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SOME FRACTIONAL DIFFERENTIAL INEQUALITIES AND THEIR APPLICATIONS

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Abstract. We consider some inequalities involving derivatives of non-integer order. These inequalities arise naturally when investigating differential equations of fractional order. We find some bounds for solutions of these inequalities and give some applications.

1. Introduction

Investigating differential equations of fractional order often leads to inequalities involving derivatives of non-integer order. Many inequalities are available in the literature for derivatives of integer order [1, 4]. To the contrary, differential inequalities with fractional derivatives are not well developed. This paper is an attempt to fill this gap. In particular we are interested in the following inequalities which may be found in [1], p. 138–141.

THEOREM 1. *Let* $a(t)$, $q(t)$, $b_j(t)$, $u^{(j)}(t)$, $j=0,\ldots,k$, *be nonnegative continuous functions for* $t \geqslant 0$, *and suppose that*

$$u^{(k)}(t) \leqslant a(t) + q(t)\sum_{j=0}^{k}\int_0^t b_j(s)u^{(j)}(s)ds, \ t \geqslant 0,$$

where $k \geqslant 0$ *is an integer. Then,*

$$u^{(k)}(t) \leqslant a(t) + q(t)\int_0^t \phi_1(s)\exp\left(\int_s^t \phi_2(\tau)d\tau\right)ds, \ t \geqslant 0,$$

where

$$\phi_1(t) = a(t)b_k(t) + \sum_{j=0}^{k-1}\sum_{i=0}^{j} u^{(j)}(0)b_i(t)\frac{t^{j-i}}{(j-i)!}, + \sum_{j=0}^{k-1}\frac{b_{k-j-1}(t)}{j!}\int_0^t (t-x)^j a(x)dx,$$

$$\phi_2(t) := q(t)b_k(t) + \sum_{j=0}^{k-1}\frac{b_{k-j-1}(t)}{j!}\int_0^t (t-x)^j q(x)dx.$$

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THEOREM 2. *Let $a(t)$, $b_j(t)$, $u^{(j)}(t)$, $j = 0,...,k$, be nonnegative continuous functions for $t \geqslant 0$, with $a(t)$ nondecreasing, and suppose that*

$$u^{(k)}(t) \leqslant a(t) + \sum_{j=0}^{k} \int_0^t b_j(s) u^{(k)}(s) u^{(j)}(s) ds, \ t \geqslant 0,$$

where $k \geqslant 1$ is an integer. Then,

$$u(t) \leqslant \frac{a(t) \exp\left(\int_0^t \psi_1(s) ds\right)}{1 - \int_0^t \psi_2(s) \exp\left(\int_0^s \psi_1(\tau) d\tau\right) ds},$$

while

$$1 - \int_0^t \psi_2(s) \exp\left(\int_0^s \psi_1(\tau) d\tau\right) ds > 0,$$

where

$$\psi_1(t) := \sum_{j=0}^{k-1} \sum_{i=0}^{j} u^{(j)}(0) b_i(t) \frac{t^{j-i}}{(j-i)!},$$

$$\psi_2(t) := a(t) \sum_{j=0}^{k} b_{k-j}(t) \frac{t^j}{j!}.$$

THEOREM 3. *Let $b_j(t)$, $u^{(j)}(t)$, $j = 0,...,k$, be nonnegative continuous functions for $t \geqslant 0$, let $a(t)$ be a positive nondecreasing function and suppose that*

$$u^{(k)}(t) \leqslant a(t) + \sum_{j=0}^{k} \int_0^t b_j(s) u^{(l)}(s) u^{(j)}(s) ds, \ t \geqslant 0,$$

where $k - 1 \geqslant l \geqslant 0$. Then,

$$u^{(k)}(t) \leqslant \frac{a(t) \exp\left(\int_0^t \psi_3(s) ds\right)}{1 - \int_0^t \psi_4(s) \exp\left(\int_0^s \psi_3(\tau) d\tau\right) ds},$$

while

$$1 - \int_0^t \psi_4(s) \exp\left(\int_0^s \psi_3(\tau) d\tau\right) ds > 0,$$

where

$$\psi_3(t) := \sum_{j=0}^{k-1} \sum_{i=0}^{j} u^{(j)}(0) b_i(t) \frac{t^{j-i}}{(j-i)!} \left(\sum_{i=l}^{k-1} \frac{u^{(i)}(0)}{a(t)} \frac{t^{i-l}}{(i-l)!} + \frac{t^{k-l}}{(k-l)!} \right),$$

$$\psi_4(t) := \frac{t^{k-l}}{(k-l)!} \sum_{j=0}^{k} b_{k-j}(t) \frac{t^j}{j!}.$$

In this paper, we prove the non-integer order analogues of these inequalities as well as several other inequalities. Some examples illustrating the applications of these inequalities are also provided. The reader will notice that our results improve and/or extend some situations even in the integer order case.

The rest of the paper is organized as follows. In Section 2. we introduce some preliminaries. In Section 3. we present our results and their proofs. Section 4. is devoted to some applications.

2. Preliminaries

In this section we introduce some notations, definitions and lemmas which will be needed later. For more details, we refer the reader to [1] and [5].

We denote by L_p, $1 \leqslant p \leqslant \infty$, the usual Lebesgue spaces, and by $AC([a,b])$ the space of all absolutely continuous functions on $[a,b]$.

DEFINITION 1. Let $f(x) \in L_1(a,b)$, the integral

$$(I^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > a,$$

where $\alpha > 0$, is called the Riemann-Liouville fractional integral of order α of the function f.

We also use f_α to denote $I^\alpha f$.

DEFINITION 2. The expression

$$(D^\alpha f)(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} dt,$$

where $0 < \alpha < 1$, is called the Riemann-Liouville fractional derivative of order α of f provided the right-hand side is pointwise defined on (a,b).

Notice that $D^\alpha f(x) = \frac{d}{dx} I^{1-\alpha} f(x)$. For convenience, we use the notation $I^{-\alpha}$ to denote D^α for $\alpha \geqslant 0$.

DEFINITION 3. Let $0 < \alpha < 1$. A function $f(x) \in L_1(a,b)$ is said to have a summable fractional derivative $D^\alpha f$ on (a,b) if $f_{1-\alpha} \in AC([a,b])$.

DEFINITION 4. We define the space $I^\alpha(L_p(a,b))$, $\alpha > 0$, $1 \leqslant p < \infty$, to be the space of all functions f such that $f = I^\alpha \varphi$ for some $\varphi \in L_p(a,b)$.

PROPOSITION 4. *If $f(x)$ has a summable fractional derivative $D^\beta f$, $0 \leqslant \beta < 1$, on (a,b), then for $\alpha \geqslant 0$,*

$$I^\alpha D^\beta f(x) = f_{\alpha-\beta}(x) - \frac{f_{1-\beta}(a)}{\Gamma(\alpha)} (x-a)^{\alpha-1}.$$

See ([5], p. 48).

COROLLARY 5. *If $f(x)$ has a summable fractional derivative $D^{\alpha}f$, $0 \leqslant \alpha < 1$, on (a,b), then for $0 \leqslant \beta \leqslant \alpha < 1$ we have*

$$D^{\beta}f(x) = I^{\alpha-\beta}D^{\alpha}f(x) + \frac{f_{1-\alpha}(a)}{\Gamma(\alpha-\beta)}(x-a)^{\alpha-\beta-1}.$$

Proof. In Proposition 4, replace α by $\alpha-\beta$, and replace β by α.

