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PHRAGMÉN–LINDELÖF TYPE ALTERNATIVE RESULTS FOR THE STOKES FLOW EQUATION

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Abstract. In this paper,we derive estimates for weighted energy expression for the solution of the Stokes flow equation in a semi-infinite plane channel by means of a second order differential inequality.From the estimates ,we establish Phragmén-Lindelöf alternative that the solutions either grow or decay exponentially.In the case of decay,we also show how to bound the total weighted energy.

1. Introduction

In 1856,B.de Saint-Venant [24] formulated and conjectured a famous mathematical and mechanical principle which came to be known in subsequent literatures as Saint-Venant's principle and led to an extensive investigation in the framework of applied mathematics. Early work on Saint-Venant's principle primarily focused on the initial-boundary value problems involving elliptic equations. It is Boley [3] who, in the 1950s,first pointed out the validity of a Saint-Venant's principle for the heat equations.Since then, an extensive attention has been paid to the parabolic problems, (see [5], [6], [7], [9], [25]). These studies are motivated by a desire to establish,for parabolic equation,spatial decay estimates analogous to those obtained for elliptic equation in the investigation of Saint-Venant's principle in elasticity theory. For a review of recent advance on Saint-Venant's principle, one may refer to [10], [12], [15] and the references cited therein.

Making use of explicit upper bounds for solutions of the transient heat conduction equation in a half space,Boley [3] was the first to assert that the spatial influence of the transient effects was even more localized than that of steady state. Then Boley [4] considered the more traditionary initial-boundary value problem for cylindrical domains or semi-infinite strips subject to non-zero boundary conditions on the ends only and obtained some illustrative results, which showed, for instance, that the spatial decay of end effect at any time t in the transient problems is faster than that for the steady state case.

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The work which described by Edelstein [5], Sigillito [25], Knowles [16], Horgan et al. [11], Ames et al. [2] is concerned with initial-boundary value problems for parabolic equations on cylindrical type domains subject to non-zero boundary conditions on the end only. Several methods, such as energy estimate,maximum principle and so on, are employed to establish the exponential decay of solution for such problems with distance from the ends. All of the results display that the spatial decay of end effects at any time t in the transient problem is either as fast as or even faster than that for the steady state case.

Knowles [17] established exponential decay estimates for solution of the biharmonic equation in his study of Saint-Venant's principle in plane isotropic elastostatics for bounded,simply connected domain of general shape. Since then, many authors have investigated the same problems in a semi-infinite strip (see [13],[12] and the references cited therein). Their common goal has been to try to establish an energy decay rate which is close to the exact one (an exact decay rate $\kappa = 4.20/h$ is founded by Horgan et al. [16] making use of a eigenfunction expansion for the solution of the biharmonic equation). It is well known that another interprelation for the biharmonic equation in the plane is that of the stream function in two dimensional Stokes flow, hence the results of the Saint-Venant's principle in plane elastostatics are also relevant to the study of the spatial evolution of stationary Stokes flows in a semi-infinite paralled plate channel. Numerous authors have dealt with Saint-Venant type decay estimate for solutions of the biharmonic equations in a semi-infinite channel in R^2 ,we mention in particular the papers of Ames et al. [1], Flavin [6], Oleinic et al. [21], [22], Horgan [13] and Flavin et al. [8]. Then Lin [18] established a spatial decay bound for the transient Stokes flow in a semi-infinite strip. Recently, Song [26] investigated the same time-dependent Stokes flow problem and obtained an analogous result with an improved decay rate. The papers concerning with biharmonic equations may be viewed as a version of Saint-Venant's principle in steady state or transient stokes flow, but common to all was the assumption that the solution must satisfy some a priori decay criterion at infinity.

The classical Phragmén-Lindelöf theorem states that harmonic function which vanishes on the cylindrical surface must either grow or decay exponentially with distance from the finite end of the cylinder. Phragmén-Lindelöf type Alternative results were obtained by Flavin el. at [8] for semi-linear second order elliptic equations in the half cylinder and by Horgan et al [14] for harmonic functions with non-linear boundary conditions on the lateral surface of a semi-infinite cylinder. Particularly, Payne et al. [23] established the Phragmén-Lindelöf type results in three type special domains in R^2. Additional references for Phragmén-Lindelöf type results may be found in [19], [20] and [15] therein.

In the present paper, we are concerned with the flow of an incompressible viscous which is governed by the transient Stokes flow equation in a semi-infinite channel and establish a Phragmén-Lindelöf type growth-decay estimate for the problem. We formulate the problem in section 2 and derive a basic differential inequality in section 3. We then obtain growth-decay estimate for the same weighted energy expressions in section 4. We finally show how the total energy in the decay results is bounded explicitly in terms of data in section 5.

2. Formulation

We consider the time-dependent Stokes equations governing the transient slow flow of an incompressible slow viscous fluid on an unbounded region Ω_0 defined by

$$\Omega_0 := \{(x_1, x_2) \mid x_1 > 0, 0 < x_2 < h\}, \tag{2.1}$$

Where h is a fixed constant,and we introduce the notation

$$L_z = \{(x_1, x_2) \mid x_1 = z \geqslant 0, 0 \leqslant x_2 \leqslant h\}. \tag{2.2}$$

The velocity field $v_\alpha(x_1, x_2, t)$ and the pressure $p(x_1, x_2, t)$ $(\alpha = 1, 2)$ for the transient Stokes flow of an incompressible viscous fluid are to be classical solutions of the initial-boundary value problem:

$$v_{\alpha,t} = \nu \Delta v_\alpha - p_{,\alpha}, \quad in \quad \Omega_0 \times [0, \infty), \tag{2.3}$$

$$v_{\alpha,\alpha} = 0, \quad in \quad \Omega_0 \times [0, \infty), \tag{2.4}$$

$$v_\alpha(x_1, 0, t) = v_\alpha(x_1, h, t) = 0, \quad \alpha = 1, 2, \tag{2.5}$$

$$v_\alpha(0, x_2, t) = f_\alpha(x_2, t), \quad \alpha = 1, 2, \tag{2.6}$$

$$v_\alpha(x_1, x_2, 0) = 0, \quad (x_1, x_2) \in \Omega_0. \tag{2.7}$$

where Δ is the two dimensional Laplace operator and ν is the constant kinematic viscosity. The functions $f_\alpha(x_2, t)$ are assumed to satisfy the compatibility $f_\alpha(0, t) = f_\alpha(h, t) = 0$. For simplicity, it is assumed that $\int_{L_0} f_1 dx_2 = 0$ for all $t > 0$.

In order to eliminate the pressure term $p_{,\alpha}$, we introduce the stream function $u(x_1, x_2, t)$ such that

$$v_1 = u_{,2}, v_2 = -u_{,1}. \tag{2.8}$$

The problem $(2.3) - (2.7)$ is then transformed into the following fourth order initial-boundary value problem:

$$\Delta^2 u = (\Delta u)_{,t}, \quad in \quad \Omega_0, \tag{2.9}$$

$$u(x_1, 0, t) = u_n(x_1, 0, t) = 0, \quad x_1 > 0, t > 0, \tag{2.10}$$

$$u(x_1, h, t) = u_n(x_1, h, t) = 0, \quad x_1 > 0, t > 0, \tag{2.11}$$

$$u(0, x_2, t) = g_1(x_2, t) = \int_0^{x_2} f_1(s, t) ds, \quad 0 \leqslant x_2 \leqslant h, t > 0, \tag{2.12}$$

$$u_{,1}(0, x_2, t) = g_2(x_2, t) = -f_2(x_2, t), \quad 0 \leqslant x_2 \leqslant h, t > 0, \tag{2.13}$$

$$u_{,\alpha}(x_1, x_2, 0) = 0, \quad x_1 > 0, 0 < x_2 < h, \tag{2.14}$$

where we make no assumption on u as $x_1 \to \infty$. Here Δ is the harmonic operator, and Δ^2 is the biharmonic operator, u_n is the outward normal derivative, and we adapt the standard notations , i. e. $u_{,i} = \frac{\partial u}{\partial x_i}, u_{,t} = \frac{\partial u}{\partial t}$. The differentiable function g_1 and g_2 are prescribed and assumed to satisfy appropriate compatibility conditions: $g_1(0, t) = g_1(h, t) = g_2(0, t) = g_2(h, t) = g_1'(0, t) = g_1'(h, t) = 0$, where g_1' denotes the partial differentiation with respect to x_2.

We define energy expressions of the form:

$$
\begin{aligned}
E_1(z,t) = &\int_0^t \int_0^z \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA d\eta + \frac{1}{2} \int_0^z \int_{L_\xi} u_{,\alpha} u_{,\alpha} dA|_{\eta=t} \\
&+ \frac{2h^2}{\pi^2} \int_0^t \int_0^z \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dA d\eta + \frac{h^2}{\pi^2} \int_0^t \int_0^z \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dA d\eta \qquad (2.15) \\
&+ \frac{h^2}{\pi^2} \int_0^z \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t} + \frac{1}{2} \frac{h^2}{\pi^2} \int_0^z \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t},
\end{aligned}
$$

$$
\begin{aligned}
E_2(z,t) = &\int_0^t \int_z^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA d\eta + \frac{1}{2} \int_z^\infty \int_{L_\xi} u_{,\alpha} u_{,\alpha} dA|_{\eta=t} \\
&+ \frac{2h^2}{\pi^2} \int_0^t \int_z^\infty \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dA d\eta + \frac{h^2}{\pi^2} \int_0^t \int_z^\infty \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dA d\eta \qquad (2.16) \\
&+ \frac{h^2}{\pi^2} \int_z^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t} + \frac{1}{2} \frac{h^2}{\pi^2} \int_z^\infty \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t},
\end{aligned}
$$

where we use the summation convention on repeated indices and the Greek subscripts range over 1, 2. Our purpose is to determine the alternative that either $E_1(z,t)$ grows exponentially or $E_2(z,t)$ decays exponentially as $z \to \infty$.

