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HIGHER DIMENSIONAL COMPACTNESS OF HARDY OPERATORS INVOLVING OINAROV–TYPE KERNELS

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Abstract. The compactness of the higher dimensional generalized Hardy operator $(\mathcal{K}f)(x) = \int_{S_x} k(x,y)f(y)dy$ and its conjugate operator $\mathcal{K}^*$ has been characterized for the case $1 < p$, $q < \infty$. This is done by reducing the problem to the corresponding one dimensional situation.

1. Introduction

The L^p-L^q boundedness and compactness of the generalized Hardy operator $(Lf)(s) = \int_0^s l(s,t)f(t)dt$ involving the so called "Oinarov kernel" $l(s,t)$ has been a subject of investigation during the last decades. A good account of such work can be found in [6], [8], [9], [10] and the references therein. Also, the boundedness and compactness of L has been studied in the framework of general Banach function spaces defined over $\mathbb{R}^+$, see, e.g., [7].

Our aim, in this paper, is to study the L^p-L^q compactness of an N-dimensional analogue of the operator L defined by

$$(\mathcal{K}f)(x) = \int_{S_x} k(x,y)f(y)dy, \quad x \in E$$

where E and S_x are certain cones in $\mathbb{R}^N$ (defined below) and show that the compactness of $\mathcal{K}$ can be characterized in terms of the compactness of the one dimensional operator L. We also study the corresponding conjugate operator $\mathcal{K}^*$. Such reduction for some other operators can be found in [3], [5]. In [12], the author works with smoothly star-shaped regions and studies the boundedness of $\mathcal{K}$ in terms of the boundedness of L under the special case when $l(s,t) \equiv 1 \equiv k(x,y)$. The class of smoothly star-shaped regions is larger than the one considered here. However, in our case, we dispense with the smoothness condition. Further, if there is no confusion, we use the same notations E and S_x for cones as done by Sinnamon [12] for star-shaped regions. In the general case the boundedness of $\mathcal{K}$ has been studied in [13].

The paper is organized in the following manner: In Section 2, we collect certain preliminaries which is required for the main results in this paper. The reduction of the

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compactness of $\mathcal{K}$ in terms of the compactness of L for $1 < p, q < \infty$ has been done in Section 3 and subsequently in this section, the precise weight characterization for the compactness of $\mathcal{K}$ in the case $1 < p \leqslant q < \infty$ is given. Finally, in Section 4, the case $1 < q < p < \infty$ has been discussed as well as the conjugate operator $\mathcal{K}^*$ has been studied. There is no ambiguity in the symbol L being used for the space as well as for the operator. It is clear with the context. Moreover, in the case of a space, the symbol L is followed by a superscript, e.g., L^p, L^q etc.

2. Preliminaries

Let $\Omega \subseteq \mathbb{R}^N$. For a weight function u on Ω , we shall denote by $L^p(\Omega, u)$, $1 \leqslant p < \infty$, the weighted Lebesgue space which is the set of all measurable functions f defined on Ω such that

$$\|f\|_{p,\Omega,u} := \left(\int_\Omega |f(x)|^p \, u(x)dx \right)^{\frac{1}{p}} < \infty.$$

It is known that for $1 \leqslant p < \infty$, $L^p(\Omega, u)$ is a Banach space and for $1 < p < \infty$, it is reflexive too. If the duality on the weighted Lebesgue space $L^p(\Omega, u)$, $1 < p < \infty$, is defined by

$$\langle f, g \rangle = \int_\Omega f(x)g(x)dx, \quad g \in L^p(\Omega, u)$$

then we can identify the conjugate space of $L^p(\Omega, u)$ by $L^{p'}(\Omega, u^{1-p'})$, $p' = \frac{p}{p-1}$ being the conjugate index of p, i.e.

$$[L^p(\Omega, u)]^* = L^{p'}(\Omega, u^{1-p'}).$$

For a bounded linear operator T between two normed linear spaces X and Y, we denote by T^*, the conjugate of T acting between Y^* and X^*.

