformly prox-regular set K becomes the convex set K and consequently problem (5.1) reduces to finding $u \in K$ such that

$$F(u,Tu,v) \geqslant 0, \quad \forall v \in K,$$

which is the equilibrium problems (2.1) considered in Section 3 using the auxiliary principle technique.

If $F(u,Tu,v) = \langle Tu, v-u\rangle$, then problem (5.1) is equivalent to fining $u \in K$ such that

$$\langle Tu, v-u\rangle + \gamma\|u-v\|^2 \geqslant 0, \quad \forall u,v \in K, \tag{5.2}$$

which is known as the regularized variational inequalities. Noor [30] has used the auxiliary principle technique to suggest and analyze some iterative schemes for solving the regularized variational inequalities (5.2). In particular, for suitable and appropriate choice of the operators and the spaces, one can obtain a number of new and previously known classes of equilibrium problems and variational inequalities as special cases of problem (5.1).

We use the auxiliary principle technique to suggest and analyze some iterative methods for solving the regularized equilibrium problems (5.1).

For a given $u \in K$, where K is a prox-regular set in H, consider the problem of finding $w \in K$ such that

$$\rho F(w, Tw, v) + \langle w - u, v - w \rangle + \gamma \|v - w\|^2 \geqslant 0, \quad \forall v \in K, \tag{5.3}$$

where $\rho > 0$ is a constant. Inequality of type (5.3) is called the auxiliary regularized equilibrium problem. Note that if $w = u$, then w is a solution of (5.1). This simple observation enables us to suggest the following iterative method for solving (5.1).

Algorithm 5.1. For a given $u_0 \in K$, compute the approximate solution $u_{n+1} \in K$ by the iterative scheme

$$\rho F(u_{n+1}, Tu_{n+1}, v) + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geqslant -\gamma \|u_{n+1} - u_n\|^2 \quad \forall v \in K. \tag{5.4}$$

Algorithm 5.1 is called the proximal point algorithm for solving regularized equilibrium problems (5.1). In particular, if $\gamma = 0$, then the r-prox-regular set K becomes the standard convex set K, and consequently Algorithm 5.1 reduces to Algorithm 3.2 for solving the equilibrium problem (2.1).

For $F(u, Tu, v) = \langle Tu, v - u \rangle$, Algorithm 5.1 reduces to:

Algorithm 5.2. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle Tu_{n+1} + u_{n+1} - u_n, v - u_{n+1} \rangle + \gamma \|v - u_{n+1}\|^2 \geqslant 0, \quad \forall v \in K,$$

which is called the proximal method for solving the regularized variational inequalities (5.2). In particular, if $\gamma = 0$, then the r-prox-regular set K becomes the standard convex set K, and consequently Algorithm 5.2 reduces to Algorithm 3.3 for solving the variational inequalities (2.3) which is considered in Section 3.

We now suggest another method by using the auxiliary principle technique, the convergence of which requires only the partially relaxed strongly jointly monotonicity, which is a weaker condition than cocoercivity.

For a given $u \in K$, consider the problem of finding $w \in K$ such that

$$\rho F(u, Tu, v) + \langle w - u, v - w \rangle + \gamma \|v - w\|^2 \geqslant 0, \quad \forall v \in K, \tag{5.5}$$

which is also called the auxiliary uniformly regularized equilibrium problem. Note that problems (5.3) and (5.5) are quite different. If $w = u$, then clearly w is a solution of the regularized equilibrium problem (5.1). This fact enables us to suggest and analyze the following iterative method for solving (5.1).

Algorithm 5.3. For a given $u_0 \in K$, compute the approximate solution $u_{n+1} \in K$ by the iterative scheme

$$\rho F(u_n, Tu_n, v) + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geqslant -\gamma \|v - u_{n+1}\|^2, \quad \forall v \in K. \tag{5.6}$$

Note that for $\gamma = 0$, the r-prox-regular set K becomes a convex set K and Algorithm 5.4 reduces to Algorithm 3.4 for solving the variational inequalities (2.3).

For $F(u, Tu, v) = \langle Tu, v - u \rangle$, Algorithm 5.3 reduces to:

Algorithm 5.4. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle Tu_n + u_{n+1} - u_n, v - u_{n+1} \rangle + \gamma \|v - u_{n+1}\|^2 \geqslant 0, \quad \forall v \in K,$$

for solving the regularized variational inequalities (5.2). In particular, if $r = \infty$, then the r-prox-regular set K becomes the standard convex set K, and consequently Algorithm 5.4 reduces to Algorithm 3.5 for solving the variational inequalities (2.2) which is considered in Section 3. In a similar way, for suitable and appropriate choice of the operators and spaces, one can obtain a number of new and known algorithms for solving various classes of (regularized) variational inequalities and related optimization problems. Using essentially the technique developed in Section 3, one can study the convergence analysis of Algorithm 5.1 and Algorithm 5.3.

6. Hemiequilibrium problems

Variational inequalities have been extended and generalized in several direction using novel and new techniques. There are significant developments of variational inequalities related with multivalued, nonmonotone, nonconvex optimization and structural analysis. An important and useful generalization of variational inequalities is a class of variational inequalities, which is known as hemivariational inequalities. The hemivariational inequalities were introduced and investigated by Panagiotopoulos [48, 49] by using the concept of the generalized directional derivatives of nonconvex and nondifferentiable functions. This class has important applications in structural analysis and nonconvex optimization. In particular, it has been shown [5] that if a nonsmooth and nonconvex superpotential of a structure is quasidifferentiable then these problems can be studied via hemivariational inequalities. The solution of the hemivariational inequalities gives the position of the state equilibrium of the structure. It is worth mentioning that hemivariational inequalities can be viewed as a special case of mildly nonlinear variational inequalities, considered and introduced by Noor [22]. However, numerical techniques considered for solving mildly nonlinear variational inequalities can not be extended for hemivariational inequalities due to the presence of nonlinear and nondifferentiable terms. For the applications and formulation of the hemivariational inequalities, see [5, 21, 39, 4-41, 43, 46, 48, 49] and the references therein.

Thus it is clear that hemivariational inequalities and equilibrium problems are different generalizations of variational inequalities. It is natural to consider the unification of these two generalized problems. Motivated and inspired by this fact, we consider another class of equilibrium problems which is called the *hemiequilibrium problems*. The class of the hemiequilibrium problems includes the hemivariational inequalities and equilibrium problems as special cases. In this Section, we show that the auxiliary principle technique can be used to suggest some iterative schemes for hemiequilibrium problems. We also prove that the convergence of these methods require either pseudomonotonicity or partially relaxed strongly monotonicity. These are weaker conditions than monotonicity. As a special case, we obtain new iterative schemes for solving hemivariational inequalities and related optimization problems. The comparison of these methods with other methods is a subject of future research.