PROPOSITION 6. *A function $f(x)$ is in $I^{\alpha}(L_1)$ if and only if $f_{1-\alpha}$ is absolutely continuous on $[a,b]$ and $f_{1-\alpha}(a) = 0$.*
See ([5], Theorem 2.3, p. 43).

The next four lemmas are proven in [1] (Lemma 1.1, Lemma 4.1, Corollary 5.3, Theorem 10.3, respectively).

LEMMA 7. *Let $g(t)$ and $f(t)$ be continuous functions for $t \geqslant a$, let $v(t)$ be a differentiable function for $t \geqslant a$, and suppose that*

$$\begin{aligned} v'(t) &\leqslant f(t) + g(t)v(t), \quad t \geqslant a, \\ v(a) &\leqslant v_0. \end{aligned}$$

Then, for $t \geqslant a$,

$$v(t) \leqslant v_0 \exp\left(\int_a^t g(s)ds\right) + \int_a^t f(s)\exp\left(\int_s^t g(\tau)d\tau\right)ds.$$

LEMMA 8. *Let $v(t)$ be a positive differentiable function satisfying the inequality*

$$v'(t) \leqslant h(t)v(t) + k(t)v^p(t), \quad t \in [a,b],$$

where the functions h and k are continuous functions in $[a,b]$, and $p \geqslant 0$, $p \neq 1$, is a constant. Then,

$$v(t) \leqslant \exp\left(\int_a^t h(s)ds\right)\left[v^q(a) + q\int_a^t k(s)\exp\left(-q\int_a^s h(\tau)d\tau\right)ds\right]^{1/q},$$

for $t \in [a,T)$, where $q = 1-p$ and T is chosen so that the expression between the brackets is positive in the subinterval $[a,T)$.

LEMMA 9. *Let v, f, g and k be non-negative continuous functions in $[a,b]$. Let ω be a continuous, non-negative and non-decreasing function in $[0,\infty)$, with $\omega(0) = 0$ and $\omega(u) > 0$ for $u > 0$, and let $F(t) := \max_{0 \leqslant s \leqslant t} f(s)$ and $G(t) := \max_{0 \leqslant s \leqslant t} g(s)$. Assume that*

$$v(t) \leqslant f(t) + g(t)\int_a^t k(s)\omega(v(s))ds, \quad t \in [a,b].$$

Then

$$v(t) \leqslant H^{-1}\left[H(F(t)) + G(t)\int_a^t k(s)ds\right], \quad t \in [a,T),$$

where $H(v) := \int_{v_0}^{v} \frac{d\tau}{\omega(\tau)}$, $0 < v_0 \leqslant v$, H^{-1} is the inverse of H and $T > a$ is such that $H(F(t)) + G(t)\int_a^t k(s)ds \in D(H^{-1})$ for all $t \in [a,T)$.

Let $I \subset \mathbb{R}$, and let $g_1, g_2 : I \to \mathbb{R} \setminus \{0\}$. We write $g_1 \propto g_2$ if g_2/g_1 is nondecreasing in I.

LEMMA 10. *Let* $f(t)$ *be a positive continuous function in* $[a,b]$, $k_j(t,s)$, $j=1,...,n$, *are nonnegative continuous functions for* $a \leqslant s \leqslant t < b$ *which are nondecreasing in* t *for any fixed* s, $g_j(u)$, $j=1,...,n$, *are nondecreasing continuous functions in* $[0,\infty)$, *with* $g_j(u) > 0$ *for* $u > 0$ *and* $u(t)$ *is a nonnegative continuous functions in* $[a,b]$. *If* $g_1 \propto g_2 \propto ... \propto g_n$ *in* $(0,\infty)$, *then the inequality*

$$u(t) \leqslant f(t) + \sum_{j=1}^{n} \int_a^t k_j(t,s) g_j(u(s)) ds, \quad t \in [a,b],$$

implies that

$$u(t) \leqslant c_n(t), \quad a \leqslant t < T$$

where $c_0(t) := \max_{0 \leqslant s \leqslant t} f(s)$,

$$c_j(t) := G_j^{-1}\left[G_j(c_{j-1}(t)) + \int_a^t k_j(t,s) ds\right], \quad j=1,...,n,$$

$$G_j(u) := \int_{u_j}^{u} \frac{dx}{g_j(x)}, \quad u > 0, \quad u_j > 0,$$

and T *is chosen so that the functions* $c_j(t)$, $j=1,...,n$, *are defined for* $a \leqslant t < T$.

LEMMA 11. (Generalized Young's Inequality) *We have, for positive* a_i, $i = 1,\ldots,k$, *the inequality*

$$\left(\sum_{i=1}^{k} a_i\right)^n \leqslant k^{n-1} \sum_{i=1}^{k} a_i^n,$$

where k *and* n *are integers.*

3. Inequalities with derivatives of fractional order

In this section we establish, among other results, the fractional order analogues of the inequalities presented in Section 2..

THEOREM 12. *Assume that* $a(t)$, $b(t)$ *and* $c(t)$ *are nonnegative continuous functions on* $[0,T]$, $0 < T \leqslant \infty$. *Let* $u(t)$ *be a nonnegative function having a summable nonnegative fractional derivative* $D^\alpha u$ *and satisfying*

$$D^\alpha u(t) \leqslant a(t) + b(t) \int_0^t c(s) \left(\sum_{j=0}^{k} D^{\beta_j} u(s)\right)^n ds, \quad t \in (0,T), \tag{1}$$

where $n \geqslant 1$, $\beta_0 = 0$, $0 < \beta_j \leqslant \alpha < 1$, $1 \leqslant j \leqslant k$. *Then*

$$D^\alpha u(t) \leqslant a(t) + b(t) \left\{ \left(\int_0^t g(s) ds\right)^{1-n} - (n-1) \int_0^t h(s) ds \right\}^{-\frac{1}{n-1}} \tag{2}$$

provided that $g(t) \in L_1(0,T)$ *and*

$$\left(\int_0^t g(s) ds\right)^{1-n} \int_0^t h(s) ds < \frac{1}{(n-1)},$$

where

$$g(t) := 2^{n-1}c(t)\left(\sum_{j=0}^{k}\left[\frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)}t^{\alpha-\beta_j-1}+a_{\alpha-\beta_j}(t)\right]\right)^n, \tag{3}$$

and

$$h(t) := 2^{n-1}c(t)\left(\sum_{j=0}^{k}b_{\alpha-\beta_j}(t)\right)^n. \tag{4}$$

Proof. Let us set

$$\varphi(t) := \int_0^t c(s)\left(\sum_{j=0}^{k}D^{\beta_j}u(s)\right)^n ds. \tag{5}$$

Then, clearly $\varphi(0)=0$,

$$\varphi'(t) = c(t)\left(\sum_{j=0}^{k}D^{\beta_j}u(t)\right)^n, \tag{6}$$

and

$$D^{\alpha}u(t) \leqslant a(t)+b(t)\varphi(t), \quad t\in(0,T). \tag{7}$$

Moreover, it is clear from Corollary 5 that $D^{\beta_j}u(t)$, $j=1,\ldots,k$, are nonnegative, and thus $\varphi(t)$ is nonnegative and nondecreasing.