Consider the generalized Hardy operator $L : L^p((0,\infty), v) \to L^q((0,\infty), u)$ defined by

$$(Lf)(s) := \int_0^s l(s,t)f(t)dt, \quad s > 0,$$

where the kernel $l(s,t)$ is defined for $0 < t < s < \infty$ and $l(s,t) \geqslant 0$. The kernel $l(s,t)$ is called *Oinarov* if

(i) $l(s,t)$ is increasing in the first variable, i.e.,

$$l(s_1, t) \leqslant l(s_2, t), \quad for \quad 0 < s_1 < s_2; \tag{2.1}$$

(ii) $l(s,t)$ is decreasing in the second variable, i.e.,

$$l(s, t_1) \leqslant l(s, t_2), \quad for \quad 0 < t_2 < t_1; \tag{2.2}$$

(iii) there exist positive constants c_1, c_2 such that

$$c_1[l(s,r) + l(r,t)] \leqslant l(s,t) \leqslant c_2[l(s,r) + l(r,t)], \quad 0 < t < r < s. \tag{2.3}$$

Such kernels were introduced by Bloom and Kermen [2]. However, because of the considerable work done with these kernels by Oinarov [9], [10], these are named after him.

The conjugate operator L^* to L is given by

$$(L^*g)(s) := \int_s^\infty l(t,s)g(t)dt, \quad s>0.$$

Let $\sum_N$ be the unit sphere in $\mathbb{R}^N$, i.e., $\sum_N = \left\{x \in \mathbb{R}^N : \mid x \mid = 1\right\}$, where $\mid x \mid$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^N$. Let A be a measurable subset of $\sum_N$ .We denote by $E,$ a measurable cone in $\mathbb{R}^N$ and is defined by

$$E = \left\{x \in \mathbb{R}^N : x = s\sigma; 0 \leqslant s < \infty, \sigma \in A\right\}.$$

Let $S_x, x \in \mathbb{R}^N$ denote part of E with 'radius' $\leqslant \mid x \mid$, i.e.,

$$S_x = \left\{y \in \mathbb{R}^N : y = s\sigma, 0 \leqslant s \leqslant \mid x \mid, \sigma \in A\right\}.$$

Let E be a cone in $\mathbb{R}^N$. We consider the $N-$dimensional generalized Hardy operator

$$(\mathcal{K}f)(x) = \int_{S_x} k(x,y)f(y)dy, \quad x \in E$$

where the kernel $k(x,y)$ is defined on $E \times E$ for $|y| \leqslant |x|$ and is such that $k(x,y) \geqslant 0$. Following the one dimensional case, the kernel $k(x,y)$ is called *Oinarov* if the following are satisfied:

(i) k is increasing in the first argument, i.e.,

$$k(x_1,y) \leqslant k(x_2,y), \quad \mid x_1 \mid \leqslant \mid x_2 \mid, y \in E; \tag{2.4}$$

(ii) k is decreasing in the second argument, i.e.,

$$k(x,y_1) \geqslant k(x,y_2), \quad x \in E, \mid y_1 \mid \leqslant \mid y_2 \mid; \tag{2.5}$$

(iii) there exist positive constants c_1, c_2 such that

$$c_1[k(x,y)+k(y,z)] \leqslant k(x,z) \leqslant c_2[k(x,y)+k(y,z)], \quad \mid z \mid \leqslant \mid y \mid \leqslant \mid x \mid. \tag{2.6}$$

REMARK 1 . If k is a positive kernel satisfying (2.4) and (2.5) then it only depends on the radial part. Indeed, let $x_i = s\sigma_i, y_i = t\tau_i,\ s,t>0,\ \sigma_i,\tau_i \in A,\ i=1,2.$ Note that $\mid x_1 \mid = \mid x_2 \mid$ and $\mid y_1 \mid = \mid y_2 \mid$. Then using (2.4) and (2.5) we obtain

$$k(x_1,y_1) = k(x_2,y_1) = k(x_2,y_2).$$

Thus, if we set

$$l(s,t) = k(s\sigma, t\tau), \tag{2.7}$$

then l is a positive kernel defined on $(0,\infty)\times(0,\infty)$ corresponding to the kernel $k(s\sigma,t\tau)$ defined on $E \times E$. Clearly, k is Oinarov if and only if l is so.