First of all we recall the following well known concepts from the nonsmooth analysis, see [3].

DEFINITION 6.1. Let f be locally Lipschitz continuous at a given point $x \in H$ and v be any other vector in H. The Clarke's generalized directional derivative of f at x in the direction v, denoted by $f^0(x,v)$, is defined as

$$f^0(x,v) = \lim_{t \longrightarrow 0^+} \sup_{h \longrightarrow 0} \frac{f(x+h+tv) - f(x+h)}{t}.$$

The generalized gradient of f at x, denoted $\partial f(x)$, is defined to be subdifferential of the function $f^0(x;v)$ at 0. That is

$$\partial f(x) = \{w \in H : \langle w, v \rangle \leqslant f^0(x;v), \quad \forall v \in H\}.$$

LEMMA 6.1. *Let f be locally Lipschitz continuous at a given point $x \in H$ with a constant L. Then*

(i) $\partial f(x)$ is a none-empty compact subset of H and $\|\xi\| \leqslant L$ for each $\xi \in \partial f(x)$.

(ii) For every $v \in H$, $f^0(x;v) = max\{\langle \xi, c \rangle : \xi \in \partial f(x)\}$.

(iii) The function $v \longrightarrow f^0(x;v)$ is finite, positively homogeneous, subadditive, convex and continuous.

(iv) $f^0(x;-v) = (-f)^0(x;v)$.

(v) $f^0(x;v)$ is upper semicontinuous as a function of $(x;v)$.

(vi) $\forall x \in H$, there exists a constant $\alpha > 0$ such that

$$|f^0(x;v)| \leqslant \alpha \|v\|, \quad \forall v \in H.$$

If f is convex on K and locally Lipschitz continuous at $x \in K$, then $\partial f(x)$ coincides with the subdifferential $f'(x)$ of f at x in the sense of convex analysis , and $f^0(x;v)$ coincides with the directional derivative $f'(x;v)$ for each $v \in H$. That is, $f^0(x;v) = \langle f'(x), v \rangle, \forall v \in H$.

For a given nonlinear continuous trifunction $F(.,.,.)$, consider the problem of finding $u \in K$ such that

$$F(u, Tu, v) + \int_\Omega f^0(u; v-u) d\Omega \geqslant 0, \quad \forall v \in K. \tag{6.1}$$

Here $f^0(u;v-u) := f^0(x, u(x); v(x) - u(x))$ denotes the generalized directional derivative of the function $f(x,.)$ at $u(x)$ in the direction $v(x) - u(x)$. Problem of type (6.1) is called the *hemiequilibrium problems* introduced and studied by Noor [39, 41].

If $F(u, Tu, v) = \langle Tu, v-u \rangle$, then problem (6.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u \rangle + \int_\Omega f^0(u; v-u) d\Omega \geqslant 0, \quad \forall v \in K, \tag{6.2}$$

which is known as the hemivariational inequalities introduced and studied by Panagiotopoulos [48, 49] in order to formulate variational principles connected to energy functions which are neither convex nor smooth. It is has been shown that the technique of hemivariational inequalities is very efficient to describe the behaviour of complex structure arising in enginnering and industrial sciences, see the references.

If the nonlinear continuous function is differentiable, then $f^0(u,v) = \langle f'(u), v\rangle$, $\forall v \in H,$ and consequently the hemivariationa inequality (6.2) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle + \langle f'(u), v-u\rangle \geqslant 0, \quad \forall v \in K,$$

which is known as the mildly nonlinear variational inequality. It is worth mentioning that mildly nonlinear variational inequalities were first introduced and studied by Noor [22] in 1975. Mildly nonlinear variational inequalities have been generalized and extended in several directions. Mildly nonlinear variational inequalities have important applications in various branches of pure and applied sciences. Thus we conclude the hemiequilibrium problem is more general and includes the several classes of equilibrium problems, variational inequalities and related optimization problems as special cases.

We again use the auxiliary principle technique to suggest and analyze a class of iterative methods for solving the hemiequilibrium problems (6.1). The analysis is in the spirit of Section 3. However, to convey an idea of the technique and for the sake of completeness, we sketch the main points.

For a given $u \in K,$ consider the auxiliary problem of finding $w \in K$ such that

$$\rho F(w, Tw, v) + \langle E'(w) - E'(u), v-w\rangle + \rho \int_\Omega f^0(w; v-w)d\Omega \geqslant 0, \quad \forall v \in K, \tag{6.3}$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly convex function $E(u)$ at $u \in K.$ Since $E(u)$ is a strongly convex function, problem (6.3) has an unique solution. We note that if $w = u,$ then clearly w is solution of the hemiequilibrium problem (6.1). This observation enables us to suggest and analyze the following iterative method for solving (6.1).

Algorithm 6.1. For a given $u_0 \in H,$ compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}\rho F(u_{n+1}, Tu_{n+1}, v) &+ \langle E'(u_{n+1}) - E'(u_n), v-u_{n+1}\rangle \\ &+ \rho \int_\Omega f^0(u_{n+1}; v-u_{n+1})d\Omega \geqslant 0, \quad \forall v \in K,\end{aligned} \tag{6.4}$$

where $\rho > 0$ is a constant. Algorithm 6.1 is called the proximal method for solving the hemiequilibrium problems (6.1).

If $F(u, Tu, v) = \langle Tu, v-u\rangle$, then Algorithm 6.1 reduces to:

Algorithm 6.2. For a given $u_0 \in H,$ calculate the approximate solution u_{n+1} by the iterative schemes

$$\langle \rho Tu_{n+1} + E'(u_{n+1}) - E'(u_n), v-u_{n+1}\rangle + \rho \int_\Omega f^0(u_{n+1}; v-u_{n+1})d\Omega \geqslant 0, \quad \forall v \in K,$$

is called the proximal point method for solving hemivariational inequalities (6.2) and is due to Noor [40]. In brief, for suitable and appropriate choice of the operators and the spaces, one can obtain a number of known and new algorithms for solving variational-like inequalities and related problems.

