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CLASSES OF CONVEX FUNCTIONS ASSOCIATED WITH BERNSTEIN OPERATORS OF SECOND KIND

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Abstract. The ordinary convex functions satisfy some well-known inequalities involving the classical Bernstein operators. We describe some classes of generalized convex functions satisfying similar inequalities with respect to Bernstein operators of second kind.

1. Introduction

Consider the classical Bernstein operators (B_n) on $C[0,1]$. For $f \in C[0,1]$ the following statements are equivalent (see, e.g., [1], Cor. 6. 3. 8):

(a) f is convex;

(b) $B_nf \geqslant B_{n+1}f$, $n \geqslant 1$;

(c) $B_nf \geqslant f$, $n \geqslant 1$.

In [8] Paolo Soardi introduced the so-called Bernstein operators of second kind, β_n, which have the same relation with Chebyshev polynomials of second kind as the classical Bernstein operators have with Chebyshev polynomials of first kind. He raised the problem to find the properties of f which are inherited by β_nf. Some answers and a Voronovskaja-type formula can be found in [7].

In this paper we study some classes of generalized convex functions f related to the inequalities

(b') $\beta_nf \geqslant \beta_{n+1}f$, $n \geqslant 1$;

(c') $\beta_nf \geqslant f$, $n \geqslant 1$.

2. Bernstein operators of second kind

Following [8], we present a brief description of the Bernstein operators of second kind.

Details about hypergroups and polynomial hypergroups can be found in [4], [5], [6], [8] and the references given there.

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Denote by δ_t the unit mass at $t \in \mathbb{R}$. Let $\mathbb{Z}$ be the group of the integers with convolution defined by $\delta_m * \delta_n = \delta_{m+n}$, $m,n \in \mathbb{Z}$. For a given $x \in [-1,1]$ let Y_n be the random walk on $\mathbb{Z}$ with law

$$\mu = \frac{1-x}{2}\delta_{-1} + \frac{1+x}{2}\delta_1.$$

Then the distribution of Y_n is described by

$$\mu^n = \left(\frac{1-x}{2}\delta_{-1} + \frac{1+x}{2}\delta_1\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1+x}{2}\right)^{n-k} \left(\frac{1-x}{2}\right)^k \delta_{n-2k}.$$

Let $(B_n)_{n\geqslant 1}$ be the classical (first kind) Bernstein operators on $C[-1,1]$. Then for $f \in C[-1,1]$ we have

$$B_n f(x) = Ef\left(\frac{Y_n}{n}\right) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1+x}{2}\right)^{n-k} \left(\frac{1-x}{2}\right)^k f\left(\frac{n-2k}{n}\right).$$

Let $T_n(x) = \cos(n \arccos x)$ denote the Chebyshev polynomials of first kind. Define the convolution on $\mathbb{N} = \{0,1,2,\dots\}$ by

$$\delta_n * \delta_m = \frac{T_{m+n}(1/\sqrt{1-x^2})\delta_{m+n} + T_{|m-n|}(1/\sqrt{1-x^2})\delta_{|m-n|}}{2T_n(1/\sqrt{1-x^2})T_m(1/\sqrt{1-x^2})}.$$

This hypergroup is denoted by $\mathbb{N}^1$. Let Z_n be the random walk on $\mathbb{N}^1$ with law δ_1.

If $f \in C[-1,1]$ is even, we have (see [8])

$$B_n f(x) = Ef\left(\frac{Z_n}{n}\right).$$

Let now $P_n(x) = \frac{1}{\sqrt{1-x^2}} \sin((n+1)\arccos x)$ denote the Chebyshev polynomials of second kind. On $\mathbb{N}$ the convolution will be defined by

$$\delta_n * \delta_m = \sum_{k=|m-n|}^{m+n} \frac{P_k(1/\sqrt{1-x^2})}{P_m(1/\sqrt{1-x^2})P_n(1/\sqrt{1-x^2})}\delta_k$$

where the summation is taken over all k such that $k-|m-n|$ is even.

This hypergroup is denoted by $\mathbb{N}^2$. Let V_n be the random walk on $\mathbb{N}^2$ with law δ_1.