3. Basic inequality

We will make frequently use of Schwarz's inequality, the arithmetic-geometric mean inequality and the following well-known Wirtinger-type inequalities [23]:

$$\int_{L_z} (u_{,1})^2 dx_2 \leqslant \frac{h^2}{\pi^2} \int_{L_z} (u_{,12})^2 dx_2, \qquad (3.1)$$

$$\int_{L_z} (u_{,2})^2 dx_2 \leqslant \frac{1}{4} \frac{h^2}{\pi^2} \int_{L_z} (u_{,22})^2 dx_2, \qquad (3.2)$$

$$\int_{L_z} u^2 dx_2 \leqslant (\frac{2}{3})^4 \frac{h^4}{\pi^4} \int_{L_z} (u_{,22})^2 dx_2. \qquad (3.3)$$

In this section, our goal is to establish a basic inequality

$$|\Phi(z,t)| \leqslant c \frac{\partial^2}{\partial z^2} \Phi(z,t).$$

where c is a constant.

Firstly, we must define the expression of $\Phi(z,t)$, the definition of $\Phi(z,t)$ is divided into three steps.

Step 1: The definition of $\Phi_1(z,t)$.

For a solution u of the problem $(2.9)-(2.14)$, we define

$$f(z,t) = \int_0^t \int_{L_z} u_{,\alpha} u_{,\alpha} dx_2 d\eta - \int_0^t \int_{L_z} u u_{,11} dx_2 d\eta. \qquad (3.4)$$

Making use of the initial-boundary value conditions, Green's theorem, and integrating by parts, we can write

$$\begin{aligned} f(z,t) &= \int_0^t \int_{L_0} u_{,\alpha} u_{,\alpha} dx_2 d\eta - \int_0^t \int_{L_0} uu_{,11} dx_2 d\eta \\ &\quad + \int_0^t \int_0^z \int_{L_\xi} u_{,\alpha} u_{,\alpha 1} dA d\eta - \int_0^t \int_0^z \int_{L_\xi} uu_{,\alpha\alpha 1} dA d\eta \\ &= \int_0^t \int_{L_0} u_{,\alpha} u_{,\alpha} dx_2 d\eta - \int_0^t \int_{L_0} uu_{,11} dx_2 d\eta \\ &\quad - \int_0^t \int_0^z \int_{L_\xi} (z-\xi)_{,\beta} u_{,\alpha} u_{,\alpha\beta} dA d\eta + \int_0^t \int_0^z \int_{L_\xi} (z-\xi)_{,\beta} uu_{,\alpha\alpha\beta} dA d\eta \\ &= \int_0^t \int_{L_0} u_{,\alpha} u_{,\alpha} dx_2 d\eta - \int_0^t \int_{L_0} uu_{,11} dx_2 d\eta + \int_0^t \int_0^z \int_{L_\xi} (z-\xi) u_{,\alpha\beta} u_{,\alpha\beta} dA d\eta \\ &\quad - \int_0^t \int_0^z \int_{L_\xi} (z-\xi) uu_{,\alpha\alpha\beta\beta} dA d\eta + z \int_0^t \int_{L_0} (u_{,\alpha} u_{,\alpha 1} - uu_{,\alpha\alpha 1}) dx_2 d\eta. \end{aligned} \tag{3.5}$$

We now process the fourth term on the right of (3.5).

$$\begin{aligned} &\int_0^t \int_0^z \int_{L_\xi} (z-\xi) uu_{,\alpha\alpha\beta\beta} dA d\eta = \int_0^t \int_0^z \int_{L_\xi} uu_{,1\eta} dA d\eta \\ &\quad - \int_0^t \int_0^z \int_{L_\xi} (z-\xi) u_{,\alpha} u_{,\alpha\eta} dA d\eta - z \int_0^t \int_{L_0} uu_{,1\eta} dx_2 d\eta \\ &= - \int_0^t \int_0^z \int_{L_\xi} u_{,1} u_{,\eta} dA d\eta + \int_0^t \int_{L_z} uu_{,\eta} dx_2 d\eta - \int_0^t \int_{L_0} uu_{,\eta} dx_2 d\eta \\ &\quad - \frac{1}{2} \int_0^z \int_{L_\xi} (z-\xi) u_{,\alpha} u_{,\alpha} dA|_{\eta=t} - z \int_0^t \int_{L_0} uu_{,1\eta} dx_2 d\eta. \end{aligned} \tag{3.6}$$

Now we combine (3.5), (3.6) and denote

$$\Phi_1(z,t) = \int_0^t \int_{L_z} u_{,\alpha} u_{,\alpha} dx_2 d\eta - \int_0^t \int_{L_z} uu_{,11} dx_2 d\eta + \int_0^t \int_{L_z} uu_{,\eta} dx_2 d\eta. \tag{3.7}$$

From (3.5), (3.6) and (3.7), we get

$$\begin{aligned} \Phi_1(z,t) &= \lambda_1(z,t) + \int_0^t \int_0^z \int_{L_\xi} (z-\xi) u_{,\alpha\beta} u_{\alpha\beta} dA d\eta \\ &\quad + \frac{1}{2} \int_0^z \int_{L_\xi} (z-\xi) u_{,\alpha} u_{,\alpha} dA|_{\eta=t} + \int_0^t \int_0^z \int_{L_\xi} u_{,1} u_{,\eta} dA d\eta, \end{aligned} \tag{3.8}$$

where we have defined

$$\lambda_1(z,t) = \int_0^t \int_{L_0} (u_{,\alpha} u_{,\alpha} - uu_{,11} + uu_{,\eta}) dx_2 d\eta + z \int_0^t \int_{L_0} (u_{,\alpha} u_{,\alpha 1} - uu_{,\alpha\alpha 1} + uu_{,1\eta}) dx_2 d\eta.$$

Step 2: The definition of $\Phi_2(z,t)$.

We consider the following integral :

$$\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta.$$

Upon integrating by parts, using the divergence theorem and the initial-boundary conditions, we obtain

$$\begin{aligned}
&\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta\\
=&\frac{1}{2}\int_0^t\int_{L_z}(u_{,\eta})^2dx_2d\eta-\frac{1}{2}\int_0^t\int_{L_0}(u_{,\eta})^2dx_2d\eta+2\int_0^t\int_0^z\int_{L_\xi}u_{,\alpha\eta}u_{,\alpha 1}dAd\eta\\
&-\int_0^t\int_{L_z}u_{,\eta}u_{,11}dx_2d\eta+\int_0^t\int_{L_0}u_{,\eta}u_{,11}dx_2d\eta-\frac{1}{2}\int_0^z\int_{L_\xi}(z-\xi)u_{,\alpha\beta}u_{,\alpha\beta}dA|_{\eta=t}\\
&-z\int_0^t\int_{L_0}u_{,\beta\eta}u_{,1\beta}dx_2d\eta+z\int_0^t\int_{L_0}u_{,\eta}u_{,\alpha\alpha 1}dx_2d\eta-z\int_0^t\int_{L_0}u_{,\eta}u_{,1\eta}dx_2d\eta.
\end{aligned}\tag{3.9}$$

Next we treat the third term on the right side of equality (3.9)

$$\begin{aligned}
&\int_0^t\int_0^z\int_{L_\xi}u_{,\alpha\eta}u_{,1\alpha}dAd\eta\\
&\quad=\int_0^t\int_{L_z}u_{,1\eta}u_{,1}dx_2d\eta-\int_0^t\int_{L_0}u_{,1\eta}u_{,1}dx_2d\eta-\int_0^t\int_0^z\int_{L_z}u_{,\alpha\alpha\beta\beta}u_{,1}dAd\eta,
\end{aligned}$$

In order to get an expression similar to (3.8), we must deal with

$$\begin{aligned}
&\int_0^t\int_0^z\int_{L_\xi}u_{,1}u_{,\alpha\alpha\beta\beta}dAd\eta\\
=&-\int_0^t\int_0^z\int_{L_\xi}u_{,1\alpha}u_{,\alpha\beta\beta}dAd\eta+\int_0^t\int_{L_z}u_{,1}u_{,1\beta\beta}dx_2d\eta-\int_0^t\int_{L_0}u_{,1}u_{,1\beta\beta}dx_2d\eta\\
=&\frac{1}{2}\int_0^t\int_{L_z}u_{,\alpha\beta}u_{,\alpha\beta}dx_2d\eta-\frac{1}{2}\int_0^t\int_{L_0}u_{,\alpha\beta}u_{,\alpha\beta}dx_2d\eta-\int_0^t\int_{L_z}u_{,1\alpha}u_{,1\alpha}dx_2d\eta\\
&+\int_0^t\int_{L_0}u_{,1\alpha}u_{,1\alpha}dx_2d\eta+\int_0^t\int_{L_z}u_{,1}u_{,1\beta\beta}dx_2d\eta-\int_0^t\int_{L_0}u_{,1}u_{,1\beta\beta}dx_2d\eta.
\end{aligned}\tag{3.10}$$

where , we have used the condition

$$u_{,\alpha 1}(x_1,0,t)=u_{,\alpha 1}(x_1,h,t)=0.$$

As shown in *Step* 1, we define

$$\begin{aligned}\Phi_2(z,t) =&\frac{1}{2}\int_0^t\int_{L_z}(u_{,\eta})^2dx_2d\eta+2\int_0^t\int_{L_z}u_{,1}u_{,1\eta}dx_2d\eta\\&-\int_0^t\int_{L_z}u_{,\eta}u_{,11}dx_2d\eta-\int_0^t\int_{L_z}u_{,\alpha\beta}u_{,\alpha\beta}dx_2d\eta\\&+2\int_0^t\int_{L_z}u_{,1\alpha}u_{,1\alpha}dx_2d\eta-2\int_0^t\int_{L_z}u_{,1}u_{,1\beta\beta}dx_2d\eta.\end{aligned}\tag{3.11}$$

By(3.9), (3.10) and (3.11), $\Phi_2(z,t)$ may be written as

$$\begin{aligned}\Phi_2(z,t) = \lambda_2(z,t)&+\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta\\&+\frac{1}{2}\int_0^z\int_{L_\xi}(z-\xi)u_{,\alpha\beta}u_{,\alpha\beta}dA|_{\eta=t}.\end{aligned}\tag{3.12}$$

where

$$\begin{aligned}\lambda_2(z,t) =&\int_0^t\int_{L_0}(\frac{1}{2}(u_{,\eta})^2+2u_{,1}u_{,1\eta}-u_{,\eta}u_{,11}-u_{,\alpha\beta}u_{,\alpha\beta}+2u_{,1\alpha}u_{,1\alpha}-2u_{,1}u_{,1\beta\beta})dx_2d\eta\\&+z\int_0^t\int_{L_0}(u_{,\eta}u_{,\alpha\alpha1}-u_{,\beta\eta}u_{,1\beta}-u_{,\eta}u_{,1\eta})dx_2d\eta.\end{aligned}$$

Step 3: The definition of $\Phi_3(z,t)$.