Now, we would like to estimate the right hand side of (6) in terms of $\varphi(t)$. By Corollary 5 we have

$$D^{\beta_j}u(t) = \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)}t^{\alpha-\beta_j-1}+I^{\alpha-\beta_j}D^{\alpha}u(t). \tag{8}$$

Substituting (8) in (6), then using (7) and Lemma 11, we obtain

$$\begin{aligned}\varphi'(t) &= c(t)\left(\sum_{j=0}^{k}\frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)}t^{\alpha-\beta_j-1}+\sum_{j=0}^{k}I^{\alpha-\beta_j}D^{\alpha}u(t)\right)^n\\ &\leqslant c(t)\left(\sum_{j=0}^{k}\frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)}t^{\alpha-\beta_j-1}+\sum_{j=0}^{k}a_{\alpha-\beta_j}(t)+\sum_{j=0}^{k}I^{\alpha-\beta_j}(b(t)\varphi(t))\right)^n \qquad (9)\\ &\leqslant 2^{n-1}c(t)\left\{\left(\sum_{j=0}^{k}\left[\frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)}t^{\alpha-\beta_j-1}+a_{\alpha-\beta_j}(t)\right]\right)^n+\left(\sum_{j=0}^{k}I^{\alpha-\beta_j}(b(t)\varphi(t))\right)^n\right\}.\end{aligned}$$

Since $\varphi(t)$ is a nondecreasing function, we can write (9) in the form

$$\varphi'(t) \leqslant g(t)+h(t)\varphi^n(t), \tag{10}$$

where g and h are as defined by (3) and (4).

Integrating both sides of (10) over $(0,t)$, we obtain

$$\varphi(t) \leqslant l(t) + \int_0^t h(s)\varphi^n(s)ds, \tag{11}$$

where $l(t) := \int_0^t g(s)ds$. Note that $g(t)$ is nonnegative and thus $\max_{0\leqslant s\leqslant t} l(s) = l(t)$. Applying Lemma 9 (with $\omega(v) = v^n$) we infer that

$$\varphi(t) \leqslant H^{-1}\left[H(l(t)) + \int_0^t h(s)ds\right],$$

where $H(v) = \frac{v^{1-n}}{1-n} - \frac{v_0^{1-n}}{1-n}$ and $H^{-1}(z) = \left[v_0^{1-n} - (n-1)z\right]^{-\frac{1}{n-1}}$. That is

$$\varphi(t) \leqslant \left\{l(t)^{1-n} - (n-1)\int_0^t h(s)ds\right\}^{-\frac{1}{n-1}} \tag{12}$$

as long as

$$l(t)^{n-1}\int_0^t h(s)ds < \frac{1}{(n-1)}.$$

Our result follows from (7) and (12).

REMARK 1. The assumption $g(t) \in L_1(0,T)$ is added in the statement of the theorem to ensure that $\max_{0\leqslant s\leqslant t} l(s)$ exists. This is needed to apply Lemma 9. However, this condition is not really restrictive when the β_j are not "very close" to α. Indeed, since $a(t)$ is continuous then for $I^{\alpha-\beta_j}a(t) \in L_n(0,T)$ we have

(i) if $\alpha - \beta_j \geqslant 1 - 1/n$ for all $j = 1,\dots,k,$ then $n(\alpha-\beta_j-1)+1 \geqslant 0$ and thus $g(t) \in L_1(0,T)$,

(ii) if $\alpha - \beta_{\hat{\jmath}} < 1 - 1/n$ for some $1 \leqslant \hat{\jmath} \leqslant k$ then we need this condition which in fact will involve $c(t)$. This condition arises here in the noninteger case because we are allowing $u(t)$ to be singular at 0.

COROLLARY 13. *If, in addition to the hypotheses of Theorem 12, u is continuous in the right neighborhood of 0 or that $u \in I^{\alpha}(L_1)$, then $g(t)$ reduces to*

$$g(t) = 2^{n-1}c(t)\left(\sum_{j=0}^{k} a_{\alpha-\beta_j}(t)\right)^n.$$

Proof. This follows from Proposition 6.

For $n = 1$ we have the following inequality.

THEOREM 14. *Assume that $a(t)$, $b(t)$ and $c(t)$ are nonnegative continuous functions on $[0,T]$, $0 < T \leqslant \infty$. Let $u(t)$ be a nonnegative function having a summable nonnegative fractional derivative $D^{\alpha}u$ and satisfying*

$$D^{\alpha}u(t) \leqslant a(t) + b(t)\int_0^t c(s)\left(\sum_{j=0}^{k} D^{\beta_j}u(s)\right)ds, \quad t \in (0,T), \tag{13}$$

where $\beta_0 = 0,\ 0 < \beta_j \leqslant \alpha < 1,\ 1 \leqslant j \leqslant k$. Then

$$D^{\alpha}u(t) \leqslant a(t) + b(t)\int_0^t g(s)\exp\left(\int_s^t h(\tau)d\tau\right)ds \tag{14}$$

where

$$g(t) = c(t)\sum_{j=0}^{k}\left[\frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)}t^{\alpha-\beta_j-1} + a_{\alpha-\beta_j}(t)\right], \tag{15}$$

and

$$h(t) = c(t)\sum_{j=0}^{k} b_{\alpha-\beta_j}(t). \tag{16}$$

Proof. This follows by applying Lemma 7 to (10).

For the next theorem we need the following notation

$$M_1(t) := c(t)\sum_{j=1}^{k}\frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)}t^{\alpha-\beta_j-1},$$

$$M_2(t) := c(t)\sum_{j=1}^{k} I^{\alpha-\beta_j}a(t),$$

$$M_3(t) := c(t)\sum_{j=1}^{k}\frac{t^{\alpha-\beta_j}}{\Gamma(\alpha-\beta_j+1)}.$$

THEOREM 15. *Let $a(t)$ and $c(t)$ be nonnegative continuous functions on $[0,T]$. Let $u(t)$ be a nonnegative function having a summable nonnegative fractional derivative $D^{\alpha}u$ and*

$$D^{\alpha}u(t) \leqslant a(t) + \int_0^t c(s)D^{\alpha}u(s)\sum_{j=1}^{k} D^{\beta_j}u(s)ds, \quad t \in (0,T), \tag{17}$$

where $0 < \beta_j \leqslant \alpha < 1$ and k is an integer.
(a) If $a(t)$ is nondecreasing, then

$$D^{\alpha}u(t) \leqslant a(t)\exp\left(\int_0^t M_1(s)ds\right)\left[1 - \int_0^t M_2(s)\exp\left(\int_0^s M_1(\tau)d\tau\right)ds\right]^{-1},$$

for $t \in (0, T_1)$, *where* T_1 *is the largest value of* t *for which*

$$1 - \int_0^t M_2(s) \exp\left(\int_0^s M_1(\tau) d\tau\right) ds > 0.$$

(b) *If* $a(t)$ *is nonincreasing, then*

$$D^\alpha u(t) \leqslant a_0 \exp\left(\int_0^t M_1(s) ds\right) \left[1 - a_0 \int_0^t M_3(s) \exp\left(\int_0^s M_1(\tau) d\tau\right) ds\right]^{-1},$$

for $t \in (0, T_2)$, *where* $a_0 := a(0)$ *and* T_2 *is the largest value of* t *for which*

$$1 - a_0 \int_0^t M_3(s) \exp\left(\int_0^s M_1(\tau) d\tau\right) ds > 0.$$

Proof. (a) Suppose first that $a(t)$ is positive and nondecreasing. Then,

$$\frac{D^\alpha u(t)}{a(t)} \leqslant 1 + \int_0^t c(s) \frac{D^\alpha u(s)}{a(s)} \sum_{j=1}^k D^{\beta_j} u(s) ds. \tag{18}$$