The operator $\mathcal{K}^* : L^p(E,v) \to L^q(E,u)$, conjugate to $\mathcal{K}$ is defind by

$$(\mathcal{K}^*g)(x) = \int_{E\setminus S_x} k(y,x)g(y)dy, \quad x \in E. \tag{2.8}$$

Let X be a normed linear space and X^* denote its conjugate space. We say that a sequence $\{x_n\}$ in X is strongly convergent (or simply convergent) to $x \in X$, written

$x_n \to x$, if $\| x_n - x \| \to 0$ as $n \to \infty$. A sequence $\{x_n\}$ in X is said to converge weakly to $x \in X$, written $x_n \xrightarrow{w} x$, if $f(x_n) \to f(x)$, for each $f \in X^*$. A sequence $\{f_n\}$ in X^* is said to be weak* convergent to $f \in X^*$, written $f_n \xrightarrow{w^*} f$, if $f_n(x) \to f(x)$ for each $x \in X$. Note that the strong convergence implies the weak convergence which in turn implies the weak* convergence. The implications in the reverse direction do not hold in general. However, if X is a reflexive space then the weak* convergence implies the weak convergence.

The proofs of the theorems presented in this paper require some well known assertions which are collected in the following :

THEOREM 2.A. *Let X and Y be Banach spaces.*

(i) A bounded linear operator $T : X \to Y$ is compact if and only if its conjugate $T^* : Y^* \to X^*$ is compact.

(ii) If $T : X \to Y$ is compact and $\{x_n\}$ is a sequence in X such that $\{x_n\} \xrightarrow{w} x$, for some $x \in X$, then $Tx_n \to Tx$

(iii) An operator $T : X \to Y$ is compact if $T^* : Y^* \to X^*$ is weak* -norm sequentially continuous i.e. for each sequence $\{f_n\}$ in Y^* with $\{f_n\} \xrightarrow{w^*} f$, for some $f \in Y^*$, we have $T^*(f_n) \to T^*f$.

3. The results

For the sake of convenience we shall use the following notations. We denote for $n \geqslant 0$

$$(L_n f)(s) := \int_0^s l^n(s,t) f(t)\,dt,$$

$$(L_n^* g)(s) := \int_s^\infty l^n(t,s) g(t)\,dt.$$

$$(\mathcal{K}_n h)(x) := \int_{S_x} k^n(x,y) h(y)\,dy,$$

and

$$(\mathcal{K}_n^* h)(x) := \int_{E \setminus S_x} k^n(y,x) h(y)\,dy.$$

For example, L_0 is the standard Hardy operator $\int_0^s f(t)\,dt$.

In [7], Lomakina and Stepanov studied the compactness of the operator L in the framework of general Banach function spaces defined on $\mathbb{R}^+$. In terms of L^p - L^q compactness, their result reads as

THEOREM 3.A. *Let $1 < p \leqslant q < \infty$ and U, V be weight functions on $(0,\infty)$. Then the operator $L : L^p((0,\infty),V) \to L^q((0,\infty),U)$ involving the Oinarov kernel l is compact if and only if*

$$\max(A_0, A_1) < \infty$$

and

$$\lim_{s \to a_i} A_i(s) = \lim_{s \to b_i} A_i(s) = 0, \quad i = 0, 1$$

where

$$A_0 = \sup_{s>0} A_0(s) = \sup_{s>0} (L_0^* U)^{1/q}(s) \left(L_{p'} V^{1-p'} \right)^{1/p'} (s),$$

$$A_1 = \sup_{s>0} A_1(s) = \sup_{s>0} (L_q^* U)^{1/q}(s) \left(L_0 V^{1-p'} \right)^{1/p'} (s),$$

$$a_i = \inf \{s > 0 : A_i(s) > 0\} \quad i = 0, 1$$

and

$$b_i = \sup \{s > 0 : A_i(s) > 0\} \quad i = 0, 1.$$

The aim is to extend the above result in the higher dimensional setting. More precisely, we characterize the compactness of the operators $\mathcal{K}$ and $\mathcal{K}^*$ defined in Section 2. The following is the key result which characterizes the compactness of $\mathcal{K}$ in terms of the compactness of the one dimensional operator L.