In [8] the Bernstein operators of second kind $\beta_n : C[0,1] \longrightarrow C[0,1]$ are defined by

$$\beta_n f(x) = Ef\left(\frac{V_n}{n}\right),\ f \in C[0,1],\ x \in [0,1], n \geqslant 1.$$

Let $n \geqslant 1$, $m = [n/2]$, $0 \leqslant k \leqslant m$, $0 \leqslant x \leqslant 1$. Consider the polynomials

$$w_{n,k}(x) = \frac{n+1-2m+2k}{(n+1)2^{n+1}x}\binom{n+1}{m-k}((1-x)^{m-k}(1+x)^{n+1-m+k} - (1-x)^{n+1-m+k}(1+x)^{m-k}).$$

THEOREM 2.1. ([8]) *For* $f \in C[0,1]$, $x \in [0,1]$ *and* $n \geqslant 1$,

$$\beta_n f(x) = \sum_{k=0}^{m} f\left(\frac{n-2m+2k}{n}\right) w_{n,k}(x).$$

3. Classes of generalized convex functions

Let $K_1 := \{f \in C[0,1] : f \text{ is increasing and convex}\}$. Let $[x_1,\ldots,x_n;f]$ be the divided difference of the function f at the points $x_1 < \cdots < x_n$.

We shall denote by K_2 the set of all functions $f \in C[0,1]$ such that

(i) for all $m \geqslant 1$ and $k \in \{0,1,\ldots,m-1\}$, $\quad f\left(\frac{1}{2m-1}\right) \geqslant f(0)$ and

$$\frac{2m-k}{2m-k+1}\left[\frac{2m-2k-1}{2m+1}, \frac{m-k}{m}; f\right] \geqslant \frac{2m-2k-1}{2m-2k+1}\left[\frac{m-k-1}{m}, \frac{2m-2k-1}{2m+1}; f\right];$$

(ii) for all $m \geqslant 2$ and $k \in \{0,1,\ldots,m-2\}$,

$$\frac{m-k}{m-k+1}\left[\frac{m-k-1}{m}, \frac{2m-2k-1}{2m-1}; f\right] \geqslant \frac{2m-k}{2m-k-1}\left[\frac{2m-2k-3}{2m-1}, \frac{m-k-1}{m}; f\right].$$

PROPOSITION 3.1. *K_1 is contained in K_2.*

Proof. Let $f \in C[0,1]$ be increasing and convex, $m \geqslant 1$, $k \in \{0,1,\ldots,m-1\}$. Then

$$\frac{2m-k}{2m-k+1} > \frac{2m-2k-1}{2m-2k+1} > 0$$

and

$$\left[\frac{2m-2k-1}{2m+1}, \frac{m-k}{m}; f\right] \geqslant \left[\frac{m-k-1}{m}, \frac{2m-2k-1}{2m+1}; f\right] \geqslant 0.$$

Now (i) follows immediately; (ii) can be proved similarly.

PROPOSITION 3.2. *The function $f \in C[0,1]$ is in K_2 if and only if*

(iii) *for all $m \geqslant 1$ and $k \in \{0,1,\ldots,m-1\}$,* $\quad f\left(\frac{1}{2m-1}\right) \geqslant f(0)$ *and*

$$\frac{2m-2k-1}{2m+1}\left[\frac{m-k-1}{m}, \frac{2m-2k-1}{2m+1}, \frac{m-k}{m}; f\right] + \frac{m}{2m-k+1}\left[\frac{2m-2k-1}{2m+1}, \frac{m-k}{m}; f\right] \geqslant 0;$$

(iv) *for all $m \geqslant 2$ and $k \in \{0,1,\ldots,m-2\}$,*

$$\frac{2m-2k-2}{2m-1}\left[\frac{2m-2k-3}{2m-1}, \frac{m-k-1}{m}, \frac{2m-2k-1}{2m-1}; f\right] + \frac{m}{2m-k}\left[\frac{m-k-1}{m}, \frac{2m-2k-1}{2m-1}; f\right] \geqslant 0.$$

Proof. By using the recurrence relation for divided differences

$$[x,y,z;f] = \frac{[y,z;f]-[x,y;f]}{z-x}$$

it is not difficult to verify that (iii) is equivalent to (i), and (iv) to (ii); this proves our assertion.