Let $\psi(t)$ denote the right hand side of (18). Then $\psi(0) = 1$,

$$D^\alpha u(t) \leqslant a(t) \psi(t),$$

and

$$\psi'(t) = c(t) \frac{D^\alpha u(t)}{a(t)} \sum_{j=1}^k D^{\beta_j} u(t) \leqslant c(t) \psi(t) \sum_{j=1}^k D^{\beta_j} u(t). \tag{19}$$

Since ψ is nondecreasing, by Corollary 5 we have

$$\begin{aligned} D^{\beta_j} u(t) &= I^{\alpha-\beta_j} D^\alpha u(t) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)} t^{\alpha-\beta_j-1} \\ &\leqslant I^{\alpha-\beta_j} (a(t)\, \psi(t)) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)} t^{\alpha-\beta_j-1} \\ &\leqslant \psi(t)\, I^{\alpha-\beta_j} a(t) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)} t^{\alpha-\beta_j-1}. \end{aligned} \tag{20}$$

Inserting (20) into (19) we obtain

$$\begin{aligned} \psi'(t) &\leqslant c(t)\, \psi(t) \sum_{j=1}^k \left(\psi(t) I^{\alpha-\beta_j} a(t) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)} t^{\alpha-\beta_j-1}\right) \\ &\leqslant M_1(t)\psi(t) + M_2(t)\psi^2(t). \end{aligned}$$

Using Lemma 8 with $p = 2$ we deduce that

$$\psi(t) \leqslant \exp\left(\int_0^t M_1(s) ds\right) \left[1 - \int_0^t M_2(s) \exp\left(\int_0^s M_1(\tau) d\tau\right) ds\right]^{-1}$$

as long as $1-\int_0^t M_2(s)\exp\left(\int_0^s M_1(\tau)d\tau\right)ds>0$. Consequently,

$$D^\alpha u(t)\leqslant a(t)\exp\left(\int_0^t M_1(s)ds\right)\left[1-\int_0^t M_2(s)\exp\left(\int_0^s M_1(\tau)d\tau\right)ds\right]^{-1},$$

for $t\in(0,T_1)$ where T_1 is the largest value of t for which

$$1-\int_0^t M_2(s)\exp\left(\int_0^s M_1(\tau)d\tau\right)ds>0.$$

If $a(t)$ is nonnegative then we carry out the same argument with $a(t)+\varepsilon$, for some $\varepsilon>0$, and then let ε tend to zero.

(b) If $a(t)$ is nonincreasing function and $a(0)=a_0$, then (17) can be written in the form

$$D^\alpha u(t)\leqslant a_0+\int_0^t c(s)D^\alpha u(s)\sum_{j=1}^k D^{\beta_j}u(s)ds. \tag{21}$$

Denoting the right hand side of (21) by $\varphi(t)$, we have $D^\alpha u(t)\leqslant\varphi(t)$ and $\varphi(0)=a_0$. By differentiation we get

$$\varphi'(t)=c(t)D^\alpha u(t)\sum_{j=1}^k D^{\beta_j}u(t)\leqslant c(t)\varphi(t)\sum_{j=1}^k D^{\beta_j}u(t)$$

and we proceed as in the first part of the proof.

For the next theorem we need the following notation

$$\begin{aligned}
M_1(t)&=\frac{u_{1-\alpha}^2(0)}{\Gamma(\alpha-\gamma)}\frac{c(t)}{a(t)}t^{\alpha-\gamma-1}\left(\sum_{j=1}^k\frac{t^{\alpha-\beta_j-1}}{\Gamma(\alpha-\beta_j)}\right)\\
M_2(t)&=\frac{u_{1-\alpha}(0)\,c(t)}{a(t)}\left\{I^{\alpha-\gamma}a(t)\sum_{j=1}^k\frac{t^{\alpha-\beta_j-1}}{\Gamma(\alpha-\beta_j)}+\frac{t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\sum_{j=1}^k I^{\alpha-\beta_j}a(t)\right\},\\
M_3(t)&=\frac{c(t)}{a(t)}I^{\alpha-\gamma}a(t)\sum_{j=1}^k I^{\alpha-\beta_j}a(t),
\end{aligned}\tag{22}$$

and

$$\begin{aligned}
M_4(t)&=\frac{u_{1-\alpha}^2(0)}{\Gamma(\alpha-\gamma)}c(t)t^{\alpha-\gamma-1}\left(\sum_{j=1}^k\frac{t^{\alpha-\beta_j-1}}{\Gamma(\alpha-\beta_j)}\right)=a(t)M_1(t),\\
M_5(t)&=u_{1-\alpha}(0)\,c(t)\left\{\frac{t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\sum_{j=1}^k\frac{t^{\alpha-\beta_j-1}}{\Gamma(\alpha-\beta_j)}+\frac{t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\sum_{j=1}^k\frac{t^{\alpha-\beta_j}}{\Gamma(\alpha-\beta_j+1)}\right\},\\
M_6(t)&=\frac{c(t)t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\sum_{j=1}^k\frac{t^{\alpha-\beta_j}}{\Gamma(\alpha-\beta_j+1)}.
\end{aligned}\tag{23}$$

THEOREM 16. *Let $a(t)$ and $c(t)$ be nonnegative continuous functions on $[0,T]$. Let $u(t)$ be a nonnegative function having a summable nonnegative fractional derivative $D^{\alpha}u$ and*

$$D^{\alpha}u(t) \leqslant a(t) + \int_0^t c(s)D^{\gamma}u(s)\sum_{j=1}^{k} D^{\beta_j}u(s)ds, \quad t \in (0,T), \tag{24}$$

where $0<\gamma\leqslant\alpha<1$ and $0<\beta_j\leqslant\alpha<1$, $j=1,\dots,k$.
(a) If $a(t)$ is nondecreasing, then

$$D^{\alpha}u(t) \leqslant a(t)\left(1+\int_0^t M_1(s)ds\right)\exp\left(\int_0^t M_2(s)ds\right)$$
$$\left[1-\left(1+\int_0^t M_1(s)ds\right)\exp\left(\int_0^t M_2(s)ds\right)\int_0^t M_3(s)ds\right]^{-1}$$

for $t\in(0,T_1)$, where T_1 is the largest value of t for which the bracket is positive.
(b) If $a(t)$ is nonincreasing, then

$$D^{\alpha}u(t) \leqslant \left(a_0+\int_0^t M_4(s)ds\right)\exp\left(\int_0^t M_5(s)ds\right)$$
$$\left[1-\left(a_0+\int_0^t M_4(s)ds\right)\exp\left(\int_0^t M_5(s)ds\right)\int_0^t M_6(s)ds\right]^{-1}$$

for $t\in(0,T_2)$, where T_2 is the largest value of t for which the bracket is positive.