As in *Step* 2, we first consider

$$\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta.$$

After using the divergence theorem and the initial-boundary conditions, we are led to

$$\begin{aligned}&\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta\\=&\frac{1}{2}\int_0^t\int_{L_z}u_{,1\alpha}u_{,1\alpha}dx_2d\eta-\frac{1}{2}\int_0^t\int_{L_0}u_{,1\alpha}u_{,1\alpha}dx_2d\eta-\int_0^t\int_0^z\int_{L_\xi}u_{,1\alpha}u_{,\alpha\beta\beta}dAd\eta\\&+\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,11\alpha}u_{,\alpha\beta\beta}dAd\eta+z\int_0^t\int_{L_0}u_{,1\alpha}u_{,\alpha\beta\beta}dx_2d\eta-z\int_0^t\int_{L_0}u_{,1\alpha}u_{,1\alpha1}dx_2d\eta\\=&\frac{1}{2}\int_0^t\int_{L_z}u_{,1\alpha}u_{,1\alpha}dx_2d\eta-\frac{1}{2}\int_0^t\int_{L_0}u_{,1\alpha}u_{,1\alpha}dx_2d\eta+\frac{1}{2}\int_0^t\int_{L_z}u_{,\alpha\beta}u_{,\alpha\beta}dx_2d\eta\\&-\frac{1}{2}\int_0^t\int_{L_0}u_{,\alpha\beta}u_{,\alpha\beta}dx_2d\eta-\int_0^t\int_{L_z}u_{,1\alpha}u_{,1\alpha}dx_2d\eta+\int_0^t\int_{L_0}u_{,1\alpha}u_{,1\alpha}dx_2d\eta\\&+\int_0^t\int_0^z\int_{L_\xi}u_{,11}u_{,1\beta\beta}dAd\eta-\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,11}u_{,\alpha\alpha\eta}dAd\eta-z\int_0^t\int_{L_0}u_{,11}u_{,1\beta\beta}dx_2d\eta\\&+z\int_0^t\int_{L_0}u_{,1\alpha}u_{,\alpha\beta\beta}dx_2d\eta-z\int_0^t\int_{L_0}u_{,1\alpha}u_{,1\alpha1}dx_2d\eta.\end{aligned}\tag{3.13}$$

Upon integrating by parts, we obtain

$$\begin{aligned}&\int_0^t\int_0^z\int_{L_\xi} u_{,11}u_{,1\beta\beta}dAd\eta\\&=\frac{1}{2}\int_0^t\int_{L_z}((u_{,11})^2-(u_{,12})^2)dx_2d\eta-\frac{1}{2}\int_0^t\int_{L_0}((u_{,11})^2-(u_{,12})^2)dx_2d\eta.\end{aligned}\tag{3.14}$$

thus , what we need to do is to deal with the eighth term on the right of (3.13).

$$\begin{aligned}&\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,11}u_{,\alpha\alpha\eta}dAd\eta\\&=\frac{1}{2}\int_0^z\int_{L_\xi}(z-\xi)(u_{,11})^2dA|_{\eta=t}-\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,112}u_{,2\eta}dAd\eta\\&=\frac{1}{2}\int_0^z\int_{L_\xi}(z-\xi)(u_{,11})^2dA|_{\eta=t}+z\int_0^t\int_{L_0}u_{,12}u_{,2\eta}dx_2d\eta\\&\quad-\int_0^t\int_0^z\int_{L_\xi}u_{,12}u_{,2\eta}dAd\eta+\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,12}u_{,12\eta}dAd\eta\\&=\frac{1}{2}\int_0^z\int_{L_\xi}(z-\xi)u_{,1\alpha}u_{,1\alpha}dA|_{\eta=t}+z\int_0^t\int_{L_0}u_{,12}u_{,2\eta}dx_2d\eta-\int_0^t\int_0^z\int_{L_\xi}u_{,12}u_{,2\eta}dAd\eta.\end{aligned}\tag{3.15}$$

Combining (3.13), (3.14), and(3.15), we define:

$$\begin{aligned}\Phi_3(z,t)&=\frac{1}{2}\int_0^t\int_{L_z}((u_{,11})^2+(u_{,22})^2)dx_2d\eta\\&=\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta+\frac{1}{2}\int_0^z\int_{L_\xi}(z-\xi)u_{,1\alpha}u_{,1\alpha}dA|_{\eta=t}\\&\quad-\int_0^t\int_0^z\int_{L_\xi}u_{,12}u_{,2\eta}dAd\eta+\lambda_3(z,t),\end{aligned}\tag{3.16}$$

where

$$\lambda_3(z,t)=\int_0^t\int_{L_0}((u_{,11})^2+(u_{,22})^2)dx_2d\eta+z\int_0^t\int_{L_0}(u_{,11}u_{,111}-u_{,12}u_{,221}+u_{,12}u_{,2\eta})dx_2d\eta.$$

A combination of (3.6), (3.11) and (3.16), we now define a new expression which is fundamental to our method.

$$\begin{aligned}\Phi(z,t)&=\Phi_1(z,t)+\frac{h^2}{\pi^2}[2\Phi_2(z,t)+\Phi_3(z,t)]\\&=\int_0^t\int_{L_z}u_{,\alpha}u_{,\alpha}dx_2d\eta-\int_0^t\int_{L_z}uu_{,11}dx_2d\eta+\int_0^t\int_{L_z}uu_{,\eta}dx_2d\eta\end{aligned}$$

$$
\begin{aligned}
&+\frac{h^2}{\pi^2}[\int_0^t\int_{L_z}(u_{,\eta})^2dx_2d\eta+2\int_{L_z}(u_{,1})^2dx_2|_{\eta=t}-2\int_0^t\int_{L_z}u_{,\eta}u_{,11}dx_2d\eta \\
&+\frac{5}{2}\int_0^t\int_{L_z}(u_{,11})^2dx_2d\eta+4\int_0^t\int_{L_z}(u_{,12})^2dx_2d\eta \\
&-\frac{3}{2}\int_0^t\int_{L_z}(u_{,22})^2dx_2d\eta-4\int_0^t\int_{L_z}u_{,1}u_{,111}dx_2d\eta].
\end{aligned}
\tag{3.17}
$$

In deriving (3.17), we have used $\int_0^t\int_{L_z}u_{,1}u_{,122}dx_2d\eta=-\int_0^t\int_{L_z}(u_{,12})^2dx_2d\eta$.

Obviously $\Phi(z,t)$ may be written in additional form

$$
\begin{aligned}
\Phi(z,t)=&\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,\alpha\beta}u_{,\alpha\beta}dAd\eta+\frac{1}{2}\int_0^z\int_{L_\xi}(z-\xi)u_{,\alpha}u_{,\alpha}dA|_{\eta=t} \\
&+\int_0^t\int_0^z\int_{L_\xi}u_{,1}u_{,\eta}dAd\eta+\frac{h^2}{\pi^2}[2\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta \\
&+\int_0^z\int_{L_\xi}(z-\xi)u_{,\alpha\beta}u_{,\alpha\beta}dA|_{\eta=t}+\int_0^t\int_0^z\int_{L_\xi}(z-\xi)u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta \\
&+\frac{1}{2}\int_0^z\int_{L_\xi}(z-\xi)u_{,1\alpha}u_{,1\alpha}dA|_{\eta=t}-\int_0^t\int_0^z\int_{L_\xi}u_{,12}u_{,2\eta}dAd\eta] \\
&+\lambda_1(z,t)+\frac{h^2}{\pi^2}[2\lambda_2(z,t)+\lambda_3(z,t)].
\end{aligned}
\tag{3.18}
$$

From (3.18), clearly, we have

$$
\begin{aligned}
\frac{\partial^2}{\partial z^2}\Phi(z,t)=&\int_0^t\int_{L_z}u_{,\alpha\beta}u_{,\alpha\beta}dx_2d\eta+\frac{1}{2}\int_{L_z}u_{,\alpha}u_{,\alpha}dx_2|_{\eta=t} \\
&+\int_0^t\int_{L_z}u_{,11}u_{,\eta}dx_2d\eta+\int_0^t\int_{L_z}u_{,1}u_{,1\eta}dx_2d\eta \\
&+\frac{h^2}{\pi^2}[2\int_0^t\int_{L_z}u_{,\alpha\eta}u_{,\alpha\eta}dx_2d\eta+\int_{L_z}u_{,\alpha\beta}u_{,\alpha\beta}dx_2|_{\eta=t} \\
&+\int_0^t\int_{L_z}u_{,1\alpha\beta}u_{,1\alpha\beta}dx_2d\eta-\int_0^t\int_{L_z}u_{,121}u_{,2\eta}dx_2d\eta \\
&-\frac{1}{2}\int_{L_z}(u_{,12})^2dx_2|_{\eta=t}+\frac{1}{2}\int_{L_z}u_{,1\alpha}u_{,1\alpha}dx_2|_{\eta=t}].
\end{aligned}
\tag{3.19}
$$