THEOREM 3.1. *Let E be a cone in $\mathbb{R}^N$ and k be a kernel on $E \times E$ depending on the radial variables, i.e., $k(x,y) = k(|x|, |y|)$. Suppose that $1 < p, q < \infty$ and u,v be weight functions on E. Then the operator $\mathcal{K} : L^p(E, v) \to L^q(E, u)$ is compact if and only if the operator $L : L^p((0,\infty), V) \to L^q((0,\infty), U)$ is compact with*

$$U(t) = \int_A u(t\tau) t^{N-1} d\tau, \quad t \in (0, \infty) \tag{3.1}$$

and

$$V(t) = \left(\int_A v^{1-p'}(t\tau) t^{N-1} d\tau \right)^{1-p}, \quad t \in (0, \infty). \tag{3.2}$$

Proof. Let $x, y \in E$. Using the polar coordinates, $x = s\sigma$, $y = t\tau$, $\sigma, \tau \in A$, and the fact that $k(x,y)$ depends on radial variables, we can set

$$k(x,y) = k(|x|, |y|) = l(s,t),$$

where $l(s,t)$ is the kernel involved in the one dimensional operator L.

First assume that $L : L^p((0,\infty), V) \to L^q((0,\infty), U)$ is compact. In order to show that $\mathcal{K}$ is compact, it suffices to show that the conjugate operator $\mathcal{K}^* : L^{q'}\left(E, u^{1-q'}\right) \to L^{p'}\left(E, v^{1-p'}\right)$

$$(\mathcal{K}^* g)(x) = \int_{E \setminus S_x} k(y,x) g(y) dy, \quad x \in E,$$

is weak*-norm sequentially continuous since then the result follows from Theorem 2.A(iii). Let $\{f_n\}$ be a sequence in $L^{q'}\left(E, u^{1-q'}\right)$ such that $\{f_n\} \xrightarrow{w^*} 0$. Without any loss of generality, we may assume that each f_n is non-negative. Define

$$F_n(t) = \int_A f_n(t\tau) t^{N-1} d\tau, \quad n \in \mathbb{N},\ t \in (0, \infty). \tag{3.3}$$

Then

$$F_n(t)= \int_A f_n(t\tau)\, u^{-\frac{1}{q}}(t\tau)\left(t^{N-1}\right)^{\frac{1}{q'}} u^{\frac{1}{q}}(t\tau)\left(t^{N-1}\right)^{\frac{1}{q}} d\tau$$
$$\leqslant \left(\int_A f_n^{q'}(t\tau)\, u^{1-q'}(t\tau)\, t^{N-1} d\tau\right)^{\frac{1}{q'}} \left(\int_A u(t\tau)\, t^{N-1} d\tau\right)^{\frac{1}{q}},$$

and therefore using (3.1) and making change of variable $t\tau = y,$ we have

$$\left(\int_0^\infty F_n^{q'}(t)\, U^{1-q'}(t)\, dt\right)^{\frac{1}{q'}} \leqslant \left(\int_0^\infty \int_A f_n^{q'}(t\tau)\, u^{1-q'}(t\tau)\, t^{N-1} d\tau dt\right)^{\frac{1}{q'}}$$
$$= \left(\int_E f_n^{q'}(y)\, u^{1-q'}(y)\, dy\right)^{\frac{1}{q'}}$$
$$< \infty,$$

which gives that $\{F_n\}$ is a sequence in $L^{q'}\left((0,\infty), U^{1-q'}\right)$. Next we note that if G is any function in $L^q((0,\infty), U)$ and $g : E \to \mathbb{R}$ is defined by

$$g(x) = G(t), \quad x = t\tau$$

then $g \in L^q(E,u)$, since by using (3.1) and making change of variable $x = t\tau$, we have

$$\int_E g^q(x)\, u(x)\, dx = \int_0^\infty \int_A g^q(t\tau)\, u(t\tau)\, t^{N-1} d\tau dt$$
$$= \int_0^\infty G^q(t)\, U(t)\, dt$$
$$< \infty.$$