Let us remark that Proposition 3.2 can be used in order to give another proof of Proposition 3.1.

EXAMPLE 1. The function $f : [0,1] \longrightarrow \mathbb{R}$, $f(x) = -x^2+4x$ is in K_2, but not in K_1. So the inclusion described in Proposition 3.1 is strict.

In what follows, we shall consider the function $\omega : (0,1] \longrightarrow \mathbb{R}$,

$$\omega(x) = x - \frac{1}{x} + 2\log x,\ x \in (0,1].$$

A function f defined on $(0,1]$ is called $(1,\omega)$-convex (see [2]) if

$$\begin{vmatrix} f(x) & f(y) & f(z) \\ 1 & 1 & 1 \\ \omega(x) & \omega(y) & \omega(z) \end{vmatrix} \geqslant 0$$

whenever $0 < x < y < z \leqslant 1$.

Let $K_3 := \{f \in C^2[0,1] \ : \ x(1+x)f''(x) + 2f'(x) \geqslant 0,\ x \in [0,1]\}$.

THEOREM 3.1.

1. $K_2 \cap C^2[0,1] \subset K_3$.
2. *For* $f \in C^2[0,1]$ *the following statements are equivalent:*
   (a) $f \in K_3$*;*
   (b) f *is* $(1,\omega)$ *- convex on* $(0,1]$*;*
   (c) f *satisfies the inequality*

$$\frac{[y,z;f]}{[y,z;\omega]} \geqslant \frac{[x,y;f]}{[x,y;\omega]}$$

for all $0 < x < y < z \leqslant 1$.

Proof.

1. Let $f \in K_2 \cap C^2[0,1]$ and $x \in [0,1]$.
   Choose $k_m \in \{0,1,\dots,m-1\}$ such that

$$\lim_{m\to\infty} \frac{k_m}{m} = 1-x.$$

According to the mean-value property of divided differences, there exist

$$t_m \in \left[\frac{m-k_m-1}{m}, \frac{m-k_m}{m}\right]$$

and

$$s_m \in \left[\frac{2m-2k_m-1}{2m+1}, \frac{m-k_m}{m}\right]$$

such that

$$\left[\frac{m-k_m-1}{m}, \frac{2m-2k_m-1}{2m+1}, \frac{m-k_m}{m}; f\right] = \frac{1}{2} f''(t_m)$$

and

$$\left[\frac{2m-2k_m-1}{2m+1}, \frac{m-k_m}{m}; f\right] = f'(s_m).$$

Then $\lim_{m\to\infty} t_m = \lim_{m\to\infty} s_m = x.$ Using Proposition 3.2 we get

$$x\frac{f''(x)}{2} + \frac{1}{1+x} f'(x) =$$
$$\lim_{m\to\infty} \Big(\frac{2m-2k_m-1}{2m+1} \left[\frac{m-k_m-1}{m}, \frac{2m-2k_m-1}{2m+1}, \frac{m-k_m}{m}; f\right] +$$
$$\frac{m}{2m-k_m+1} \left[\frac{2m-2k_m-1}{2m+1}, \frac{m-k_m}{m}; f\right] \Big) \geqslant 0.$$

Thus $x(1+x)f''(x) + 2f'(x) \geqslant 0,\ x \in [0,1]$, which means that $f \in K_3$.

2. Let $f \in C^2[0,1]$. By a result of Bonsall [3] (see also [2]), f is $(1,\omega)$-convex on $(0,1]$ if and only if

$$\begin{vmatrix} f(x) & f'(x) & f''(x) \\ 1 & 0 & 0 \\ \omega(x) & \omega'(x) & \omega''(x) \end{vmatrix} \geqslant 0$$

for all $x \in (0,1]$. Thus (b) is equivalent to $x(1+x)f''(x) + 2f'(x) \geqslant 0,\ x \in (0,1]$, which is equivalent to (a).
Finally, (b) is equivalent to (c) by virtue of Theorem 4 [2].