In virtue of (3.17), and making use of Wirtinger's inequalities and Schwarz's inequality, we obtain

$$\begin{aligned}|\Phi(z,t)| \leqslant &\frac{h^2}{\pi^2}[(\frac{2}{3})^4\frac{h^2}{\pi^2}\int_0^t\int_{L_z}(u_{,122})^2dx_2d\eta+\frac{1}{4}\int_0^t\int_{L_z}(u_{,22})^2dx_2d\eta\\&+\frac{2}{9}\varepsilon_1\int_0^t\int_{L_z}(u_{,22})^2dx_2d\eta+\frac{2}{9\varepsilon_1}\frac{h^2}{\pi^2}\int_0^t\int_{L_z}(u_{,112})^2dx_2d\eta\\&+\frac{2}{9}\varepsilon_2\int_0^t\int_{L_z}(u_{,22})^2dx_2d\eta+\frac{2}{9\varepsilon_2}\frac{h^2}{\pi^2}\int_0^t\int_{L_z}(u_{,2\eta})^2dx_2d\eta\\&+\frac{h^2}{\pi^2}\int_0^t\int_{L_z}(u_{,2\eta})^2dx_2d\eta+2\frac{h^2}{\pi^2}\int_{L_z}(u_{,12})^2dx_2|_{\eta=t}\\&+\varepsilon_3\frac{h^2}{\pi^2}\int_0^t\int_{L_z}(u_{,112})^2dx_2d\eta+\frac{1}{\varepsilon_3}\frac{h^2}{\pi^2}\int_0^t\int_{L_z}(u_{,2\eta})^2dx_2d\eta\\&+\frac{1}{2}\frac{h^2}{\pi^2}\int_0^t\int_{L_z}(u_{,112})^2dx_2d\eta+2\int_0^t\int_{L_z}(u_{,11})^2dx_2d\eta\\&+4\int_0^t\int_{L_z}(u_{,12})^2dx_2d\eta+\frac{3}{2}\int_0^t\int_{L_z}(u_{,22})^2dx_2d\eta\\&+\frac{2}{9}\varepsilon_4\frac{h^2}{\pi^2}\int_0^t\int_{L_z}(u_{,122})^2dx_2d\eta+\frac{2}{9\varepsilon_4}\frac{h^2}{\pi^2}\int_0^t\int_{L_z}(u_{,111})^2dx_2d\eta].\end{aligned}\tag{3.20}$$

for some positive constants $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$.

We choose $\varepsilon_1 = 1, \varepsilon_2 = \frac{1}{8}, \varepsilon_3 = 1, \varepsilon_4 = \frac{1}{9}$, and use (3.19), then we can obtain

$$\begin{aligned}|\Phi(z,t)| \leqslant\ &2\frac{h^2}{\pi^2}[\int_0^t\int_{L_z}u_{,\alpha\beta}u_{,\alpha\beta}dx_2d\eta+2\frac{h^2}{\pi^2}\int_0^t\int_{L_z}u_{,\alpha\eta}u_{,\alpha\eta}dx_2d\eta\\&+\frac{h^2}{\pi^2}\int_0^t\int_{L_z}u_{,1\alpha\beta}u_{,1\alpha\beta}dx_2d\eta+\frac{1}{2}\frac{h^2}{\pi^2}\int_{L_z}u_{,\alpha}u_{,\alpha}dx_2|_{\eta=t}\\&+\frac{h^2}{\pi^2}\int_{L_z}u_{,\alpha\beta}u_{,\alpha\beta}dx_2|_{\eta=t}]\\&\leqslant 2\frac{h^2}{\pi^2}[\frac{\partial^2}{\partial z^2}\Phi(z,t)+|\int_0^t\int_{L_z}(u_{,11}u_{,\eta}+u_{,1}u_{,1\eta}-\frac{h^2}{\pi^2}u_{,121}u_{,2\eta})dx_2d\eta\\&+\frac{1}{2}\frac{h^2}{\pi^2}\int_{L_z}(u_{,12})^2dx_2|_{\eta=t}|].\end{aligned}\tag{3.21}$$

Now we denote

$$y(z,t)=\int_0^t\int_{L_z}(u_{,11}u_{,\eta}+u_{,1}u_{,1\eta}-\frac{h^2}{\pi^2}u_{,121}u_{,2\eta})dx_2d\eta+\frac{1}{2}\frac{h^2}{\pi^2}\int_{L_z}(u_{,12})^2dx_2|_{\eta=t}.$$

By using Schwarz's inequality and Wirtinger's inequality (3. 1), we can get

$$\begin{aligned} |y(z,t)| \leqslant\ & \frac{1}{2}\int_0^t\int_{L_z} u_{,\alpha\beta}u_{,\alpha\beta}dx_2d\eta + \frac{h^2}{\pi^2}\int_0^t\int_{L_z} u_{,\alpha\eta}u_{,\alpha\eta}dx_2d\eta \\ +\ & \frac{1}{2}\frac{h^2}{\pi^2}\int_0^t\int_{L_z} u_{,1\alpha\beta}u_{,1\alpha\beta}dx_2d\eta + \frac{1}{2}\frac{h^2}{\pi^2}\int_{L_z}(u_{,12})^2dx_2|_{\eta=t} \qquad (3.22) \\ \leqslant\ & \frac{1}{2}\frac{\partial^2}{\partial z^2}\Phi(z,t) + \frac{1}{2}|y(z,t)|. \qquad (3.23) \end{aligned}$$

In view of (3. 23), we have

$$|y(z,t)| \leqslant \frac{\partial^2}{\partial z^2}\Phi(z,t). \tag{3.24}$$

Then, combining (3.21) and (3.24), it follows

$$|\Phi(z,t)| \leqslant 4\frac{h^2}{\pi^2}\frac{\partial^2}{\partial z^2}\Phi(z,t). \tag{3.25}$$

From (3.19) and (3.22), we have

$$\begin{aligned} \frac{\partial^2}{\partial z^2}\Phi(z,t) \geqslant & \frac{1}{2}\{\int_0^t\int_{L_z} u_{,\alpha\beta}u_{,\alpha\beta}dx_2d\eta + \frac{1}{2}\int_{L_z} u_{,\alpha}u_{,\alpha}dx_2|_{\eta=t} \\ & + \frac{h^2}{\pi^2}[2\int_0^t\int_{L_z} u_{,\alpha\eta}u_{,\alpha\eta}dx_2d\eta + \int_{L_z} u_{,\alpha\beta}u_{,\alpha\beta}dx_2|_{\eta=t} \\ & + \int_0^t\int_{L_z} u_{,1\alpha\beta}u_{,1\alpha\beta}dx_2d\eta + \frac{1}{2}\int_{L_z} u_{,1\alpha}u_{,1\alpha}dx_2|_{\eta=t}]\} \geqslant 0. \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial z^2}\Phi(z,t) \leqslant & \frac{3}{2}\{\int_0^t\int_{L_z} u_{,\alpha\beta}u_{,\alpha\beta}dx_2d\eta + \frac{1}{2}\int_{L_z} u_{,\alpha}u_{,\alpha}dx_2|_{\eta=t} \\ & + \frac{h^2}{\pi^2}[2\int_0^t\int_{L_z} u_{,\alpha\eta}u_{,\alpha\eta}dx_2d\eta + \int_{L_z} u_{,\alpha\beta}u_{,\alpha\beta}dx_2|_{\eta=t} \\ & + \int_0^t\int_{L_z} u_{,1\alpha\beta}u_{,1\alpha\beta}dx_2d\eta + \frac{1}{2}\int_{L_z} u_{,1\alpha}u_{,1\alpha}dx_2|_{\eta=t}]\}. \end{aligned} \tag{3.27}$$

From (3.25), we have the following two inequalities:

$$\frac{\partial^2}{\partial z^2}\Phi(z,t) + \frac{\pi^2}{4h^2}\Phi(z,t) \geqslant 0, \tag{3.28}$$

$$\frac{\partial^2}{\partial z^2}\Phi(z,t) - \frac{\pi^2}{4h^2}\Phi(z,t) \geqslant 0. \tag{3.29}$$

which will be utilized in the next section.

4. Results of the Phragmén-Lindelöf type alternative

We confine our problems to the region Ω_0 defined by (2.1), and we consider two possible cases for the derivative of $\Phi(z,t)$.

Case 1: There exists a $z_0 \geqslant 0$ in which $\frac{\partial}{\partial z}\Phi(z_0,t) > 0$

Since $\frac{\partial^2}{\partial z^2}\Phi(z,t) \geqslant 0$ for all $z,t > 0$, we have $\frac{\partial}{\partial z}\Phi(z,t) > 0$ for all $z \geqslant z_0$. Moreover, since $\Phi(z,t) \geqslant \Phi(z_0,t) + \frac{\partial}{\partial z}\Phi(z_0,t)(z-z_0)$, $z \geqslant z_0, t > 0$, it follows that $\Phi(z,t)$ must eventually become positive, hence there must be a z_1, at which both $\frac{\partial}{\partial z}\Phi(z_1,t) > 0$ and $\Phi(z_1,t) > 0$ $(z_1 \geqslant z_0)$.

We may rewrite (3.29) as

$$\left(e^{-kz}[\frac{\partial}{\partial z}\Phi(z,t) + k\Phi(z,t)]\right)' \geqslant 0, \tag{4.1}$$

or

$$\left(e^{kz}[\frac{\partial}{\partial z}\Phi(z,t) - k\Phi(z,t)]\right)' \geqslant 0. \tag{4.2}$$

Where $k = \frac{\pi}{2h}$ and $'$ denotes the partial differentiation with respect to z.