Proof. (a) Suppose first that $a(t)$ is positive and nondecreasing. Then,

$$\frac{D^{\alpha}u(t)}{a(t)} \leqslant 1 + \int_0^t c(s)\frac{D^{\gamma}u(s)}{a(s)}\sum_{j=1}^{k} D^{\beta_j}u(s)ds. \tag{25}$$

Let $\psi(t)$ denote the right hand side of (25). Then $\psi(0)=1$,

$$D^{\alpha}u(t) \leqslant a(t)\psi(t), \tag{26}$$

and

$$\psi'(t) = c(t)\frac{D^{\gamma}u(t)}{a(t)}\sum_{j=1}^{k} D^{\beta_j}u(t). \tag{27}$$

Since ψ is nondecreasing, by Corollary 5 we have

$$\begin{aligned} D^{\gamma}u(t) &= I^{\alpha-\gamma}D^{\alpha}u(t) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\gamma)}t^{\alpha-\gamma-1} \\ &\leqslant I^{\alpha-\gamma}(a(t)\,\psi(t)) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\gamma)}t^{\alpha-\gamma-1} \\ &\leqslant \psi(t)\; I^{\alpha-\gamma}a(t) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\gamma)}t^{\alpha-\gamma-1}. \end{aligned} \tag{28}$$

Similarly,

$$D^{\beta_j}u(t) \leqslant \psi(t)\ I^{\alpha-\beta_j}a(t) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)}t^{\alpha-\beta_j-1}, \quad j = 1,\dots,k. \tag{29}$$

Inserting (28) and (29) into (27) we obtain

$$\begin{aligned} \psi'(t) &\leqslant \frac{c(t)}{a(t)}\left(\psi(t)I^{\alpha-\gamma}a(t) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\gamma)}t^{\alpha-\gamma-1}\right) \\ &\qquad \times \sum_{j=1}^{k}\left(\psi(t)I^{\alpha-\beta_j}a(t) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)}t^{\alpha-\beta_j-1}\right) \qquad (30) \\ &\leqslant M_1(t) + M_2(t)\psi(t) + M_3(t)\psi^2(t), \qquad (31) \end{aligned}$$

where M_1, M_2, and M_3 are as in (22).

By integrating (30) we obtain

$$\psi(t) \leqslant 1 + \int_0^t M_1(s)ds + \int_0^t M_2(s)\psi(s)ds + \int_0^t M_3(s)\psi^2(s)ds.$$

Applying Lemma 10 with

$$\begin{aligned} c_0 &= 1 + \int_0^t M_1(s)ds, \\ c_1(t) &= c_0(t)\exp\int_0^t M_2(s)ds, \\ c_2(t) &= \left[c_1^{-1}(t) - \int_0^t M_3(s)ds\right]^{-1}, \end{aligned}$$

we get our result.

If $a(t)$ is nonnegative then we carry out the same argument with $a(t)+\varepsilon$, for some $\varepsilon > 0$, and then let ε tend to zero.

(b) If $a(t)$ is nonincreasing function and $a(0) = a_0$, then

$$D^{\alpha}u(t) \leqslant a_0 + \int_0^t c(s)D^{\gamma}u(s)\sum_{j=1}^{k}D^{\beta_j}u(s)ds. \tag{32}$$

Denoting the right hand side of (32) by $\varphi(t)$, we have $D^{\alpha}u(t) \leqslant \varphi(t)$, $\varphi(0) = a_0$ and

$$\varphi'(t) = c(t)D^{\gamma}u(t)\sum_{j=1}^{k}D^{\beta_j}u(t).$$

Next, we proceed as in the first part of the proof.

REMARK 2. If we apply Bihari's theorem instead with $\omega(x) = x + x^2$ and

$$K(s) = \sup\{M_2(s), M_3(s)\},$$

then we find

$$\psi(t) \leqslant \frac{c_0(t)}{1+c_0(t)} \exp\left(\int_0^t \sup\{M_2(s), M_3(s)\}\ ds\right) \times \left[1 - \frac{c_0(t)}{1+c_0(t)} \exp\left(\int_0^t \sup\{M_2(s), M_3(s)\}\ ds\right)\right]^{-1}, \quad 0 \leqslant t < T_2$$

where T_2 is the largest value of t for which

$$1 - \frac{c_0(t)}{1+c_0(t)} \exp\left(\int_0^t \sup\{M_2(s), M_3(s)\}\ ds\right) > 0.$$

We can take $T = \max\{T_1, T_2\}$ with the appropriate bound for $\psi(t)$. This gives a bound for D^α through (26). The rest of the proof (that is the other case) is similar to that of Theorem 16.

REMARK 3. Using Lemma 10 we can generalize the inequality (24) to

$$D^\alpha u(t) \leqslant a(t) + \int_0^t c(s) \sum_{i=1}^m (D^{\gamma_i} u(s))^{r_i} \sum_{j=1}^k \left(D^{\beta_j} u(s)\right)^{n_j} ds$$

with $r_i, n_j > 1$, $i = 1, ..., m$, $j = 1, ..., k$, or further to

$$D^\alpha u(t) \leqslant a(t) + \int_0^t c(s) \sum_{i=1}^m \omega_i (D^{\gamma_i} u(s)) \sum_{j=1}^k g_j \left(D^{\beta_j} u(s)\right) ds$$

with nondecreasing functions ω_i, $i = 1, ..., m$, and g_j, $j = 1, ..., k$, satisfying the hypotheses of Lemma 10. Some of the orders γ_i and β_j may be equal to α. This is going to be established for a slightly different inequality in the next result.

Let us now prove some nonlinear versions of the preceeding inequalities. We consider the case of several nonlinear integral terms. In particular we will look at different nonlinearities of power type. Let us first prepare some notation. We denote by

$$A(t,s) := \sum_{j=1}^k 2^{n_j-1} k_j(t,s) \left(I^{\alpha-\beta_j} a(s) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)} s^{\alpha-\beta_j-1}\right)^{n_j},$$

$$B(t) := \int_0^t A(t,s) ds,$$

$$l_j(t,s) := \frac{2^{n_j-1}}{(\Gamma(\alpha-\beta_j+1))^{n_j}} k_j(t,s) s^{n_j(\alpha-\beta_j)},$$

$$c_0 = \max_{0 \leqslant z \leqslant t} B(z),$$

$$c_j(t) = \left\{c_{j-1}^{1-n_j}(t) - (n_j-1)\int_0^t l_j(t,s) ds\right\}^{-\frac{1}{n_j-1}}.$$

THEOREM 17. *Assume that* $a(t)$ *is a positive continuous function in* $[0,T)$ *and* $k_j(t,s)$, $j=1,\dots,k$, *are nonnegative continuous functions for* $0 \leqslant s \leqslant t < T$. *For positive integers* $1 < n_1 \leqslant n_2 \leqslant \dots \leqslant n_k$, *and* $0 < \beta_j \leqslant \alpha < 1$, *we suppose that* $k_j(t,s)s^{n_j(\alpha-\beta_j-1)} \in L_1(0,\infty)$, $j=1,\dots,k$, *for each fixed* t. *Let* $u(t)$ *be a nonnegative function having a summable nonnegative fractional derivative* $D^\alpha u$ *and satisfying*

$$D^\alpha u(t) \leqslant a(t) + \sum_{j=1}^{k} \int_0^t k_j(t,s) \left(D^{\beta_j} u(s)\right)^{n_j} ds, \quad t \in (0,T). \tag{33}$$