EXAMPLE 2. We construct a function $f \in C^2[0,1]$ which is in K_3 but not in K_2. Let $\varepsilon \in (0, \frac{1}{3})$. Define

$$f(x) = \begin{cases} \log x, & x \in [\varepsilon, 1], \\ -\frac{1}{2\varepsilon^2}x^2 + \frac{2}{\varepsilon}x - \frac{3}{2} + \log\varepsilon, & x \in [0,\varepsilon]. \end{cases}$$

Then $f \in C^2[0,1] \cap K_3$. On the other hand, the conditions (i) and (ii) are not satisfied for $m = 2,\ k = 0$, which means that f is not in K_2.

The next results show how the functions in K_2 and K_3 are related to the Bernstein operators of second kind.

THEOREM 3.2.

1. *If $f \in K_2$, then*

$$\beta_n f \geqslant \beta_{n+1} f \geqslant f,\ n \geqslant 1.$$

2. *If $f \in C^2[0,1]$ and $\beta_n f \geqslant f$, $n \geqslant 1$, then $f \in K_3$.*

Proof.

1. Let $f \in K_2$. According to equations (24) and (25) in [7] we have

$$\beta_{2m}f(x) - \beta_{2m+1}f(x) =$$
$$\frac{1-x^2}{xm(2m+1)2^{2m+1}}\sum_{k=0}^{m-1}\binom{2m-1}{k}\left((1+x)^{2m-k}(1-x)^k - (1+x)^k(1-x)^{2m-k}\right)\times$$
$$\times\left(\frac{2m-2k-1}{2m-k}\left[\frac{m-k-1}{m},\frac{2m-2k-1}{2m+1},\frac{m-k}{m};f\right]+\right.$$
$$\left.+\frac{m(2m+1)}{(2m-k)(2m-k+1)}\left[\frac{2m-2k-1}{2m+1},\frac{m-k}{m};f\right]\right)$$

and

$$\beta_{2m-1}f(x) - \beta_{2m}f(x) =$$
$$2\left(\frac{1-x^2}{4}\right)^m\binom{2m-2}{m-1}\frac{2m-1}{m(m+1)}\left(f\left(\frac{1}{2m-1}\right)-f(0)\right)+$$
$$\frac{1-x^2}{xm(2m-1)4^m}\sum_{k=0}^{m-2}\binom{2m-2}{k}\left((1+x)^{2m-1-k}(1-x)^k-(1+x)^k(1-x)^{2m-1-k}\right)\times$$
$$\times\left(\frac{2m-2k-2}{2m-k-1}\left[\frac{2m-2k-3}{2m-1},\frac{m-k-1}{m},\frac{2m-2k-1}{2m-1};f\right]+\right.$$
$$\left.+\frac{m(2m-1)}{(2m-k)(2m-k-1)}\left[\frac{m-k-1}{m},\frac{2m-2k-1}{2m-1};f\right]\right).$$

Now, by Proposition 3.2,

$$\beta_n f \geqslant \beta_{n+1}f\ ,\ n\geqslant 1.$$

This implies

$$\beta_n f \geqslant \beta_{n+m}f\ ,\ n\geqslant 1,\ m\geqslant 1.$$

We have also (see [8], Theorem 2)

$$\lim_{m\to\infty}\beta_{n+m}f = f$$

uniformly on $[0,1]$, which entails $\beta_n f \geqslant f$.

2. Let $f \in C^2[0,1]$ and $\beta_n f \geqslant f\ ,\ n\geqslant 1$. By Theorem 4.1 [7] we have for $x\in(0,1]$

$$\lim_{n\to\infty} n(\beta_n f(x) - f(x)) = \frac{1-x^2}{2}f''(x) + \frac{1-x}{x}f'(x).$$

This implies $x(1+x)f''(x) + 2f'(x) \geqslant 0$, $x\in[0,1]$, which concludes the proof.

EXAMPLE 3. Let f be the function from Example 2. Then $f \in K_3$ and for $1/\sqrt{\log 3} < x < 1$ we have

$$\beta_3 f(x) = \frac{1-x^2}{2}\log\frac{1}{3} < \log x = f(x).$$

So the converse of the second assertion in Theorem 3.2 is not true.

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