Integrating (4.1) and (4.2), we have the following results

$$\frac{\partial}{\partial z}\Phi(z,t) + k\Phi(z,t) \geqslant [\frac{\partial}{\partial z}\Phi(z_1,t) + k\Phi(z_1,t)]e^{k(z-z_1)}, \tag{4.3}$$

$$\frac{\partial}{\partial z}\Phi(z,t) - k\Phi(z,t) \geqslant [\frac{\partial}{\partial z}\Phi(z_1,t) - k\Phi(z_1,t)]e^{-k(z-z_1)}. \tag{4.4}$$

for $z \geqslant z_1$, and hence we further get

$$\frac{\partial}{\partial z}\Phi(z,t) \geqslant \frac{\partial}{\partial z}\Phi(z_1,t)\cosh k(z-z_1) + k\Phi(z_1,t)\sinh k(z-z_1). \tag{4.5}$$

If we integrate (3.19) from z_1 to z and use (4.5), then we have

$$\begin{aligned}
&\int_0^t\int_{z_1}^z\int_{L_\xi} u_{,\alpha\beta}u_{,\alpha\beta}dAd\eta + \frac{1}{2}\int_{z_1}^z\int_{L_\xi} u_{,\alpha}u_{,\alpha}dA|_{\eta=t} \\
&\quad + \int_0^t\int_{z_1}^z\int_{L_\xi} u_{,11}u_{,\eta}dAd\eta + \int_0^t\int_{z_1}^z\int_{L_\xi} u_{,1}u_{,1\eta}dAd\eta \\
&\quad + \frac{h^2}{\pi^2}\int_0^t\int_{z_1}^z\int_{L_\xi} 2u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta + \frac{h^2}{\pi^2}\int_0^t\int_{z_1}^z\int_{L_\xi} u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta \\
&\quad + \frac{h^2}{\pi^2}\int_{z_1}^z\int_{L_\xi} u_{,\alpha\beta}u_{,\alpha\beta}dA|_{\eta=t} - \frac{h^2}{\pi^2}\int_0^t\int_{z_1}^z\int_{L_\xi} u_{,121}u_{,2\eta}dAd\eta \\
&\quad - \frac{1}{2}\frac{h^2}{\pi^2}\int_{z_1}^z\int_{L_\xi} (u_{,12})^2dA|_{\eta=t} + \frac{1}{2}\frac{h^2}{\pi^2}\int_{z_1}^z\int_{L_\xi} u_{,1\alpha}u_{,1\alpha}dA|_{\eta=t} \\
&\geqslant \frac{\partial}{\partial z}\Phi(z_1,t)[\cosh k(z-z_1) - 1] + k\Phi(z_1,t)\sinh k(z-z_1).
\end{aligned} \tag{4.6}$$

Integrating (3.17) from z_1 to z, we get

$$\begin{aligned}&\frac{3}{2}[\int_0^t\int_{z_1}^z\int_{L_\xi}u_{,\alpha\beta}u_{,\alpha\beta}dAd\eta+\frac{1}{2}\int_{z_1}^z\int_{L_\xi}u_{,\alpha}u_{,\alpha}dA|_{\eta=t}\\&\quad+2\frac{h^2}{\pi^2}\int_0^t\int_{z_1}^z\int_{L_\xi}u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta+\frac{h^2}{\pi^2}\int_{z_1}^z\int_{L_\xi}u_{,\alpha\beta}u_{,\alpha\beta}dA|_{\eta=t}\\&\quad+\frac{h^2}{\pi^2}\int_0^t\int_{z_1}^z\int_{L_\xi}u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta+\frac{1}{2}\frac{h^2}{\pi^2}\int_{z_1}^z\int_{L_\xi}u_{,1\alpha}u_{,1\alpha}dA|_{\eta=t}]\\&\geqslant\int_{z_1}^z\frac{\partial^2}{\partial s^2}\Phi(s,t)ds.\end{aligned}\tag{4.7}$$

Combining (4.6) and (4.7), we conclude that

$$\begin{aligned}&\lim_{z\longrightarrow\infty}\{e^{-kz}[\int_0^t\int_0^z\int_{L_\xi}u_{,\alpha\beta}u_{,\alpha\beta}dAd\eta+\frac{1}{2}\int_0^z\int_{L_\xi}u_{,\alpha}u_{,\alpha}dA|_{\eta=t}\\&\quad+2\frac{h^2}{\pi^2}\int_0^t\int_0^z\int_{L_\xi}u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta+\frac{h^2}{\pi^2}\int_0^z\int_{L_\xi}u_{,\alpha\beta}u_{,\alpha\beta}dA|_{\eta=t}\\&\quad+\frac{h^2}{\pi^2}\int_0^t\int_0^z\int_{L_\xi}u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta+\frac{1}{2}\frac{h^2}{\pi^2}\int_0^z\int_{L_\xi}u_{,1\alpha}u_{,1\alpha}dA|_{\eta=t}]\}\\&\geqslant c_1(t).\end{aligned}\tag{4.8}$$

Where $c_1(t)$ is a positive expression depending on z_1 , i. e.

$$c_1(t)=\frac{1}{3}e^{-kz_1}[\frac{\partial}{\partial z}\Phi(z_1,t)+k\Phi(z_1,t)].$$

We can conclude from (4.8) that

$$\lim_{z\longrightarrow\infty}\{e^{-kz}E_1(z,t)\}\geqslant c_1(t).\tag{4.9}$$

Case 2: $\frac{\partial}{\partial z}\Phi(z,t)\leqslant 0$ for all $z\geqslant 0$.

We first show that in this case, $\Phi(z,t)\geqslant 0$ for all $z\geqslant 0$. If we suppose that there is a $z_0>0$, such that $\Phi(z_0,t)<0$, then by our assumption $\frac{\partial}{\partial z}\Phi(z,t)\leqslant 0$. We have $\Phi(z,t)\leqslant\Phi(z_0,t)$ for all $z\geqslant z_0$. But by (3.28), we have $\frac{\partial}{\partial z}\Phi(z,t)-\frac{\partial}{\partial z}\Phi(z_0,t)\geqslant -k^2\Phi(z_0,t)(z-z_0)$ and hence $\frac{\partial}{\partial z}\Phi(z,t)$ cannot remain nonpositive for all z. By this contradiction, we conclude that if $\frac{\partial}{\partial z}\Phi(z,t)\leqslant 0$ for all $z\geqslant 0$, then $\Phi(z,t)\geqslant 0$ for all $z\geqslant 0$.

In order to derive our estimates in this case, we integrate (4. 2) from 0 to z and obtain

$$-\frac{\partial}{\partial z}\Phi(z,t)+k\Phi(z,t)\leqslant c_2(t)e^{-kz}.\tag{4.10}$$

Where $c_2(t)=-\frac{\partial}{\partial z}\Phi(0,t)+k\Phi(0,t)$.

Since $\Phi(z,t)\geqslant 0$ for all $z\geqslant 0$, from (4.10), we see that $\Phi(z,t)$ and $-\frac{\partial}{\partial z}\Phi(z,t)$ decay exponentially as $z\to\infty$, so we can write

$$\begin{aligned}
-\frac{\partial}{\partial z}\Phi(z,t) = &\int_0^t\int_z^\infty\int_{L_\xi} u_{,\alpha\beta}u_{,\alpha\beta}dAd\eta + \frac{1}{2}\int_z^\infty\int_{L_\xi} u_{,\alpha}u_{,\alpha}dA|_{\eta=t}\\
&+\int_0^t\int_z^\infty\int_{L_\xi} u_{,11}u_{,\eta}dAd\eta + \int_0^t\int_z^\infty\int_{L_\xi} u_{,1}u_{,1\eta}dAd\eta\\
&+\frac{h^2}{\pi^2}\int_0^t\int_z^\infty\int_{L_\xi}(2u_{,\alpha\eta}u_{,\alpha\eta}+u_{,1\alpha\beta}u_{,1\alpha\beta})dAd\eta\\
&+\frac{h^2}{\pi^2}\int_z^\infty\int_{L_\xi}(u_{,\alpha\beta}u_{,\alpha\beta}+\frac{1}{2}u_{,1\alpha}u_{,1\alpha}-\frac{1}{2}(u_{,12})^2)dA|_{\eta=t}\\
&-\frac{h^2}{\pi^2}\int_0^t\int_z^\infty\int_{L_\xi} u_{,121}u_{,2\eta}dAd\eta.
\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}
\Phi(z,t) = &\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)u_{,\alpha\beta}u_{,\alpha\beta}dAd\eta+\frac{1}{2}\int_z^\infty\int_{L_\xi}(\xi-z)u_{,\alpha}u_{,\alpha}dA|_{\eta=t}\\
&+\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)u_{,11}u_{,\eta}dAd\eta+\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)u_{,1}u_{,1\eta}dAd\eta\\
&+\frac{h^2}{\pi^2}\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)(2u_{,\alpha\eta}u_{,\alpha\eta}+u_{,1\alpha\beta}u_{,1\alpha\beta})dAd\eta\\
&+\frac{h^2}{\pi^2}\int_z^\infty\int_{L_\xi}(\xi-z)(u_{,\alpha\beta}u_{,\alpha\beta}+\frac{1}{2}u_{,1\alpha}u_{,1\alpha}-\frac{1}{2}(u_{,12})^2)dA|_{\eta=t}\\
&-\frac{h^2}{\pi^2}\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)u_{,121}u_{,2\eta}dAd\eta\geqslant 0.
\end{aligned}\tag{4.12}$$