Thus by using (3.3), we have

$$\int_0^\infty F_n(t)\, G(t)\, dt = \int_0^\infty \left(\int_A f_n(t\tau)\, t^{N-1} d\tau\right) G(t)\, dt$$
$$= \int_0^\infty \int_A f_n(t\tau)\, g(t\tau)\, t^{N-1} d\tau dt$$
$$= \int_E f_n(x)\, g(x)\, dx$$
$$\to 0 \quad as \quad n \to \infty,$$

i.e. $F_n \xrightarrow{w} 0$. Further since L is compact, by Theorem 2.A $((i)$ and $(ii))$

$$\| L^* F_n \|_{p',(0,\infty),V^{1-p'}} \to 0 \quad as \quad n \to \infty.$$

Now making change of variables $y = t\tau$, $x = s\sigma$ so that for $\sigma \in A$, $|x| = s$ and using (3.2), (3.3), we have

$$
\begin{aligned}
\| \mathcal{K}^* f_n \|_{p',E,v^{1-p'}} &= \left(\int_E \left(\int_{E \setminus S_x} k(y,x) f_n(y)\, dy \right)^{p'} v^{1-p'}(x)\, dx \right)^{\frac{1}{p'}} \\
&= \left(\int_0^\infty \int_A \left(\int_s^\infty \int_A k(t\tau, s\sigma) f_n(t\tau)\, t^{N-1} d\tau dt \right)^{p'} v^{1-p'}(s\sigma)\, s^{N-1} d\sigma ds \right)^{\frac{1}{p'}} \\
&= \left(\int_0^\infty \left(\int_s^\infty l(t,s) F_n(t)\, dt \right)^{p'} V^{1-p'}(s)\, ds \right)^{\frac{1}{p'}} \\
&= \| L^* F_n \|_{p',(0,\infty),V^{1-p'}},
\end{aligned}
$$

and we are done.

Conversely, assume that $\mathcal{K} : L^p(E,v) \to L^q(E,u)$ is compact. Let $\{F_n\}$ be a sequence in $L^{q'}\left((0,\infty), U^{1-q'}\right)$ such that $F_n \xrightarrow{w^*} 0$. Without any loss of generality we may assume that each F_n is non-negative. Define

$$
f_n(t\tau) = F_n(t)\, u(t\tau)\, U^{-1}(t), \quad n \in \mathbb{N},\ t \in (0,\infty), \tau \in A. \tag{3.4}
$$

Then

$$
\int_A f_n(t\tau)\, t^{N-1} d\tau = F_n(t), \quad n \in \mathbb{N}, t \in (0,\infty). \tag{3.5}
$$

Now using (3.1) and (3.4), we have

$$
\begin{aligned}
\left(\int_E f_n^{q'}(x)\, u^{1-q'}(x)\, dx \right)^{\frac{1}{q'}} &= \left(\int_0^\infty \int_A f_n^{q'}(t\tau)\, u^{1-q'}(t\tau)\, t^{N-1} d\tau dt \right)^{\frac{1}{q'}} \\
&= \left(\int_0^\infty F_n^{q'}(t) \left(\int_A u^{q'}(t\tau)\, u^{1-q'}(t\tau)\, t^{N-1} d\tau \right) U^{-q'}(t)\, dt \right)^{\frac{1}{q'}} \\
&= \left(\int_0^\infty F_n^{q'}(t)\, U^{1-q'}(t)\, dt \right)^{\frac{1}{q'}} \\
&< \infty,
\end{aligned}
$$

which means that $\{f_n\}$ is a sequence in $L^{q'}\left(E, u^{1-q'}\right)$. Thus (3.2) and (3.5) yield

$$
\| L^* F_n \|_{p',(0,\infty),V^{1-p'}} = \| \mathcal{K}^* f_n \|_{p',E,v^{1-p'}} .
$$