THEOREM 6.1. *Let $F(.,.,.)$ and T be jointly psudomonotone with respect to $\int_\Omega f^0(u;v-u)d\Omega$. Let E be differentiable strongly convex function with module $\beta > 0$. Then the approximate solution u_{n+1} obtained from Algorithm 6.1 converges to a solution $u \in K$ satisfying (6.1).*

Proof. Let $u \in K$ be a solution of (6.1). Then

$$F(u, Tu, v) + \int_\Omega f^0(u; v-u)d\Omega \geqslant 0, \quad \forall v \in K,$$

implies that

$$-F(v, Tv, u) - \int_\Omega f^0(v; v-u)d\Omega \geqslant 0, \quad \forall v \in K, \tag{6.5}$$

since $F(.,.,.)$ and T are jointly pseudomonotone with respect to $\int_\Omega f^0(u; v-u)d\Omega$.
Taking $v = u$ in (6.4) and $v = u_{n+1}$ in (6.5), we have

$$\rho F(u_{n+1}, Tu_{n+1}, u) + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \geqslant -\rho \int_\Omega f^0(u_{n+1}, ; u - u_{n+1})d\Omega. \tag{6.6}$$

and

$$-F(u_{n+1}, Tu_{n+1}, u) - \int_\Omega f^0(u_{n+1}, ; u_{n+1} - u)d\Omega \geqslant 0. \tag{6.7}$$

Now we consider the function

$$\begin{aligned} B(u, w) &= E(u) - E(w) - \langle E'(w), u - w\rangle \\ &\geqslant \beta\|u - w\|^2, \quad \text{(using strong convexity of E).} \end{aligned} \tag{6.8}$$

Combining (6.8), (6.6) and (6.7), we have

$$\begin{aligned} B(u, u_n) - B(u, u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_{n+1}), u_{n+1} - u_n\rangle \\ &\quad + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \\ &\geqslant \beta\|u_{n+1} - u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \\ &\geqslant \beta\|u_{n+1} - u_n\|^2 - \rho F(u_{n+1}, Tu_{n+1}, u) - \rho\int_\Omega f^0(u_{n+1}; u - u_{n+1})d\Omega \\ &\geqslant \beta\|u_{n+1} - u_n\|^2. \end{aligned}$$

If $u_{n+1} = u_n$, then clearly u_n is a solution of the hemiequilibrium problems (6.1). Otherwise, it follows that $B(u, u_n) - B(u, u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

Now using the technique of Zhu and Marcotte [56], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the hemiequilibrium problems (6.1). □

We again use the auxiliary principle technique to suggest and analyze another iterative method for solving the hemiequilibrium problems (6.1).

For a given $u \in K$, find $w \in K$ such that

$$\rho F(u,Tu,v) + \langle E'(w) - E'(u), v - w\rangle + \rho \int_\Omega f^0(u; v-w)d\Omega, \forall v \in K, \tag{6.9}$$

where $E'(u)$ is the differential of a strongly convex function $E(u)$ at $u \in K$. Problem (6.9) has a unique solution, since E is strongly convex function. Note that problems (6.3) and (6.9) are quite different problems. It is clear that for $w = u$, w is a solution of (6.1). This fact allows us to suggest and analyze another iterative method for solving the hemiequilibrium problem (6.1).

Algorithm 6.3. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_n,Tu_n,v) + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \geqslant -\rho \int_\Omega f^0(u_n; v-u_{n+1})d\Omega, \forall v \in K, \tag{6.10}$$

Note that for $F(u,Tu,v) = \langle Tu, v-u\rangle$, Algorithm 6.3 reduces to:

Algorithm 6.4. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Tu_n + E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \geqslant -\rho \int_\Omega f^0(u_n; v-u_{n+1})d\Omega, \forall v \in K,$$

for solving the hemivariational inequalities (6.2), which is due to Noor [40]. Similarly for suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving equilibrium problems and variational inequalities.

We now consider the convergence analysis of Algorithm 6.3 using essentially the technique of Theorem 6.1.

THEOREM 6.2. *Let $F(.,.,.)$ and $\int_\Omega f^0(u; v-u)d\Omega$ be partially relaxed strongly monotone with constants $\gamma > 0$ and $\alpha > 0$ respectively. If E is strongly convex function with modulus $\beta > 0$ and $0 < \rho < \beta/(\alpha+\gamma)$, then the approximate solution u_{n+1} obtained from Algorithm 6.3 converges to a solution of (6.1).*

Proof. Let $u \in K$ be solution of (6.1). Setting $v = u_{n+1}$ in (6.1) and $v = u$ in (6.10), we have

$$F(u,Tu,u_{n+1}) + \int_\Omega f^0(u; u_{n+1}-u)d\Omega \geqslant 0. \tag{6.11}$$

and

$$\rho F(u_n,Tu_n,u) + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \geqslant -\rho \int_\Omega f^0(u_n; u-u_{n+1})d\Omega. \tag{6.12}$$

As in Theorem 6.1 and from (6.11) and (6.12), we have

$$\begin{aligned} B(u,u_n) - B(u,u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_{n+1}), u_{n+1} - u_n\rangle \\ &\quad + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \\ &\geqslant \beta\|u_{n+1} - u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \\ &\geqslant \beta\|u_{n+1} - u_n\|^2 - \rho F(u_n, Tu_n, u) - \rho\int_\Omega f^0(u_n; u - u_{n+1})d\Omega \\ &\geqslant \beta\|u_{n+1} - u_n\|^2 - \rho\{F(u_n, Tu_n, u) + F(u, Tu, u_{n+1})\} \\ &\quad - \rho\{\int_\Omega f^0(u; u_{n+1} - u)d\Omega + \int_\Omega f^0(u_n; u - u_{n+1})d\Omega\} \\ &\geqslant \beta\|u_{n+1} - u_n\|^2 - \rho(\alpha + \gamma)\|u_{n+1} - u_n\|^2, \end{aligned}$$

where we have used the fact that $F(.,.,.)$ and T and $\int_\Omega f^0(x,.;.)d\Omega$ are partially relaxed strongly jointly monotone with constants $\alpha > 0$ and $\gamma > 0$ respectively. If $u_{n+1} = u_n$, then clearly u_n is a solution of the hemiequilibrium problems (6.1). Otherwise, for $0 < \rho < \frac{\beta}{\alpha+\gamma}$, it follows that $B(u,u_n) - B(u,u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

Now using the technique of Zhu and Marcotte [56], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the hemiequilibrium problem (6.1). □

7. Nonconvex equilibrium problems

A significant generalization of the convex functions is the introduction of g-convex functions. It is well-known that the g-functions and g-convex sets may not be convex functions and convex sets, see [4]. However, it has been shown that the class of g-convex function have some nice properties, which the convex functions have. In particular, it been shown [34] that the minimum of the g-functions over the g-convex sets can be characterized by a class of variational inequalities, which is called the *nonconvex (g-convex) variational inequality*. Inspired and motivated by the recent research work going in this field, we consider a new class of equilibrium problems, which is called *nonconvex equilibrium problems,* where the convex set is replaced by the so-called g-convex set. We again use the auxiliary principle technique to suggest a class of iterative methods for solving nonconvex equilibrium problems. The convergence of these methods requires only that the operator is partially relaxed strongly g-monotone, which is weaker than g-monotonicity. We also use the auxiliary principle technique to suggest and analyze a proximal method for solving equilibrium problem, which was introduced as a regularization of convex optimization in Hilbert space. We prove that the convergence of proximal method requires only g-pseudomonotonicity, which is a weaker condition.

First of all, we recall the following concepts and results.