If $k_j(t,s)$ *are*

(a) *nondecreasing in* t *for any fixed* s,

or

(b) *differentiable with respect to the first variable* t *and are nonincreasing in* t *for any fixed* s,

then

$$D^\alpha u(t) \leqslant a(t) + c_k(t), \quad 0 \leqslant t < T^*,$$

where T^* *is the largest value of* t *for which*

$$c_{j-1}^{1-n_j}(t) \int_0^t l_j(t,s)ds < \frac{1}{n_j - 1}.$$

Proof. (a) It is clear from Corollary 5 that $D^{\beta_j}u(t)$, $j=1,\dots,k$, are nonnegative. Let $\hat{t} < T$ be fixed. As $k_j(t,s)$ are nondecreasing in t for any fixed s, we obtain from (33) that

$$D^\alpha u(t) \leqslant a(t) + \sum_{j=1}^{k} \int_0^t k_j(\hat{t},s) \left(D^{\beta_j} u(s)\right)^{n_j} ds, \quad 0 \leqslant t \leqslant \hat{t} < T. \tag{34}$$

Let us set

$$\varphi(t) := \sum_{j=1}^{k} \int_0^t k_j(\hat{t},s) \left(D^{\beta_j} u(s)\right)^{n_j} ds.$$

Then, clearly $\varphi(0) = 0$, $D^\alpha u(t) \leqslant a(t) + \varphi(t)$ and

$$\varphi'(t) = \sum_{j=1}^{k} k_j(\hat{t},t) \left(D^{\beta_j} u(t)\right)^{n_j}.$$

Using Corollary 5 we obtain

$$\begin{aligned} \varphi'(t) &= \sum_{j=1}^{k} k_j(\hat{t},t) \left[I^{\alpha-\beta_j} D^\alpha u(t) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)} t^{\alpha-\beta_j-1} \right]^{n_j} \\ &\leqslant \sum_{j=1}^{k} k_j(\hat{t},t) \left[I^{\alpha-\beta_j} a(t) + I^{\alpha-\beta_j} \varphi(t) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)} t^{\alpha-\beta_j-1} \right]^{n_j}. \end{aligned}$$

By Lemma 11 and since $\varphi(t)$ is nondecreasing we can write

$$\varphi^{'}(t) \leqslant A(\hat{t},t) + \sum_{j=1}^{k} l_j(\hat{t},t)\varphi^{n_j}(t). \tag{35}$$

Integrating both sides of (35) we find

$$\varphi(t) \leqslant \int_0^t A(\hat{t},s)ds + \sum_{j=1}^{k} \int_0^t l_j(\hat{t},s)\varphi^{n_j}(s)ds. \tag{36}$$

This is true for $\hat{t} = t$, and thus

$$\varphi(t) \leqslant B(t) + \sum_{j=1}^{k} \int_0^t l_j(t,s)\varphi^{n_j}(s)ds.$$

The result follows from Lemma 10.

(b) Assume the hypotheses of part (b) in the statement of the theorem. Let

$$\psi(t) := \sum_{j=1}^{k} \int_0^t k_j(t,s) \left(D^{\beta_j}u(s)\right)^{n_j} ds.$$

Then

$$\begin{aligned}\psi'(t) &= \sum_{j=1}^{k} k_j(t,t) \left(D^{\beta_j}u(t)\right)^{n_j} + \sum_{j=1}^{k} \int_0^t \frac{\partial k_j}{\partial t}(t,s) \left(D^{\beta_j}u(s)\right)^{n_j} ds \\ &\leqslant \sum_{j=1}^{k} k_j(t,t) \left(D^{\beta_j}u(t)\right)^{n_j}.\end{aligned}$$

The rest of the proof is similar to that in the first part.

REMARK 4. The condition that $k_j(t,s)$ be nonincreasing in t for any fixed s can be relaxed. In fact we only need that the partial derivative $\left|\frac{\partial k_j}{\partial t}(t,s)\right|$, $j = 1,...,k$, be bounded in t by a continuous function.

In this case, suppose that $\left|\frac{\partial k_j}{\partial t}(t,s)\right| < K_j(s)$, $j = 1,...,k$, and let

$$\tilde{\varphi}(t) := \sum_{j=1}^{k} \int_0^t k_j(t,s) \left(D^{\beta_j}u(s)\right)^{n_j} ds.$$

Then we have

$$\begin{aligned}\tilde{\varphi}^{'}(t) &= \sum_{j=1}^{k} k_j(t,t) \left(D^{\beta_j}u(t)\right)^{n_j} + \sum_{j=1}^{k} \int_0^t \frac{\partial k_j}{\partial t}(t,s) \left(D^{\beta_j}u(s)\right)^{n_j} ds \\ &\leqslant \sum_{j=1}^{k} k_j(t,t) \left(D^{\beta_j}u(t)\right)^{n_j} + \sum_{j=1}^{k} \int_0^t K_j(s) \left(D^{\beta_j}u(s)\right)^{n_j} ds.\end{aligned}$$

Now let

$$\chi_1(t) = \sum_{j=1}^{k} k_j(t,t) \left(D^{\beta_j} u(t)\right)^{n_j},$$

and

$$\chi_2(t) = \sum_{j=1}^{k} \int_0^t K_j(s) \left(D^{\beta_j} u(s)\right)^{n_j} ds,$$

and consequently, we can write $\tilde{\varphi}'(t) \leqslant \chi_1(t) + \chi_2(t)$. For χ_1, using Corollary 5 we obtain the bound

$$\chi_1(t) \leqslant g_1(t) + \sum_{j=1}^{k} l_{j1}(t) \tilde{\varphi}^{n_j}(t),$$

where

$$g_1(t) = \sum_{j=1}^{k} 2^{n_j-1} k_j(t,t) \left[I^{\alpha-\beta_j} a(t) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)}\ t^{\alpha-\beta_j-1} \right]^{n_j},$$

and

$$l_{j1}(t) = 2^{n_j-1} k_j(t,t) \frac{t^{n_j(\alpha-\beta_j)}}{\Gamma^{n_j}(\alpha-\beta_j+1)}.$$

For χ_2 we have $\chi_2(0) = 0$ and

$$\chi_2'(t) = \sum_{j=1}^{k} K_j(t) \left(D^{\beta_j} u(t)\right)^{n_j} \leqslant g_2(t) + \sum_{j=1}^{k} l_{j2}(t) \tilde{\varphi}^{n_j}(t),$$

where

$$g_2(t) = \sum_{j=1}^{k} 2^{n_j-1} K_j(t) \left[I^{\alpha-\beta_j} a(t) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)}\ t^{\alpha-\beta_j-1} \right]^{n_j},$$

and

$$l_{j2}(t) = \frac{2^{n_j-1}}{\Gamma^{n_j}(\alpha-\beta_j+1)}\ K_j(t)\ t^{n_j(\alpha-\beta_j)}.$$

Thus we have

$$\chi_2(t) \leqslant \int_0^t g_2(s) ds + \sum_{j=1}^{k} \tilde{\varphi}^{n_j}(t) \int_0^t l_{j2}(s) ds.$$

Therefore,

$$\tilde{\varphi}'(t) \leqslant g(t) + \sum_{j=1}^{k} l_j(t) \tilde{\varphi}^{n_j}(t),$$

where

$$g(t) = g_1(t) + \int_0^t g_2(s) ds,$$

and

$$l_j(t) = l_{j1}(t) + \int_0^t l_{j2}(s) ds.$$

Now we can integrate both sides and apply Lemma 10.