We now show that $f_n \xrightarrow{w} 0$. For any function $g \in L^q(E,u)$, using (3.4), we have

$$
\begin{aligned}
\int_E f_n(x)\, g(x)\, dx &= \int_0^\infty \int_A F_n(t)\, u(t\tau)\, U^{-1}(t)\, g(t\tau)\, t^{N-1} d\tau dt \\
&= \int_0^\infty F_n(t) \left(\int_A u(t\tau)\, g(t\tau)\, t^{N-1} d\tau \right) U^{-1}(t)\, dt \\
&= \int_0^\infty F_n(t)\, G(t)\, dt \\
&\to 0 \quad as \quad n \to \infty,
\end{aligned}
$$

where

$$G(t) = \left(\int_A u(t\tau)\, g(t\tau)\, t^{N-1} d\tau \right) U^{-1}(t), \quad t \in (0,\infty)$$

and it can be easily verified that $G \in L^q((0,\infty), U)$. Indeed, using (3.1), we have

$$\begin{aligned}
\int_0^\infty G^q(t)\, U(t)\, dt &= \int_0^\infty \left(\int_A u(t\tau)\, g(t\tau)\, t^{N-1} d\tau \right)^q U^{1-q}(t)\, dt \\
&= \int_0^\infty \left(\int_A g(t\tau)\, u^{\frac{1}{q}}(t\tau) \left(t^{N-1}\right)^{\frac{1}{q}} u^{\frac{1}{q'}}(t\tau) \left(t^{N-1}\right)^{\frac{1}{q'}} d\tau \right)^q U^{1-q}(t)\, dt \\
&\leqslant \int_0^\infty \left(\int_A g^q(t\tau)\, u(t\tau)\, t^{N-1} d\tau \right) \left(\int_A u(t\tau)\, t^{N-1} d\tau \right)^{q-1} U^{1-q}(t)\, dt \\
&= \int_E g^q(x)\, u(x)\, dx < \infty.
\end{aligned}$$

Now as $\mathcal{K}$ is compact, by Theorem 2.A $((i)$ and $(ii))$, $\| \mathcal{K}^* f_n \|_{p',E,v^{1-p'}}$ and hence $\| L^* F_n \|_{p',(0,\infty),V^{1-p'}}$ converges to 0 as $n \to \infty$. Now the compactness of L follows from Theorem 2.A (iii).

REMARK 2. Theorem 3.1 can be compared with a result of the authors ([5], Theorem 4.1) where it is proved for $k(x,y) \equiv 1$ and $l(s,t) \equiv 1$. However, there the integrals are considered over star-shaped regions.

Now we can give the precise weight characteriztion of the compactness of $\mathcal{K}$:

THEOREM 3.2. *Let $1 < p \leqslant q < \infty$ and u, v be weight functions on E. Then the operator $\mathcal{K} : L^p(E,v) \to L^q(E,u)$ involving the Oinarov kernel k is compact if and only if*

$$\mathcal{A} = \max(\mathcal{A}_0, \mathcal{A}_1) < \infty \tag{3.6}$$

and

$$\lim_{x \to x_i} \mathcal{A}_i(x) = \lim_{x \to \widetilde{x}_i} \mathcal{A}_i(x) = 0, \quad i = 0,1 \tag{3.7}$$

where

$$\begin{aligned}
\mathcal{A}_0 &= \sup_{x \in E \setminus \{0\}} \mathcal{A}_0(x) = \sup_{x \in E \setminus \{0\}} (\mathcal{K}_0^* u)^{1/q}(x) \left(\mathcal{K}_{p'} v^{1-p'} \right)^{1/p'}(x), \\
\mathcal{A}_1 &= \sup_{x \in E \setminus \{0\}} \mathcal{A}_1(x) = \sup_{x \in E \setminus \{0\}} (\mathcal{K}_q^* u)^{1/q}(x) \left(\mathcal{K}_0 v^{1-p'} \right)^{1/p'}(x),
\end{aligned}$$

$$x_i = \inf\{x \in E \setminus \{0\} : \mathcal{A}_i(x) > 0\} \quad i = 0,1$$

$$\widetilde{x}_i = \sup\{x \in E \setminus \{0\} : \mathcal{A}_i(x) > 0\} \quad i = 0,1.$$