DEFINITION 7.1. Let K be any set in H. The set K is said to be g-convex, if there exists a function $g: K \longrightarrow K$ such that

$$g(u)+t(g(v)-g(u)) \in K, \quad \forall u, v \in K, t \in [0,1].$$

Note that every convex set is g-convex, but the converse is not true, see [4]. In passing, we remark that the notion of the g-convex set was introduced by Noor [27] implicitly in 1988.

From now onward, we assume that K is a g-convex set, unless otherwise specified.

DEFINITION 7.2. The function $f: K \longrightarrow H$ is said to be g-convex, if

$$f(g(u)+t(g(v)-g(u))) \leqslant (1-t)f(g(u))+tf(g(v)). \quad \forall u, v \in K, t \in [0,1].$$

Clearly every convex function is g-convex, but the converse is not true.

DEFINITION 7.3. A function f is said to be strongly g-convex on the g-convex set K with modulus $\mu > 0$, if, $\forall u, v \in K, t \in [0,1]$,

$$f(g(u)+t(g(v)-g(u)) \leqslant (1-t)f(g(u))+tf(g(v))-t(1-t)\mu\|g(v)-g(u)\|^2.$$

Using the convex analysis techniques, one can easily show that the differentiable g-convex function f is strongly g-convex function if and only if

$$f(g(v))-f(g(u)) \geqslant \langle f'(g(u)), g(v)-g(u)\rangle + \mu\|g(v)-g(u)\|^2$$

or

$$\langle f'(g(u))-f'(g(v)), g(u)-g(v)\rangle \geqslant 2\mu\|g(v)-g(u)\|^2,$$

that is, $f'(g(u))$ is a strongly monotone operator.

It is well-known that the g-convex functions are not convex function, but they have some nice properties which the convex functions have. Note that for $g = I$, the g-convex functions are convex functions and definition 7.3 is a well known result in convex analysis.

For given nonlinear continuous trifunction $F(.,.,.)$ and the operator g, we consider the problem of finding $u \in K$ such that

$$F(g(u), T(g(u), g(v)) \geqslant 0, \quad \forall v \in K, \tag{7.1}$$

which is called is called the *nonconvex equilibrium problem with trifunction.* For $g \equiv I$, where I is the identity operator, the g-convex set K becomes the convex set K and consequently, problem (7.1) is equivalent to the problem (2.1).

We note that for $F(g(u), T(g(u)), g(v)) = \langle Tg(u), g(v)-g(v)\rangle$, the problem (7.1) is equivalent to finding $u \in K$ such that

$$\langle Tg(u), g(v)-g(u)\rangle \geqslant 0, \quad \forall v \in K. \tag{7.2}$$

Inequality (7.2) is known as the *nonconvex variational inequality*, which was introduced by Noor[34]. It is worth mentioning that the nonconvex variational inequalities (7.2) are quite different from the so-called general variational inequalities, introduced and studied by Noor [27] in 1988. For the applications and numerical methods of general variational inequalities; see Noor [27, 29, 32] and the references therein.

In brief, for a suitable and appropriate choice of the operators T, g, and the space H, one can obtain a wide class of equilibrium, variational inequalities and complementarity problems as special cases of problems (7.1). This clearly shows that problem (7.1) is quite general and unifying one.

DEFINITION 7.4. The trifunction $F(.,.,.)$ and the operatos T are said to be :

(i) *partially relaxed strongly jointly g-monotone*, if there exists a constant $\alpha > 0$ such that

$$F(g(u),T(g(u))g(v)) + F(g(v),T(g(v)),g(z)) \leqslant \alpha\|g(z)-g(u)\|^2, \quad \forall u,v,z \in K.$$

(ii) *jointly g-monotone*, if

$$F(g(u),T(g(u)),g(v)) + F(g(v),T(g(v)),g(u)) \leqslant 0, \quad \forall u,v \in K.$$

(iii) *jointly g-pseudomonotone,* if

$$F(g(u),T(g(u)),g(v)) \geqslant 0 \Longrightarrow -F(g(v),T(g(v)),g(u)) \geqslant 0, \quad \forall u,v \in K.$$

(iv) *jointly g-hemicontinuous,* $\forall u,v \in K, t \in [0,1]$, if the mapping $F(g(u)+t(g(v)-g(u)),T(g(u)+t(g(v)-g(u)),g(v))$ is continuous.

We remark that if $z = u$, then partially relaxed strongly jointly g-monotonicity is exactly jointly g-monotonicity of $F(.,.,.)$. For $g \equiv I$, the indentity operator, then Definition 7.1 reduces to the standard definitions of partially relaxed jointly strongly monotonicity, jointly monotonicity and jointly pseudomonotonicity introduced in Section 3.

LEMMA 7.1. *Let $F(.,.,.)$ be jointly g-pseudomonotone, jointly g-hemicontinuous and g-convex with respect to third argument. Then the nonconvex equilibrium problem* (7.1) *is equivalent to finding $u \in K$ such that*

$$-F(g(v),T(g(v)),g(u)) \geqslant 0, \quad \forall v \in K. \tag{7.3}$$

Proof. Let $u \in K$ be a solution of (7.1). Then

$$F(g(u),T(g(u)),g(v)) \geqslant 0, \quad \forall v \in K$$

which implies

$$-F(g(v),T(g(v)),g(u)) \geqslant 0, \quad \forall v \in K,$$

since $F(.,.,.)$ and T are jointly g-pseudomonotone.

Conversely, let $u \in K$ satisfy (7.3). Since K is a g-convex set, $\forall u,v \in K$, $t \in [0,1]$,

$$g(v_t) = g(u) + t(g(v)-g(u)) \equiv (1-t)g(u) + tg(v) \in K.$$

Taking $g(v) = g(v_t)$ in (7.3), we have

$$F(g(v_t),T(g(v_t)),g(u)) \leqslant 0. \tag{7.4}$$

Now using (7.4) and g-convexity of $F(.,.,.)$ with respect to third argument, we have

$$\begin{aligned}0 &\leqslant F(g(v_t),T(g(v_t)),g(v_t))\\ &= F(g(v_t),T(g(v_t)),(1-t)g(u)+tg(v))\\ &\leqslant tF(g(v_t),T(g(v_t)),g(v))+(1-t)F(g(v_t),T(g(v_t)),g(u))\\ &\leqslant tF(g(v_t),T(g(v_t)),g(v))\end{aligned}$$

Dividing the above inequality by t and letting $t \longrightarrow 0,$ we have

$$F(g(u),T(g(u)),g(v)) \geqslant 0, \quad \forall v \in K,$$

the required (7.1). □

REMARK 7.1. Problem (7.3) is known as the *dual mixed quasi nonconvex equilibrium problem.* One can easily show that the solution set of problem (7.3) is closed and g-convex set. From Lemma 7.1, it follows that the solution set of problems (7.1) and (7.3) are the same. This inter relationship has played an important role in the study of well-posedness of equilibrium problems and variational inequalities. In fact, Lemma 7.1 can be viewed as a natural generalization and extension of a well-known Minty's Lemma in variational inequalities theory.