THEOREM 18. *Suppose the following conditions hold:*

1. $a(t)$ *is a positive continuous function in* $[0,T)$*;*

2. $k_j(t,s)$, $j=1,\dots,k$, *are nonnegative continuous functions for* $0 \leqslant s \leqslant t < T$ *which are nondecreasing in* t *for any fixed* s*;*

3. $g_j(u)$, $j=1,\dots,k$, *are nondecreasing continuous functions in* $[0,\infty)$*, with* $g_j(u)>0$ *for* $u>0$*, and* $w_1 \propto w_2 \propto \ldots \propto w_n$ *in* $(0,\infty)$*, where*

$$w_j(z(t))=g_j\left(h_j(t)z(t)+e_j(t)\right), \quad h_j(t)=\frac{t^{\alpha-\beta_j}}{\Gamma(\alpha-\beta_j+1)}, \quad e_j(t)=\frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)}t^{\alpha-\beta_j-1};$$

4. $u(t)$ *is a nonnegative function having a summable nonnegative fractional derivative* $D^{\alpha}u$ *and satisfying*

$$D^{\alpha}u(t) \leqslant a(t)+\sum_{j=1}^{k}\int_0^t k_j(t,s)g_j\left(D^{\beta_j}u(s)\right)ds, \quad t\in(0,T), \tag{37}$$

where $0<\beta_j \leqslant \alpha < 1$*,* $j=1,\dots,k$*.*

Then,

$$D^{\alpha}u(t) \leqslant c_n(t), \qquad 0 \leqslant t \leqslant T^*,$$

where

$$c_0(t)=\max_{0\leqslant s\leqslant t} a(s), \quad c_j(t)=W_j^{-1}\left[W_j\left(c_{j-1}(t)+\int_0^t k_j(t,s)ds\right)\right],$$

$$W_j(z)=\int_{z_j}^{z}\frac{dx}{w_j(x)}, \quad z>0,\ z_j>0,$$

and T^* *is chosen so that the functions* $c_j(t)$*,* $j=1,\dots,k$*, are defined for* $0\leqslant t\leqslant T^*$*.*

Proof. Let $\hat{a}(t)=\max_{0\leqslant s\leqslant t} a(s)$ and $\hat{t}<T$ be fixed. As $k_j(t,s)$ are nondecreasing in t for any fixed s, we obtain from (37) that

$$D^{\alpha}u(t) \leqslant \hat{a}(\hat{t})+\sum_{j=1}^{k}\int_0^t k_j(\hat{t},s)\ g_j\left(D^{\beta_j}u(s)\right)ds, \quad 0\leqslant t\leqslant \hat{t}<T. \tag{38}$$

Let

$$\varphi(t)=\hat{a}(\hat{t})+\sum_{j=1}^{k}\int_0^t k_j(\hat{t},s)\ g_j\left(D^{\beta_j}u(s)\right)ds.$$

Then, clearly $\varphi(0)=\hat{a}(\hat{t})>0$, $D^{\alpha}u(t)\leqslant\varphi(t)$ and

$$\varphi'(t)=\sum_{j=1}^{k}k_j(\hat{t},t)\ g_j\left(D^{\beta_j}u(t)\right).$$

Using Corollary 5 and the fact that g_j is nondecreasing, we obtain

$$\begin{aligned}\varphi^{'}(t) &= \sum_{j=1}^{k} k_j(\hat{t},t)\ g_j\left[I^{\alpha-\beta_j}D^{\alpha}u(t) + \frac{u_{1-\alpha}(0)}{\Gamma(\alpha-\beta_j)}t^{\alpha-\beta_j-1}\right] \\ &\leqslant \sum_{j=1}^{k} k_j(\hat{t},t)\ g_j\left[h_j(t)\varphi(t) + e_j(t)\right] \\ &= \sum_{j=1}^{k} k_j(\hat{t},t)\ w_j\left(\varphi(t)\right).\end{aligned}$$

By integration, we have

$$\varphi(t) \leqslant \hat{a}(\hat{t}) + \sum_{j=1}^{k}\int_0^t k_j(\hat{t},s)\ w_j(\varphi(s))ds.$$

This is true for $\hat{t} = t$, and thus

$$\varphi(t) \leqslant \hat{a}(t) + \sum_{j=1}^{k}\int_0^t k_j(t,s)\ w_j(\varphi(s))ds.$$

The result follows from Lemma 10.

REMARK 5. Similar results may be proved for functions g_j such that the corresponding w_j are in

$$\mathscr{F} = \left\{ k:\mathbb{R}_+\to\mathbb{R}_+ \;\middle|\; \begin{array}{l} k \text{ nondecreasing,} \\ k(v)>0 \text{ for } v>0, \\ k(av)\leqslant ak(v) \text{ for } v\geqslant 0,\ a\geqslant 1 \end{array} \right\},$$

or in

$$H_{rw} = \left\{ k:\mathbb{R}_+\to\mathbb{R}_+ \;\middle|\; \begin{array}{l} k \text{ nondecreasing and continuous,} \\ k(v)>0 \text{ for } v>0, \\ k(av)\leqslant r(a)w(v) \text{ for } v\geqslant 0,\ a>0, \text{ where,} \\ r(a) \text{ nonnegative continuous in } \mathbb{R}_+ \\ w(v) \text{ is nondecreasing continuous in } \mathbb{R}_+, w(v)>0 \text{ for } v>0 \end{array} \right\}.$$

See Theorem 10.2 and Corollary 10.2–10.5 in [1], p.92.

4. Applications

In this section we illustrate our previous results by some applications. In particular, we show how to use these results to prove boundedness, global existence and determine the asymptotic behavior for some families of fractional differential equations.

Let us consider the following weighted Cauchy-type problem:

$$\begin{aligned} &D^{\alpha}u(t) = f(t,u,D^{\beta_1}u,D^{\beta_2}u,\dots,D^{\beta_k}u), \quad t>0,\ 0<\beta_j\leqslant\alpha<1,\\ &t^{1-\alpha}u(t)|_{t=0} = u_0\in\mathbb{R}, \end{aligned} \tag{39}$$

where f is a continuous (linear or nonlinear) function in all its variables.

Let us define for $r>0$ the space

$$C_r^0([0,h]) := \left\{ v\in C^0((0,h]) : \lim_{t\to 0^+} t^r v(t) \text{ exists and is finite} \right\}.$$

Here $C^0((0,h])$ is the usual space of continuous functions on $(0,h]$. It turns out that the space $C_r^0([0,h])$ endowed with the norm

$$\|v\|_r := \max_{0\leqslant t\leqslant h} t^r\,|v(t)|$$

is a Banach space. Then, we define the space

$$\begin{aligned} C_{1-\alpha}^{\alpha}([0,h]) = \{v\in C_{1-\alpha}^0([0,h]) : v(t)\\ = v_0 t^{\alpha-1} + I^{\alpha}v^*(t) \text{ for some } v_0\in\mathbb{R}, v^*\in C_{1-\alpha}^0([0,h])\} \end{aligned}$$

The space $(C_{1-\alpha}^{\alpha}([0,h]), \|\cdot\|_{1-\alpha,\alpha})$, where $\|v\|_{1-\alpha,\alpha} := \|v\|_{1-\alpha} + \|D^{\alpha}v\|_{1-\alpha}$ and $\alpha>1/2$, is also a Banach space.