Proof. We use the polar coordinates $x = s\sigma$, $y = t\tau$ with $\sigma, \tau \in A$ and $s, t > 0$. The result is obtained in view of (2.7) and Theorems 3.1 and 3.A if we show that

$A_i(s) = \mathcal{A}_i(x),\ \ i = 0, 1.$ We find that

$$\begin{aligned} A_0(s) &= \left(\int_s^\infty \int_A u(t\tau)\, t^{N-1} d\tau dt \right)^{\frac{1}{q}} \left(\int_0^s l^{p'}(s,t) \left(\int_A v^{1-p'}(t\tau)\, t^{N-1} d\tau \right)^{(1-p)(1-p')} dt \right)^{\frac{1}{p'}} \\ &= \left(\int_s^\infty \int_A u(t\tau)\, t^{N-1} d\tau dt \right)^{\frac{1}{q}} \left(\int_0^s \int_A k^{p'}(s\sigma, t\tau) v^{1-p'}(t\tau)\, t^{N-1} d\tau dt \right)^{\frac{1}{p'}} \\ &= \mathcal{A}_0(x). \end{aligned}$$

Similarly, $A_1(s) = \mathcal{A}_1(x)$ and we are done.

4. Final results and remarks

REMARK 3. Following a result of Ando [1], it is known that for $1 < q < p < \infty$, the operator $L : L^p((0,\infty), u) \to L^q((0,\infty), v)$ is bounded if and only if it is compact. The same is true in higher dimension also. But the L^p - L^q boundedness of $\mathcal{K}$ is already known, see [13]. Consequently, the same are the compactness conditions of $\mathcal{K}$.

In view of Theorems 2.A (i) and 3.2, the conjugate operator $\mathcal{K}^* : L^{q'}(E, u^{1-q'}) \to L^{p'}(E, v^{1-p'})$ is compact if and only if (3.6) and (3.7) are satisfied. Replacing $p', q', u^{1-q'}$ and $v^{1-p'}$ by, respectively, q, p, v and u, we immediately obtain the following :

THEOREM 4.1. *Let $1 < p \leqslant q < \infty$ and u, v be weight functions on E. Then the operator $\mathcal{K}^* : L^p(E, v) \to L^q(E, u)$ involving the Oinarov kernel k is compact if and only if*

$$\mathcal{A}^* = \max(\mathcal{A}_0^*, \mathcal{A}_1^*) < \infty$$

and

$$\lim_{x \to x_i^*} \mathcal{A}_i^*(x) = \lim_{x \to \widetilde{x}_i^*} \mathcal{A}_i^*(x) = 0, \quad i = 0, 1$$

where

$$\begin{aligned} \mathcal{A}_0^* &= \sup_{x \in E \setminus \{0\}} \mathcal{A}_0^*(x) = \sup_{x \in E \setminus \{0\}} (\mathcal{K}_0^* v^{1-p'})^{1/p'}(x)\, (\mathcal{K}_q u)^{1/q}(x), \\ \mathcal{A}_1^* &= \sup_{x \in E \setminus \{0\}} \mathcal{A}_1^*(x) = \sup_{x \in E \setminus \{0\}} (\mathcal{K}_{p'}^* v^{1-p'})^{1/p'}(x)\, (\mathcal{K}_0 u)^{1/q}(x), \\ x_i^* &= \inf\{x \in E \setminus \{0\} : \mathcal{A}_i^*(x) > 0\} \quad i = 0, 1 \\ \widetilde{x}_i^* &= \sup\{x \in E \setminus \{0\} : \mathcal{A}_i^*(x) > 0\} \quad i = 0, 1. \end{aligned}$$

REMARK 4. In the light of Remark 3 and using the technique of Theorem 4.1, the compactness of the operator $\mathcal{K}^*$ for the case $1 < q < p < \infty$ can be obtained. For conciseness, the construction of the result and its proof is left to the reader.

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