For a given $u \in K$, consider the problem of finding $w \in K$ satisfying the auxiliary nonconvex equilibrium problem

$$\rho F(g(u),T(g(u)),g(v))+\langle E'(g(w))-E'(g(u)),g(v)-g(w)\rangle \geqslant 0 \quad \forall v \in K, \quad (7.5)$$

where $\rho > 0$ is a constant and E' is the differential of a strongly g-convex function E. Problem (7.5) has a unique solution, since the function E is strongly g-convex function. We note that if $w = u$, then clearly w is a solution of the nonconvex equilibrium problems (7.1). This observation enables us to suggest the following method for solving (7.1).

Algorithm 7.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\rho F(g(u_n),T(g(u_n)),g(v))+\langle E'(g(u_{n+1}))-E'(g(u_n)),g(v)-g(u_{n+1})\rangle \geqslant 0, \forall v \in K, \quad (7.6)$$

where $\rho > 0$ is a constant.

If $F(g(u),T(g(u)),(v)) = \langle Tg(u),g(v)-g(u)\rangle$, then Algorithm 7.1 reduces to:

Algorithm 7.2. For a given $u_0 \in H,$ compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Tg(u_n)+E'(g(u_{n+1}))-E'(g(u_n)),g(v)-g(u_{n+1})\rangle \geqslant 0, \quad \forall v \in K,$$

for solving nonconvex variational ineqaulities (7.2). For suitable and appropriate choice of the operators and the space H, one can obtain various new and known methods for solving equilibrium, variational inequalities and complementarity problems.

For the convergence analysis of Algorithm 7.1, we need the following result.

THEOREM 7.1. *Let* $E(u)$ *be a strongly* g*-convex with modulus* $\beta > 0$. *and the operator* T *is partially relaxed jointly strongly monotone with constant* $\alpha > 0$. *If* $0 < \rho < \frac{\beta}{\alpha}$, *then the approximate solution* u_{n+1} *obtained from Algorithm* 7.1 *converges to a solution of* (7.1).

Proof. Let $u \in K$ be a solution of (7.1). Then

$$F(g(u), T(g(u)), g(v)) \quad \geqslant 0, \quad \forall v \in K, \tag{7.7}$$

where $\rho > 0$ is a constant.

Now taking $v = u_{n+1}$ in (7.7) and $v = u$ in (7.6), we have

$$F(g(u), T(g(u)), g(u_{n+1})) \geqslant 0. \tag{7.8}$$

and

$$\rho F(g(u_n), T(g(u_n)), g(u)) + \langle E'(g(u_{n+1})) - E'(g(u_n)), g(u) - g(u_{n+1})\rangle \geqslant 0. \tag{7.9}$$

We consider the Bregman function

$$\begin{aligned} B(u, w) &= E(g(u)) - E(g(w)) - \langle E'(g(w)), g(u) - g(w)\rangle \\ &\geqslant \beta \|g(u) - g(w)\|^2, \quad \text{using strongly } g\text{-convexity of } E. \end{aligned} \tag{7.10}$$

Now combining (7.8), (7.9) and (7.10), we have

$$\begin{aligned} B(u, u_n) - B(u, u_{n+1}) &= E(g(u_{n+1})) - E(g(u_n)) - \langle E'(g(u_n)), g(u_{n+1}) - g(u_n) \\ &\quad + \langle E'(g(u_{n+1})) - E'(g(u_n)), g(u) - g(u_{n+1})\rangle \\ &\geqslant \beta \|g(u_{n+1}) - g(u_n)\|^2 + \langle E'(g(u_{n+1})) \\ &\quad - E'(g(u_n)), g(u) - g(u_{n+1})\rangle \\ &\geqslant \beta \|g(u_{n+1}) - g(u_n)\|^2 - \rho F(g(u_n), T(g(u_n)), g(u)) \\ &\quad - \rho F(g(u), T(g(u)), g(u_{n+1})) \\ &\geqslant \{\beta - \rho\alpha\} \|g(u_{n+1}) - g(u_n)\|^2, \end{aligned}$$

where we have used the fact that $F(.,.,.)$ and T are partially relaxed jointly strongly monotone with constant $\alpha > 0$.

If $u_{n+1} = u_n$, then clearly u_n is a solution of the nonconvex equilibrium problems (7.1). Otherwise, for $0 < \rho < \frac{\beta}{\alpha}$, it follows that $B(u, u_n) - B(u, u_{n+1})$ is nonnegative and we must have

$$\lim_{n \to \infty} \|u_{n+1} - u_n\| = 0.$$

Now using the technique of Zhu and Marcotte [56], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the nonconvex equilibrium problem (7.1). □

We now show that the auxiliary principle technique can be used to suggest and analyze a proximal method for solving nonconvex equilibrium problems (7.1). We prove that the convergence of the proximal method requires only jointly pseudomonotonicity, which is a weaker condition than monotonicity.

For a given $u \in K$ consider the auxiliary problem of finding $w \in K$ such that

$$\rho F(g(w), T(g(w)), g(v)) + \langle E'(g(w)) - E'(g(u)), g(v) - g(w) \rangle \geqslant 0, \forall v \in K, \quad (7.11)$$

where $\rho > 0$ is a constant and E' is the differential of a strongly differentiable g-convex function E. Since E is strongly differentiable g-convex function, there exists a unique solution of the auxiliary problem (7.11). Note that if $w = u$, then w is a solution of (7.1). This fact enables us to suggest the following iterative method for solving nonconvex equilibrium problems (7.1).

Algorithm 7.3. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(g(u_{n+1}), T(g(u_{n+1})), g(v)) + \langle E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1}) \rangle \geqslant 0, \forall v \in K. \quad (7.12)$$

Algorithm 7.3 is known as the proximal method for solving nonconvex equilibrium problem (7.1).

If $F(g(u), g(v)) = \langle Tg(u), g(v) - g(u) \rangle$, then Algorithm 7.3 reduces to:

Algorithm 7.4. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T(g(u_{n+1})) + E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1}) \rangle \geqslant 0, \quad \forall v \in K.$$

In a similar way, one can obtain a variant form of proximal methods for solving variational inequalities and equilibrium problems as special cases.

We now study the convergence analysis of Algorithm 3.4 using the technique of Theorem 3.1. For the sake of completeness and to convey an idea of the techniques involved, we sketch the main points only.