For functions in $C_{1-\alpha}^{\alpha}([0,h])$ we have the following

PROPOSITION 19. *Let $0<\alpha<1$. If $u\in C_{1-\alpha}^{\alpha}([0,h])$ then u has a summable derivative $D^{\alpha}u$ on $(0,h)$ (in the sense of Definition 3).*

Proof. Clearly $I^{1-\alpha}u\in AC([0,h])$ since $I^{1-\alpha}u = const + I^{1-\alpha}I^{\alpha}v^*$, for some $v^*\in C_{1-\alpha}^0([0,h])$.

So for solutions of (39) we have

THEOREM 20. *If $u\in C_{1-\alpha}^{\alpha}([0,h])$ is a solution of (39), then*

$$u(t) = u_0 t^{\alpha-1} + I^{\alpha}D^{\alpha}u.$$

Proof. From Proposition 19, u has a summable derivative. The result follows from Proposition 4 and the initial conditions in (39).

Now we prove a boundedness and global existence result for (39).

THEOREM 21. *Suppose that*

$$|f(t,u,v_1,v_2,\dots,v_k)| \leqslant a(t)+b(t)\int_0^t c(s)\left(|u(s)|+\sum_{j=1}^{k}|v_j(s)|\right)ds, \quad t>0, \tag{40}$$

with nonnegative continuous functions $a(t)$, $b(t)$ and $c(t)$. Then, any local solution of (39) in $\left(C_{1-\alpha}^{\alpha}([0,T), \|\cdot\|_{1-\alpha,\alpha}\right)$ is global in time. That is, it exists for all time $t>0$.

Proof. Following the proof of Theorem 14, we have

$$|D^{\alpha}u(t)| \leqslant a(t) + b(t)\int_0^t g(s)\exp\left(\int_s^t h(\tau)d\tau\right)ds =: L(t)$$

for all $t > 0$, where,

$$g(t) = c(t)\sum_{j=0}^{k}\left[\frac{|u_{1-\alpha}(0)|}{\Gamma(\alpha-\beta_j)}t^{\alpha-\beta_j-1} + a_{\alpha-\beta_j}(t)\right],$$

and

$$h(t) = c(t)\sum_{j=0}^{k} b_{\alpha-\beta_j}(t).$$

By Theorem 20, we have

$$|u(t)| = \left|u_0 t^{\alpha-1} + I^{\alpha}D^{\alpha}u\right| \leqslant |u_0|t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}L(s)ds.$$

Therefore $u(t)$ is bounded (away from zero) by a continuous function. Thus u can be extended for all $t > 0$.

If

$$|f(t,u,v_1,v_2,\ldots,v_k)| \leqslant a(t) + b(t)\int_0^t c(s)\left(|u(s)| + \sum_{j=1}^{k}|v_j(s)|\right)^n ds,$$

that is in the nonlinear case, then Theorem 12 gives us a bound for u provided that

$$\left(\int_0^t g(s)ds\right)^{1-n}\int_0^t h(s)ds < 1/(n-1).$$

In particular, if

$$\left(\int_0^{\infty} g(s)ds\right)^{1-n}\int_0^{\infty} h(s)ds < 1/(n-1),$$

(which may happen in case $u_{1-\alpha}(0) = 0$, which in turn occur when u is continuous in the right neighborhood of zero), then this implies that the solution u exists for all time $t > 0$.

REMARK 6. We have similar results for the case

$$|f(t,u,v_1,\ldots,v_k)| \leqslant a(t) + \int_0^t c(s)|D^{\gamma}u(s)|\sum_1^k |v_j(s)|ds,$$

with $0 < \gamma \leqslant \alpha$, and its nonlinear versions.

REMARK 7. These results are also valid for the "usual" initial condition $u(0) = u_0$ instead of the weighted one in (39).

Now we show how the results in Section 3. can provide information about the behavior of solutions for large values of t.

THEOREM 22. *Assume that $a(t)$, $b(t)$ and $c(t)$ are nonnegative continuous functions on $J=[0,T]$, $T>0$, and $g(t)\in L_1(0,T)$. If $u(t)$ has a summable nonnegative fractional derivative $D^\alpha u$, f satisfies (40), and $t^{1-\alpha}I^\alpha\psi(t)$ is bounded, where $\psi(t):=a(t)+b(t)\int_0^t g(z)\exp\left(\int_z^t h(\tau)d\tau\right)dz$, then*

$$|u(t)|\leqslant C/t^{1-\alpha},\ t>0$$

for some positive constant C.

Proof. Since u is in $(C^\alpha_{1-\alpha}([0,\infty),\|\cdot\|_{1-\alpha,\alpha})$, we have from Theorem 14 and 20

$$\begin{aligned}|u(t)|&=\left|u_0t^{\alpha-1}+I^\alpha D^\alpha u\right|\\&\leqslant|u_0|t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[a(s)+b(s)\int_0^s g(z)\exp\left(\int_z^s h(\tau)d\tau\right)dz\right]ds\\&\leqslant|u_0|t^{\alpha-1}+I^\alpha\psi(t).\end{aligned}$$

Therefore,

$$t^{1-\alpha}|u(t)|\leqslant u_0+t^{1-\alpha}I^\alpha\psi(t)<C.$$

REMARK 8. In case

$$|f(t,u,v_1,v_2,\ldots,v_k)|\leqslant a(t)+b(t)\int_0^t c(s)\left(|u(s)|+\sum_{j=1}^k|v_j(s)|\right)^n ds$$

for instance, then we have from Theorem 12 that

$$\psi(t)=a(t)+b(t)\left\{\left(\int_0^t g(s)ds\right)^{1-n}-(n-1)\int_0^t h(s)ds\right\}^{-\frac{1}{n-1}}.$$

If $\left(\int_0^\infty g(s)ds\right)^{n-1}\int_0^\infty h(s)ds<1/(n-1)$, then $\psi(t)\leqslant a(t)+Cb(t)$ for some positive constant C. Thus

$$t^{1-\alpha}I^\alpha\psi(t)\leqslant t^{1-\alpha}I^\alpha a(t)+Ct^{1-\alpha}I^\alpha b(t). \tag{41}$$

A simple condition assuring boundedness of the right hand side of (41) is

$$a(t)\leqslant C_1t^{\lambda_1-1}e^{-\omega_1t},\quad \lambda_1,\omega_1>0$$

and

$$b(t)\leqslant C_2t^{\lambda_2-1}e^{-\omega_2t},\quad \lambda_2,\omega_2>0,$$

for some positive constants C_1 and C_2. Indeed, it suffices to apply the Lemma (See [2, 3])

LEMMA 23. *If* $\alpha, \lambda, \omega > 0$*, then for any* $t > 0$*, we have*

$$t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} s^{\lambda-1} e^{-\omega s} ds \leqslant Const.$$

with $\lambda = \lambda_1, \lambda_2$ and $\omega = \omega_1, \omega_2$.

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