From (4.10), we have

$$-\frac{\partial}{\partial z}\Phi(z,t)\leqslant c_2(t)e^{-kz}.\tag{4.13}$$

As shown in (4.7), from (4.11) we have $\frac{1}{2}E_2(z,t)\leqslant -\frac{\partial}{\partial z}\Phi(z,t)$. So we have

$$E_2(z,t)\leqslant 2c_2(t)e^{-kz}.\tag{4.14}$$

Making use of Schwarz's inequality and Wirginter's inequality, we find that

$$\begin{aligned}
\Phi(z,t)\geqslant \frac{1}{2}[&\int_0^t\int_{z_1}^z\int_{L_\xi}(\xi-z)u_{,\alpha\beta}u_{,\alpha\beta}dAd\eta+\frac{1}{2}\int_{z_1}^z\int_{L_\xi}(\xi-z)u_{,\alpha}u_{,\alpha}dA|_{\eta=t}\\
&+2\frac{h^2}{\pi^2}\int_0^t\int_{z_1}^z\int_{L_\xi}(\xi-z)u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta+\frac{h^2}{\pi^2}\int_{z_1}^z\int_{L_\xi}(\xi-z)u_{,\alpha\beta}u_{,\alpha\beta}dA|_{\eta=t}\\
&+\frac{h^2}{\pi^2}\int_0^t\int_{z_1}^z\int_{L_\xi}(\xi-z)u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta+\frac{1}{2}\frac{h^2}{\pi^2}\int_{z_1}^z\int_{L_\xi}(\xi-z)u_{,1\alpha}u_{,1\alpha}dA|_{\eta=t}].
\end{aligned}\tag{4.15}$$

Consequently, in virtue of (4.10), (4.11), (4.12) and (4.15), we have decay estimates for some weighted energy expressions, namely

$$\int_0^t \int_z^\infty \int_{L_\xi} [1+k(\xi-z)]u_{,\alpha\beta}u_{,\alpha\beta}dAd\eta + \frac{1}{2}\int_z^\infty \int_{L_\xi} [1+k(\xi-z)]u_{,\alpha}u_{,\alpha}dA|_{\eta=t}$$
$$+\frac{h^2}{\pi^2}\int_0^t \int_z^\infty \int_{L_\xi} [1+k(\xi-z)](2u_{,\alpha\eta}u_{,\alpha\eta}+u_{,1\alpha\beta}u_{,1\alpha\beta})dAd\eta \tag{4.16}$$
$$+\frac{h^2}{\pi^2}\int_z^\infty \int_{L_\xi} [1+k(\xi-z)](u_{,\alpha\beta}u_{,\alpha\beta}+\frac{1}{2}u_{,1\alpha}u_{,1\alpha})dA|_{\eta=t} \leqslant 2c_2(t)e^{-kz}.$$

We summarize the above results in the following theorem.

THEOREM 4.1. *Let u be a solution of problem* $(2.9)-(2.14)$ *in the semi-infinite strip* Ω_0 *defined by* (2.1)*, then either inequality* (4.9) *or* (4.16) *holds, in which* $k=\frac{\pi}{2h}$, $c_1(t)=\frac{1}{3}e^{-kz_1}[\frac{\partial}{\partial z}\Phi(z_1,t)+k\Phi(z_1,t)]$, $c_2(t)=-\frac{\partial}{\partial z}\Phi(0,t)+k\Phi(0,t)$.

This is our result of the Phragmén-Lindelöf type alternative.

As an application of the Theorem, for the decay case, we can establish a pointwise decay estimate for the solution of problem (2.9)-(2.14), i. e.

$$\max_{x_2\in[0,h]} u^2(z,x_2,t) \leqslant \frac{h^2}{2\pi^2}\int_z^\infty \int_{L_\xi} u_{,\alpha\beta}u_{,\alpha\beta}dA \leqslant c_2(t)e^{-kz}. \tag{4.17}$$

The proof is analogous to that in [23], so we omit it.

5. Upper bound for the total energy

We now show how an explicit bound can be determined for $c_2(t)$ which occured in *Case* 2, where $\frac{\partial}{\partial z}\Phi(z,t)\leqslant 0$ for all $z\geqslant 0$. Since $c_2(t)=-\frac{\partial}{\partial z}\Phi(0,t)+k\Phi(0,t)$, and since $\Phi(z,t)$ and $-\frac{\partial}{\partial z}\Phi(z,t)$ decay exponentially as $z\longrightarrow\infty$. We need only to consider integrals of (4.11) and (4.12).

Firstly, we seek a bound for $-\frac{\partial}{\partial z}\Phi(0,t)$. Since $-\frac{\partial}{\partial z}\Phi(0,t)\leqslant\frac{3}{2}E_2(0,t)(see(4.7))$, we need only to bound $E_2(0,t)$.

Now, we write

$$E_2(z,t)=F_1(z,t)+\frac{2h^2}{\pi^2}F_2(z,t)+\frac{h^2}{\pi^2}F_3(z,t), \tag{5.1}$$

where

$$F_1(z,t)=\int_0^t \int_z^\infty \int_{L_\xi} u_{,\alpha\beta}u_{,\alpha\beta}dAd\eta+\frac{1}{2}\int_z^\infty \int_{L_\xi} u_{,\alpha}u_{,\alpha}dA|_{\eta=t},$$
$$F_2(z,t)=\int_0^t \int_z^\infty \int_{L_\xi} u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta+\frac{1}{2}\int_z^\infty \int_{L_\xi} u_{,\alpha\beta}u_{,\alpha\beta}dA|_{\eta=t},$$
$$F_3(z,t)=\int_0^t \int_z^\infty \int_{L_\xi} u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta+\frac{1}{2}\int_z^\infty \int_{L_\xi} u_{,1\alpha}u_{,1\alpha}dA|_{\eta=t}.$$

$$\begin{aligned} F_u(z,t) =& F_1(z,t) + \frac{2h^2}{\pi^2}F_2(z,t) \\ =& \int_0^t \int_z^\infty \int_{L_\xi} u_{,\alpha\beta}u_{,\alpha\beta}dAd\eta + \frac{1}{2}\int_z^\infty \int_{L_\xi} u_{,\alpha}u_{,\alpha}dA|_{\eta=t} \\ &+ 2\frac{h^2}{\pi^2}\int_0^t \int_z^\infty \int_{L_\xi} u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta + \frac{h^2}{\pi^2}\int_z^\infty \int_{L_\xi} u_{,\alpha\beta}u_{,\alpha\beta}dA|_{\eta=t}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} F_w(z,t) =& \int_0^t \int_z^\infty \int_{L_\xi} w_{,\alpha\beta}w_{,\alpha\beta}dAd\eta + \frac{1}{2}\int_z^\infty \int_{L_\xi} w_{,\alpha}w_{,\alpha}dA|_{\eta=t} \\ &+ 2\frac{h^2}{\pi^2}\int_0^t \int_z^\infty \int_{L_\xi} w_{,\alpha\eta}w_{,\alpha\eta}dAd\eta + \frac{h^2}{\pi^2}\int_z^\infty \int_{L_\xi} w_{,\alpha\beta}w_{,\alpha\beta}dA|_{\eta=t}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} F_{uw}(z,t) =& \int_0^t \int_z^\infty \int_{L_\xi} u_{,\alpha\beta}w_{,\alpha\beta}dAd\eta + \frac{1}{2}\int_z^\infty \int_{L_\xi} u_{,\alpha}w_{,\alpha}dA|_{\eta=t} \\ &+ 2\frac{h^2}{\pi^2}\int_0^t \int_z^\infty \int_{L_\xi} u_{,\alpha\eta}w_{,\alpha\eta}dAd\eta + \frac{h^2}{\pi^2}\int_z^\infty \int_{L_\xi} u_{,\alpha\beta}w_{,\alpha\beta}dA|_{\eta=t}. \end{aligned} \tag{5.4}$$

where $w(x_1,x_2,t)$ is an arbitrary smooth function defined on Ω_0 that satisfies the same boundary conditions as u.

By using Schwarz's inequality, we get

$$\sqrt{F_u(z,t)F_w(z,t)} \geqslant F_{uw}(z,t). \tag{5.5}$$

It is clear that

$$\begin{aligned} F_{uw}(0,t) =& F_u(0,t) + \int_0^t \int_0^\infty \int_{L_\xi} u_{,\alpha\beta}(w_{,\alpha\beta}-u_{,\alpha\beta})dAd\eta + \frac{1}{2}\int_0^\infty \int_{L_\xi} u_{,\alpha}(w_{,\alpha}-u_{,\alpha})dA|_{\eta=t} \\ &+ 2\frac{h^2}{\pi^2}\int_0^t \int_0^\infty \int_{L_\xi} u_{,\alpha\eta}(w_{,\alpha\eta}-u_{,\alpha\eta})dAd\eta + \frac{h^2}{\pi^2}\int_0^\infty \int_{L_\xi} u_{,\alpha\beta}(w_{,\alpha\beta}-u_{,\alpha\beta})dA|_{\eta=t} \\ =& F_u(0,t) + I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{5.6}$$

Making use of the divergence theorem, and (2.9)-(2.14), we obtain

$$\begin{aligned} I_1 =& -\int_0^t \int_0^\infty \int_{L_\xi} u_{,\alpha\beta\beta}(w_{,\alpha} - u_{,\alpha})dAd\eta = \int_0^t \int_0^\infty \int_{L_\xi} u_{,\alpha\alpha\eta}(w-u)dAd\eta \\ =& -\int_0^t \int_0^\infty \int_{L_\xi} u_{,\alpha\eta}w_{,\alpha}dAd\eta + \int_0^t \int_0^\infty \int_{L_\xi} u_{,\alpha\eta}u_{,\alpha}dAd\eta \\ \geqslant& -\frac{\varepsilon_1}{2}\frac{h^2}{\pi^2}\int_0^t \int_0^\infty \int_{L_\xi} u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta - \frac{1}{2\varepsilon_1}\int_0^t \int_0^\infty \int_{L_\xi} w_{,\alpha 2}w_{,\alpha 2}dAd\eta \\ &+ \frac{1}{2}\int_0^\infty \int_{L_\xi} u_{,\alpha}u_{,\alpha}dA|_{\eta=t}. \end{aligned} \tag{5.7}$$