THEOREM 7.2. *Let $E(u)$ be a strongly g-convex with modulus $\beta > 0$. and the trifunction $F(.,.,.)$ be jointly pseudomonotone, then the approximate solution u_{n+1} obtained from Algorithm 7.2 converges to a exact solution of (7.1).*

Proof. Let $u \in K$ be a solution of (7.1). Then

$$F(g(u), T(g(u)), g(v)) \geqslant 0, \forall v \in K,$$

which implies that

$$-F(g(v), T(g(v), g(u)) \geqslant 0, \forall v \in K, \quad (7.13)$$

since $F(.,.,.)$ and T are jointly pseudomonotone.

Taking $v = u_{n+1}$ in (7.13) and $v = u$ in (7.12), we have

$$-F(g(u_{n+1}), T(g(u_{n+1})), g(u)) \geqslant 0 \quad (7.14)$$

and

$$\rho F(g(u_{n+1}), T(g(u_{n+1})), g(u)) \geqslant -\langle E'(g(u_{n+1})) - E'(g(u_n)), g(u) - g(u_{n+1}) \rangle. \quad (7.15)$$

Now as in Theorem 7.1, from (7.10), (7.14) and (7.15), we have

$$\begin{aligned}B(u,u_n)-B(u,u_{n+1}) &= E(g(u_{n+1}))-E(g(u_n))-\langle E'(g(u_n)),g(u_{n+1})-g(u_n)\\ &\quad+\langle E'(g(u_{n+1}))-E'(g(u_n)),g(u)-g(u_{n+1})\rangle\\ &\geqslant \beta\|g(u_{n+1})-g(u_n)\|^2\\ &\quad+\langle E'(g(u_{n+1}))-E'(g(u_n)),g(u)-g(u_{n+1})\rangle\\ &\geqslant \beta\|g(u_{n+1})-g(u_n)\|^2-\rho F(g(u_{n+1}),T(g(u_{n+1})),g(u))\\ &\geqslant \beta\|g(u_{n+1})-g(u_n)\|^2.\end{aligned}$$

If $u_{n+1} = u_n$, then clearly u_n is a solution of the nonconvex equilibrium problems (7.1). Otherwise, it follows that $B(u,u_n)-B(u,u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = 0.$$

Now using the technique of Zhu and Marcotte [56], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the nonconvex equilibrium problem (7.1).

8. Well-posed equilibrium problems

In recent years, much attention has been given to introduce the concept of well-posedness for variational of variational inequalities, see [18-21] and the references therein. In this Section, we introduce the similar concepts of well-posedness for equilibrium problems of type (2.1). The results obtained can be considered as a natural generalization of previous results of Lucchetti and Patrone [15, 16], Goeleven and Mantague [11] and Noor [29, 42] For this purpose, we define the following:

For a given $\epsilon > 0$, we consider the sets

$$\begin{aligned}A(\epsilon) &= \{u\in K : F(u,Tu,v)\geqslant -\epsilon\|v-u\|,\quad \forall v\in K\}\\ B(\epsilon) &= \{u\in K : F(v,Tv,u)\leqslant \epsilon\|v-u\|,\quad \forall v\in K\}.\end{aligned}$$

For a nonempty set $X\subset H$, we define the diameter of X, denoted by $D(X)$, as

$$D(X) = \sup\{\|v-u\|;\quad \forall u,v\in X\}.$$

DEFINITION 8.1. We say that the equilibrium problem (2.1) is *well-posed,* if and only if

$$A(\epsilon)\neq\phi \quad\text{and}\quad D(A(\epsilon))\longrightarrow 0,\text{as}\quad \epsilon\longrightarrow 0.$$

For $F(u,Tu,v) = \langle Tu,v-u\rangle$, our definition of well-posedness is exactly the same as one introduced by Lucchetti and Patrone [15, 16] for variational inequalities and extended by Noor [29, 42] and Goeleven and Mantague [11] for variational-like inequalities and hemivariational inequalities respectively.

THEOREM 8.1. *Let the function $F(.,.,.)$ and T be jointly g-pseudomonotone, jointly g-hemicontinuous and convex in the third argument. Then*

$$A(\epsilon) = B(\epsilon).$$

Proof. Let $u \in K$ be such that

$$F(u, Tu, v) \geqslant -\epsilon\|v - u\|, \quad \forall v \in K,$$

which implies that

$$F(v, Tv, u) \leqslant \epsilon\|v - u\|, \quad \forall v \in K, \tag{8.1}$$

since $F(.,.,.)$ and T are jointly g-pseudomonotone.

Thus

$$A(\epsilon) \subset B(\epsilon). \tag{8.2}$$

Conversely, let $u \in K$ such that (8.1) hold. Since K is a convex set, $\forall u, v \in K$, $t \in [0, 1]$, $\quad v_t = u + t(v - u) \equiv (1 - t)u + tv \in K$.

Taking $v = v_t$ in (8.1), we have

$$F(v_t, Tv_t, u) \leqslant t\epsilon\|v - u\|. \tag{8.3}$$

Also

$$\begin{aligned} 0 &= F(v_t, Tv_t, v_t) \\ &\leqslant tF(v_t, Tv_t, v) + (1 - t)F(v_t, Tv_t, u) \\ &\leqslant tF(v_t, Tv_t, v) + (1 - t)t\epsilon\|v - u\|, \end{aligned}$$

where we have used (8.3).

Dividing the above inequality by t and letting $t \longrightarrow 0$, we have

$$F(u, Tu, v) \geqslant -\epsilon\|v - u\|, \quad \forall v \in K,$$

which implies that

$$B(\epsilon) \subset A(\epsilon). \tag{8.4}$$

Thus from (8.2) and (8.4), we have

$$A(\epsilon) = B(\epsilon).$$

□

THEOREM 8.2. *The set $B(\epsilon)$ is closed under the assumptions of Theorem* 8.1.

Proof. Let $\{u_n : n \in N\} \subset B(\epsilon)$ be such that $u_n \longrightarrow u$ in K as $n \longrightarrow \infty$. This implies that $u_n \in K$ and

$$F(v, Tv, u_n) \leqslant \epsilon\|v - u_n\|, \quad \forall v \in K.$$

Taking the limit in the above inequality as $n \longrightarrow \infty$, we have

$$F(v, Tv, u) \leqslant \epsilon\|v - u\|, \quad \forall v \in K,$$

which implies that $u \in K$, since K is a closed and convex set. Consequently, it follows that the set $B(\epsilon)$ is closed.

Using essentially the technique of Goeleven and Mantague [11], we can prove the following results. To convey an idea and for the sake of completeness, we include their proofs.