$$
\begin{aligned}
I_2 =& \frac{1}{2}\int_0^\infty \int_{L_\xi} u_{,\alpha} w_{,\alpha} dA|_{\eta=t} - \frac{1}{2}\int_0^\infty \int_{L_\xi} u_{,\alpha} u_{,\alpha} dA|_{\eta=t} \\
\geqslant& -\frac{\varepsilon_2}{2}\frac{h^2}{\pi^2}\int_0^\infty \int_{L_\xi} u_{,\alpha 2} u_{,\alpha 2} dA|_{\eta=t} - \frac{1}{8\varepsilon_2}\int_0^\infty \int_{L_\xi} w_{,\alpha} w_{,\alpha} dA|_{\eta=t} \\
&- \frac{1}{2}\int_0^\infty \int_{L_\xi} u_{,\alpha} u_{,\alpha} dA|_{\eta=t}.
\end{aligned} \tag{5.8}
$$

$$
\begin{aligned}
I_3 =& -\frac{2h^2}{\pi^2}\int_0^t \int_0^\infty \int_{L_\xi} u_{,\alpha\alpha\eta}(w_{,\eta} - u_{,\eta}) dA d\eta \\
=& \frac{2h^2}{\pi^2}\int_0^t \int_0^\infty \int_{L_\xi} u_{,\alpha\beta\beta}(w_{,\alpha\eta} - u_{,\alpha\eta}) dA d\eta \\
=& \frac{2h^2}{\pi^2}\int_0^t \int_0^\infty \int_{L_\xi} u_{,\alpha\beta}(u_{,\alpha\beta\eta} - w_{,\alpha\beta\eta}) dA d\eta \\
\geqslant& \frac{h^2}{\pi^2}\int_0^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t} - \varepsilon_3 \int_0^t \int_0^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA d\eta \\
&- \frac{1}{\varepsilon_3}\frac{h^4}{\pi^4}\int_0^t \int_0^\infty \int_{L_\xi} w_{,\alpha\beta\eta} w_{,\alpha\beta\eta} dA d\eta.
\end{aligned} \tag{5.9}
$$

$$
\begin{aligned}
I_4 =& \frac{h^2}{\pi^2}\int_0^\infty \int_{L_\xi} u_{,\alpha\beta} w_{,\alpha\beta} dA|_{\eta=t} - \frac{h^2}{\pi^2}\int_0^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t} \\
\geqslant& -\frac{\varepsilon_4}{2}\frac{h^2}{\pi^2}\int_0^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t} - \frac{1}{2\varepsilon_4}\frac{h^2}{\pi^2}\int_0^\infty \int_{L_\xi} w_{,\alpha\beta} w_{,\alpha\beta} dA|_{\eta=t} \\
&- \frac{h^2}{\pi^2}\int_0^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t}.
\end{aligned} \tag{5.10}
$$

Setting $\varepsilon_1 = 1$, $\varepsilon_2 = \frac{1}{4}$, $\varepsilon_3 = \frac{1}{4}$, $\varepsilon_4 = \frac{1}{4}$, then combining (5.6)-(5.9), we are led to

$$
\sqrt{F_u(0,t)F_w(0,t)} \geqslant \frac{3}{4}F_u(0,t) - Q(w(0,t)), \tag{5.11}
$$

where we have defined

$$
\begin{aligned}
Q(w(0,t)) =& \frac{1}{2}\int_0^t \int_0^\infty \int_{L_\xi} w_{,\alpha 2} w_{,\alpha 2} dA d\eta + \frac{1}{2}\int_0^\infty \int_{L_\xi} w_{,\alpha} w_{,\alpha} dA|_{\eta=t} \\
&+ 4\frac{h^4}{\pi^4}\int_0^t \int_0^\infty \int_{L_\xi} w_{,\alpha\beta\eta} w_{,\alpha\beta\eta} dA d\eta + 2\frac{h^2}{\pi^2}\int_0^\infty \int_{L_\xi} w_{,\alpha\beta} w_{,\alpha\beta} dA|_{\eta=t}.
\end{aligned}
$$

From (5.10), we can conclude that

$$
\frac{3}{4}(\sqrt{F_u(0,t)} - \frac{2}{3}\sqrt{F_w(0,t)})^2 \leqslant \frac{1}{3}(F_w(0,t) + 3Q(w(0,t))). \tag{5.12}
$$

which implies that

$$\sqrt{F_u(0,t)} \leqslant \frac{2}{3}\sqrt{F_w(0,t)} + \frac{2}{3}(F_w(0,t) + 3Q(w(0,t)))^{\frac{1}{2}}. \quad (5.13)$$

Squaring and making use of the arithmetic-geometric mean inequality, we obtain

$$F_u(0,t) \leqslant \frac{16}{9}F_w(0,t) + \frac{8}{3}Q(w(0,t)). \quad (5.14)$$

We now define $w(x_1,x_2,t)$ as

$$w(x_1,x_2,t)=\left[\int_0^{x_2}\frac{\partial g_1}{\partial \xi}(\xi,t)d\xi+x_1\left(g_2(x_2,t)+\int_0^{x_2}\frac{\partial g_1}{\partial \xi}(\xi,t)d\xi\right)\right]e^{-x_1}. \quad (5.15)$$

Obviously, it is easy to verify $w(x_1,x_2,t)$ satisfying the same initial-boundary value conditions as $u(x_1,x_2,t)$.

Thus, from (5.13), we have bounded $F_u(0,t)$.

The next step is to bound $F_3(0,t)$. We will use the following lemma:

LEMMA 5.1. *In Case* 2, *for arbitrary* $z \geqslant 0$, $z_0 > 0$.

$$z_0^2 F_3(z+z_0,t) \leqslant 2F_1(z,t) + 4\frac{h^2}{\pi^2}F_2(z,t). \quad (5.16)$$

Proof. We adopt the notation

$$\widetilde{F_3}(z,t)=\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)^2u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta+\frac{1}{2}\int_z^\infty\int_{L_\xi}(\xi-z)^2u_{,1\alpha}u_{,1\alpha}dA|_{\eta=t}. \quad (5.17)$$

Obviously, we have the following identity

$$\begin{aligned}0 &= \int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)^2u_{,11}\Delta^2u - \int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)^2u_{,11}\Delta u_{,\eta}dAd\eta \\ &= K_1+K_2.\end{aligned} \quad (5.18)$$

By applying the divergence theorem repeatedly and using the initial-boundary value conditions, we obtain

$$\begin{aligned}K_1 =& -\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)^2u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta + \int_0^t\int_z^\infty\int_{L_\xi}u_{,\alpha\beta}u_{,\alpha\beta}dAd\eta \\ &-2\int_0^t\int_z^\infty\int_{L_\xi}u_{,12}^2dAd\eta,\end{aligned} \quad (5.19)$$

$$K_2 = -\frac{1}{2}\int_z^\infty\int_{L_\xi}(\xi-z)^2u_{,1\alpha}u_{,1\alpha}dA|_{\eta=t}+2\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)u_{,12}u_{,2\eta}dAd\eta. \quad (5.20)$$

Inserting (5.18), (5.19) into (5.17), we find

$$\widetilde{F_3}(z,t) \leqslant \int_0^t\int_z^\infty\int_{L_\xi}u_{,\alpha\beta}u_{,\alpha\beta}dAd\eta+2\left|\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)u_{,12}u_{,2\eta}dAd\eta\right|. \quad (5.21)$$

Using Schwarz's inequality, we have

$$2\left|\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)u_{,12}u_{,2\eta}dAd\eta\right| \leqslant \varepsilon\frac{h^2}{\pi^2}\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)^2u_{,122}^2dAd\eta+\frac{1}{\varepsilon}\int_0^t\int_z^\infty\int_{L_\xi}(u_{,2\eta})^2dAd\eta, \tag{5.22}$$

making the choice $\varepsilon=\frac{\pi^2}{2h^2}$ and substituting from (5.21) and (5.16) into (5.20) to yield

$$\widetilde{F_3}(z,t)\leqslant 2F_1(z,t)+4\frac{h^2}{\pi^2}F_2(z,t). \tag{5.23}$$

We note that for arbitrary $z_0>0$

$$\widetilde{F_3}(z,t)\geqslant\int_0^t\int_{z+z_0}^\infty\int_{L_\xi}(\xi-z)^2u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta+\frac{1}{2}\int_{z+z_0}^\infty\int_{L_\xi}(\xi-z)^2u_{,1\alpha}u_{,1\alpha}dA|_{\eta=t} \geqslant z_0^2F_3(z+z_0,t). \tag{5.24}$$

So we have proved the lemma.

From the definition of $F_3(0,t)$, since $\int_0^\infty\int_{L_\xi}u_{,1\alpha}u_{,1\alpha}dA\leqslant F_2(0,t)$, we need only to bound

$$\int_0^t\int_0^\infty\int_{L_\xi}u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta =\int_0^t\int_0^1\int_{L_\xi}u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta+\int_0^t\int_1^\infty\int_{L_\xi}u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta. \tag{5.25}$$

Setting $z=0$, $z_0=1$ in (5.15) leads to

$$\int_0^t\int_1^\infty\int_{L_\xi}u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta\leqslant 2F_1(0,t)+4\frac{h^2}{\pi^2}F_2(0,t). \tag{5.26}$$

From (5.20) and (5.21), we obtain

$$\int_0^t\int_0^\infty\int_{L_\xi}\xi^2u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta\leqslant 2\int_0^t\int_0^\infty\int_{L_\xi}u_{,\alpha\beta}u_{,\alpha\beta}dAd\eta+4\frac{h^2}{\pi^2}F_2(0,t). \tag{5.27}$$

Since (5.26) is equivalent to

$$\int_0^t\int_0^\infty\int_{L_\xi}\xi^2\Psi_{,\xi}\Psi_{,\xi}dAd\eta\leqslant 2\int_0^t\int_0^\infty\int_{L_\xi}\Psi^2dAd\eta+4\frac{h^2}{\pi^2}F_2(0,t). \tag{5.28}$$

where $\Psi(x_1,x_2,t)=u_{,\alpha\beta}$.

For the first integral on the right hand side of (5.24), making a variat transformation $x_1 = \frac{1}{\xi}$ and using inequality (5.27), we deduce that

$$\begin{aligned}\int_0^t \int_0^1 \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dA d\eta &= \int_0^t \int_1^\infty \int_{L_\xi} \xi^2 u_{,\xi\alpha\beta} u_{,\xi\alpha\beta} dA d\eta \\ &\leqslant \int_0^t \int_0^\infty \int_{L_\xi} \xi^2 u_{,\xi\alpha\beta} u_{,\xi\alpha\beta} dA d\eta \\ &\leqslant 2 \int_0^t \int_0^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA d\eta + 4\frac{h^2}{\pi^2} F_2(0,t) \\ &\leqslant 2F_1(0,t) + 4\frac{h^2}{\pi^2} F_2(0,t).\end{aligned} \tag{5.29}$$

Combining (5.24), (5.25) and (5.28), we finally obtain

$$F_3(0,t) \leqslant 4F_1(0,t) + (8\frac{h^2}{\pi^2} + 1) F_2(0,t). \tag{5.30}$$

Thus, from (5.13) and (5.29), we have bounded $-\frac{\partial}{\partial z}\Phi(0,t)$.
Next, we start to bound

$$\begin{aligned}\Phi(0,t) = &\int_0^t \int_0^\infty \int_{L_\xi} \xi u_{,\alpha\beta} u_{,\alpha\beta} dA d\eta + \frac{1}{2} \int_0^\infty \int_{L_\xi} \xi u_{,\alpha} u_{,\alpha} dA|_{\eta=t} \\ &+ \int_0^t \int_0^\infty \int_{L_\xi} \xi u_{,11} u_{,\eta} dA d\eta + \int_0^t \int_0^\infty \int_{L_\xi} \xi u_{,1} u_{,1\eta} dA d\eta \\ &+ \frac{h^2}{\pi^2} \int_0^t \int_0^\infty \int_{L_\xi} \xi (2u_{,\alpha\eta} u_{,\alpha\eta} + u_{,1\alpha\beta} u_{,1\alpha\beta}) dA d\eta \\ &+ \frac{h^2}{\pi^2} \int_0^\infty \int_{L_\xi} \xi (u_{,\alpha\beta} u_{,\alpha\beta} + \frac{1}{2} u_{,1\alpha} u_{,1\alpha} - \frac{1}{2} u_{,12} u_{,12}) dA|_{\eta=t} \\ &- \frac{h^2}{\pi^2} \int_0^t \int_0^\infty \int_{L_\xi} \xi u_{,121} u_{,2\eta} dA d\eta.\end{aligned} \tag{5.31}$$

Firstly, we bound

$$J_1 = \int_0^t \int_0^\infty \int_{L_\xi} \xi u_{,\alpha\eta} u_{,\alpha\eta} dA d\eta + \frac{1}{2} \int_0^\infty \int_{L_\xi} \xi u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t}.$$

Since

$$\begin{aligned}J_1 = &- \int_0^t \int_0^\infty \int_{L_\xi} u_{,\eta} u_{,1\eta} dA d\eta - \int_0^t \int_0^\infty \int_{L_\xi} \xi u_{,\eta} u_{,\alpha\alpha\beta\beta} dA d\eta \\ &+ \frac{1}{2} \int_0^\infty \int_{L_\xi} \xi u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t}\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^t \int_0^\infty \int_{L_\xi} u_{,\eta} u_{,1\eta} dA d\eta + \int_0^t \int_0^\infty \int_{L_\xi} u_{,\eta} u_{,\alpha\alpha 1} dA d\eta \\
&\quad + \int_0^t \int_0^\infty \int_{L_\xi} \xi u_{,\beta\eta} u_{,\alpha\alpha\beta} dA d\eta + \frac{1}{2} \int_0^\infty \int_{L_\xi} \xi u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t} \\
&= -\int_0^t \int_0^\infty \int_{L_\xi} u_{,\eta} u_{,1\eta} dA d\eta + \int_0^t \int_0^\infty \int_{L_\xi} u_{,\eta} u_{,\alpha\alpha 1} dA d\eta \\
&\quad - \int_0^t \int_0^\infty \int_{L_\xi} u_{,\beta\eta} u_{,1\beta} dA d\eta.
\end{aligned} \tag{5.32}
$$

By using Schwarz's inequality, Wirtinger's inequalities and results (5.15), (5.29), we obtain

$$
J_1 \leqslant data. \tag{5.33}
$$

Next, we bound

$$
\begin{aligned}
J_2 &= \int_0^t \int_0^\infty \int_{L_\xi} \xi u_{,\alpha\beta} u_{,\alpha\beta} dA d\eta + \frac{1}{2} \int_0^\infty \int_{L_\xi} \xi u_{,\alpha} u_{,\alpha} dA|_{\eta=t} \\
&\quad + \int_0^t \int_0^\infty \int_{L_\xi} \xi u_{,1} u_{,1\eta} dA d\eta + \int_0^t \int_0^\infty \int_{L_\xi} \xi u_{,11} u_{,\eta} dA d\eta \\
&= \int_0^t \int_{L_0} u_{,\alpha} u_{,\alpha} dx_2 d\eta - \int_0^t \int_{L_0} u u_{,11} dx_2 d\eta - \int_0^t \int_0^\infty \int_{L_\xi} u u_{,1\eta} dA d\eta \\
&\quad - \int_0^t \int_0^\infty \int_{L_\xi} u_{,1} u_{,\eta} dA d\eta \\
&\leqslant data + \frac{1}{2} \int_0^t \int_{L_0} u^2 dx_2 d\eta + \frac{1}{2} \int_0^t \int_{L_0} (u_{,11})^2 dx_2 d\eta \\
&\leqslant data + \frac{1}{2} \int_0^t \int_{L_0} (u_{,11})^2 dx_2 d\eta.
\end{aligned} \tag{5.34}
$$

Since we have

$$
\begin{aligned}
\int_0^t \int_{L_0} (u_{,11})^2 dx_2 d\eta &= 2 \int_0^t \int_0^\infty \int_{L_\xi} u_{,11} u_{,111} dA d\eta \\
&\leqslant data.
\end{aligned} \tag{5.35}
$$

Combining (5.33) and (5.34), we obtain

$$
J_2 \leqslant data. \tag{5.36}
$$

Finally, we bound

$$
\begin{aligned}
J_3 &= \int_0^t \int_0^\infty \int_{L_\xi} \xi u_{,1\alpha\beta} u_{,1\alpha\beta} dA d\eta + \frac{1}{2} \int_0^\infty \int_{L_\xi} \xi u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t} \\
&\quad - \int_0^t \int_0^\infty \int_{L_\xi} \xi u_{,121} u_{,2\eta} dA d\eta - \frac{1}{2} \int_0^\infty \int_{L_\xi} \xi u_{,12} u_{,12} dA|_{\eta=t}.
\end{aligned} \tag{5.37}
$$

In light of (5.32), we have

$$\begin{aligned} J_3 \leqslant & \frac{3}{2}\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta+\frac{1}{2}\int_0^\infty\int_{L_\xi}\xi u_{,1\alpha}u_{,1\alpha}dA|_{\eta=t} \\ & +\frac{1}{2}\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,2\eta}u_{,2\eta}dAd\eta \\ \leqslant & \frac{3}{2}\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta+data. \end{aligned} \tag{5.38}$$

In order to bound (5.37), we have to seek a bound of $\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta$. We find that

$$\begin{aligned} \int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,1\alpha\beta}u_{,1\alpha\beta}dAd\eta &= -\int_0^t\int_0^\infty\int_{L_\xi}u_{,1\alpha1}u_{,1\alpha}dAd\eta-\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,1\alpha}u_{,1\alpha\beta\beta}dAd\eta \\ &\leqslant data-\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,1\alpha}u_{,1\alpha\beta\beta}dAd\eta. \end{aligned} \tag{5.39}$$

We must bound the second term on the right of (5.38).

$$\begin{aligned} & -\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,1\alpha}u_{,1\alpha\beta\beta}dAd\eta \\ & =\int_0^t\int_0^\infty\int_{L_\xi}u_{,1\alpha}u_{,\alpha\beta\beta}dAd\eta+\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,11\alpha}u_{,\alpha\beta\beta}dAd\eta \\ & =-\int_0^t\int_0^\infty\int_{L_\xi}u_{,1\alpha\beta}u_{,\alpha\beta}dAd\eta-\int_0^t\int_{L_0}u_{,1\alpha}u_{,1\alpha}dx_2d\eta \\ & \quad -\int_0^t\int_0^\infty\int_{L_\xi}u_{,11}u_{,1\beta\beta}dAd\eta-\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,11}u_{,\alpha\alpha\eta}dAd\eta \\ & \leqslant data-\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,11}u_{,\alpha\alpha\eta}dAd\eta. \end{aligned} \tag{5.40}$$

Now, we set about the last term in (5.39)

$$\begin{aligned} & |\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,11}u_{,\alpha\alpha\eta}dAd\eta| \\ & =|\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,11\alpha}u_{,\alpha\eta}dAd\eta+\int_0^t\int_0^\infty\int_{L_\xi}u_{,11}u_{,1\eta}dAd\eta| \\ & \leqslant\frac{1}{2}\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,11\alpha}u_{,11\alpha}dAd\eta+\frac{1}{2}\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta+data \\ & \leqslant\frac{1}{2}\int_0^t\int_0^\infty\int_{L_\xi}\xi u_{,11\alpha}u_{,11\alpha}dAd\eta+data. \end{aligned} \tag{5.41}$$

So, combining (5.37)-(5.40), we can conclude

$$\int_0^t \int_0^\infty \int_{L_\xi} \xi u_{,1\alpha\beta} u_{,1\alpha\beta} dA d\eta \leqslant data. \tag{5.42}$$

Further, we obtain

$$J_3 \leqslant data. \tag{5.43}$$

Inserting (5.32), (5.35) and (5.42) into (5.30) to obtain

$$\Phi(0,t) \leqslant data. \tag{5.44}$$

Thus, we have finally established an explicit upper bound in terms of prescribed data for the total energy $c_2(t)$.

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