THEOREM 8.3. *Let $F(.,.,.)$ and T be jointly g-pseudomonotone and jointly g-hemicontinuous. If the equilibrium problem (2.1) is well-posed, then equilibrium problem (2.1) has a unique solution.*

Proof. Let us define the sequence $\{u_k : k \in N\}$ by $u_k \in A(1/k)$. Let $\epsilon > 0$ be sufficiently small and let $m, n \in N$ such that $n \geqslant m \geqslant \frac{1}{\epsilon}$. Then

$$A(\frac{1}{n}) \subset A(\frac{1}{m}) \subset A(\epsilon).$$

Thus

$$\|u_n - u_m\| \leqslant D(A(\frac{1}{n})),$$

which implies that the sequence $\{u_n\}$ is a Cauchy sequence and it converges, that is, $u_k \longrightarrow u$ in K. From Theorem 8.1 and Theorem 8.2, we know that the set $A(\epsilon)$ is a closed set. Thus

$$u \in \cup_{\epsilon>0} A(\epsilon),$$

so that u is solution of the equilibrium problem (2.1). From the second condition of well-posedness, we see that the solution of the equilibrium problem (2.1) is unique. □

THEOREM 8.4. *Let $F(.,.,.)$ and T be jointly pseudomonotone and jointly hemicontinuous. If $A(\epsilon) \neq 0, \forall \epsilon > 0$. and $A(\epsilon)$ is bounded for some ϵ_0, then the equilibrium problem (2.1) has at least one solution.*

Proof. Let $u_n \in A(1/n)$. Then $A(1/n) \subset A(\epsilon)$, for n large enough. Thus for some subsequence $u_n \longrightarrow u \in K$, we have

$$F(v, Tv, u_n) \leqslant \frac{1}{n}\|v - u_n\| \leqslant \frac{1}{n}\{\|v\| + c\}, \quad \forall v \in K.$$

Taking the limit as $n \longrightarrow \infty$, we have

$$F(v, Tv, u) \leqslant 0,$$

which implies that $u \in B(0) = A(0)$, by Theorem 8.1. This shows that $u \in A(0)$, from which it follows that the equilibrium problem (2.1) has at least one solution.

REMARK 8.1.

(I) If the equilibrium problem (2.1) has a unique solution, then it is clear that $A(\epsilon) \neq 0, \forall \epsilon > 0$ and $\cap_{\epsilon>0} A(\epsilon) = \{u_0\}$.

(II) It is known that [16] if the variational inequality (2.3) has a unique solution, then it is not well-posed.

(III) From Theorem 8.3, we conclude that the unique solution of the equilibrium problem (2.1) can be computed by using the ϵ-equilibrium problem, that is, find $u_\epsilon \in K$ such that

$$F(u_\epsilon, T\epsilon, v) \geqslant -\epsilon\|v - u_\epsilon\|, \quad v \in K.$$

9. Conclusion and future research

In this paper, we have presented the state-of-the art in the theory and several computational aspects of equilibrium problems in the setting of convexity, invexity, g-convexity and uniformly prox-regular convexity. It is remarked that the concepts of invexity, g-convexity and uniformly prox-regular convexity are generalization of convexity in quite different directions and they have no interlink connections between themselves. These new concepts are very recent ones and offer great opportunities for further research. It is expected that the interplay among all these areas will certainly lead to some innovative, novel and significant results.

While our main aim in this study has been to describe the fundamental ideas and techniques , which have been used to develop the various iterative schemes and well-posedness of equilibrium problems, the foundation we have laid is quite broad, flexible and general. The study of these aspects of equilibrium problems is a fruitful and growing field of intellectual endeavour. We would like mention that many of the concepts, ideas and techniques, we have described are fundamental to all of these applications. For example, three-step and four step iterative schemes for solving equilibrium problems have been recently suggested and analyzed. In recent years, attempts have made to prove the equivalence among various one-step (Mann), two-step (Ishikawa) and three-step (Noor) iterations for solving variational inequalities and nonlinear operator equations in Banach spaces under various conditions on the operator T. Similar problems can be investigated in the theory of equilibrium problems, which is another direction of future research. In brief, the theory of the equilibrium problems does not appear to have developed to an extent that it provides a complete framework for studying various problems arising in pure and applied sciences. It is true that each of these areas of applications requires special consideration of pecularities of the physical problem at hand and the inequalities that model. The interested reader is advised to explore these interesting and fascinating fields further. It is our hope that this brief introduction may inspire and motivate the reader to discover new, innovative and novel applications of equilibrium problems in all areas of pure, regional, physical, social, industrial and engineering sciences.

REFERENCES

[1] E. BLUM, W. OETTLI, *From Optimization and Variational Inequalities to Equilibrium Problems,* Mathematics Students, **63**, (1994), 123–145.

[2] F. H. CLARKE, *Optimization and Nonsmooth Analysis*, J. Wiley and Sons, NY, 1983.

[3] F. H. CLARKE, Y. S. LEDYAEV, R. J. STERN AND P. R. WOLENSKI, *Nonsmooth Analysis and Control Theory,* Springer-Verlag, New York, NY, 1998.

[4] G. CRISTESCU, L. LUPSA, *Non-Connected Convexities and Applications*, Kluwer Academic Publishers, Dordrecht, Holland, 2002.

[5] V. F. DEM'YANOV, G. E. SAVROULAKIS, L. N. PLOYAKOVA AND P. D. PANAGIOTOPOULOS, *Quasidifferentiability and Nonsmooth Modelling in Mechanics, Engineering and Economics*, Kluwer Academic Publishers, Dordrecht, 1996.

[6] F. FLORES-BAZAN, *Existence Theorems for Generalized Noncoercive Equilibrium Problems: The Quasi-Convex Case,* SIAM J. Optim. **11**, (2000), 675–690.

[7] M. FUKUSHIMA, *Equivalent Differentiable Optimization Problems and Descent Methods for Asymmetric Variational Inequality Problems*, Math. Program., **53**, (1992), 99–110.

[8] F. GIANNESSI, A. MAUGER, *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York, NY, 1995.
[9] F. GIANNESSI, A. MAUGERI AND P. M. PARDALOS, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models,* Kluwer Academics Publishers, Dordrecht, Holland, 2001.
[10] G. GLOWINSKI, J. L. LIONS AND R. TREMOLIERES, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, Holland, 1981.
[11] D. GOELEVEN, D. MANTAGUE, *Well-posed Hemivariational Inequalities*, Numer. Funct. Anal. Optim., **16**, (1995), 909–921.
[12] D. KINDERLEHRER, G. STAMPACCHIA, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, London, England, 1980.
[13] M. A. HANSON, *On Sufficiency of the Kuhn-Tucker Conditions*, J. Math. Anal. Appl., **80**, (1981), 545–550.
[14] J. L. LIONS, G. STAMPACCHIA, *Variational Inequalities*, Comm. Pure Appl. Math., **20**, (1967), 491–512.
[15] R. LUCCHETTI, F. PATRONE, *A Characterization of Tykhonov Well-Posedness for Minimum Problems with Applications to Variational Inequalities*, Numer. Funct. Anal. Optim., **3**, (1981), 461–478.
[16] R. LUCCHETTI, F. PATRONE, *Some Properties of Well-Posed Variational Inequalities Governed by Linear Operators*, Numer. Funct. Anal. Optim., **5**, (1982-83), 349–361.
[17] B. MARTINET, *Regularisation D'inequations Variationelles par Approximations Successive*, Rev. d'Aut. Inform. Rech. Oper., **3**, (1970), 154–159.
[18] G. MASTEROENI, *Gap Functions for Equilibrium Problems*, J. Global Optim., **27**, (2003), 411–426.
[19] S. R. MOHAN, S. K. NEOGY, *On Invex Sets and Preinvex Functions*, J. Math. Anal. Appl., **189**, (1995), 901–908.
[20] U. MOSCO, *Implicit Variational Problems and Quasi Variational Inequalities*, Lecture Notes in Mathematics, Springe Verlag, Berlin, **543**, (1976), 83–156.
[21] Z. NANWIEWICZ, P. D. PANAGIOTOPOULOS, *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, Boston, 1995.
[22] M. ASLAM NOOR, *On Variational Inequalities,* Ph. D. Thesis, Brunel University, London, U. K., 1975.
[23] M. ASLAM NOOR, *Some Recent Advances in Variational Inequalities, Part 1: Basic Concepts*, New Zealand J. Math. **26**, (1997), 53–80.
[24] M. ASLAM NOOR, *Some Recent Advances in Variational Inequalities, Part 2: Other Concepts*, New Zealand J. Math., **26**, (1977), 229–255.
[25] M. ASLAM NOOR, *Mixed Quasi Invex Equilibrium Problems,* Internat. J. Math. Math. Sci., **2004**, (2004), 3057–3067.
[26] M. ASLAM NOOR, *Multivalued General Equilibrium Problems*, J. Math. Anal. Appl., **283**, (2003), 140–149.
[27] M. ASLAM NOOR, *General Variational Inequalities,* Appl. Math. Letters, **1**, (1988), 119–121.
[28] M. ASLAM NOOR, *Mixed Quasi Variational Inequalities,* Appl. Math. Comput., **146**, (2003), 553–578.
[29] M. ASLAM NOOR, *Some Development in General Variational Inequalities*, Appl. Math. Comput., **152**, (2004), 199–277.
[30] M. ASLAM NOOR, *Iterative Schemes for Nonconvex Variational Inequalities*, J. Optim. Theory Appl., **121**, (2004), 163–176.
[31] M. ASLAM NOOR, *Auxiliary Principle Technique for Equilibrium Problems*, J. Optim. Theory Appl., **122**, (2004), 371–386.
[32] M. ASLAM NOOR, *On a Class of Nonconvex Equilibrium Problems*, Appl. Math. Comput., **157**, (2004), 553–566.
[33] M. ASLAM NOOR, *Invex Equilibrium Problems*, J. Math. Anal. Appl., **302**, (2005), 463–475.
[34] M. ASLAM NOOR, *New Approximation Schemes for General Variational Inequalities*, J. Math. Anal. Appl., **251**, (2000), 217–229.
[35] M. ASLAM NOOR, *Nonconvex Functions and Variational inequalities*, J. Optim. Theory Appl., **87**, (1995), 615–630.
[36] M. ASLAM NOOR, *Variational-like Inequalities*, Optimization, **30**, (1994), 323–330.
[37] M. ASLAM NOOR, *Invex ϵ-Equilibrium Problems with Trifunction*, Int. J. Pure Appl. Math., **13**, (2004), 123–136.
[38] M. ASLAM NOOR, *Multivalued Equilibrium Problems with Trifunction*, Aust. J. Math. Anal. Appl., **1**, (2004), 1–11.
[39] M. ASLAM NOOR, *Hemiequilibrium Problems*, J. Appl. Math. Stoch. Anal., **2004**, (2004), 235–244.
[40] M. ASLAM NOOR, *Hemivariational Inequalities*, J. Appl. Math. Comput., **17**, (2005), 59–72.
[41] M. ASLAM NOOR, *Some Algorithms for Hemiequilibrium Problems*, J. Appl. Math. Comput., **19**, (2005), 135–146.
[42] M. ASLAM NOOR, *Fundamentals of Mixed Quasi Variational Inequalities*, Int. J. Pure Appl. Math., **15**, (2004), 137–258.

[43] M. ASLAM NOOR, *Hemivariational-like Inequalities,* J. Comput. Appl. Math., **182**, (2005), 316–326.
[44] M. ASLAM NOOR, *On Generalized Preinvex Functions and Monotonicites*, J. Inequal. Pure Appl. Math., **5**, (4) (2004), 1–9, Article 110.
[45] M. ASLAM NOOR, K. INAYAT NOOR, *On Equilibrium Problems,* Appl. Math. E-Notes, **4**, (2004), 125–132.
[46] M. ASLAM NOOR, K. INAYAT NOOR, *Regularized Hemivariational Inequalities*, Nonl. Funct. Anal. Appl., **10**, (2005), 495–506.
[47] M. ASLAM NOOR, W. OETTLI, *On General Nonlinear Complementarity Problems and Quasi-equilibria*, Le Matem., **49**, (1994), 313–331.
[48] P. D. PANAGIOTOPOULOS, *Nonconvex Energy Functions, Hemivariational Inequalities and Substationarity Principles*, Acta Mach. , **42**, (1983), 160–183.
[49] P. D. PANAGIOTOPOULOS, *Hemivariational inequalities, Applications to Mechanics and Engineering*, Springer Verlag, Berlin, 1993.
[50] M. PATRIKSSON, *Nonlinear Programming and Variational Inequality Problems: A Unified Approach*, Kluwer Academic Publishers, Dordrecht, Holland, 1999.
[51] R. A. POLIQUIN, R. T. ROCKAFELLAR AND L. THIBAULT, *Local Differentiability of Distance Functions*, Transactions of American Mathematical Society, **352**, (2000), 5231–5249.
[52] R. T. ROCKAFELLAR, *Monotone Operators and Proximal Point Algorithm*, SIAM J. Control Optim., **14**, (1976), 877–898.
[53] G. STAMPACCHIA, *Formes Bilineaires Coercitives sur les Ensembles Convexes*, Compt. Rend. l'Acad. Sciences, Paris, **258**, (1964), 4413–4416.
[54] T. WEIR, B. MOND, *Preinvex Functions in Multiobjective Optimization*, J. Math. Anal. Appl., **136**, (1988), 29–38.
[55] X. Q. YANG, G. Y. CHEN, *A Class of Nonconvex Functions and Pre-variational Inequalities*, J. Math. Anal. Appl., **169**, (1992), 359–373.
[56] D. L. ZHU, P. MARCOTTE, *Co-coercivity and its Role in the Convergence of Iterative Schemes for Solving Variational inequalities*, SIAM J. Optim., **6**, (1996), 714–726.

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