Mathematical
Inequalities
& Applications
Volume 10, Number 1 (2007), 1–7

A NONCOMMUTATIVE AG INEQUALITY

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(*communicated by P. S. Bullen*)

Abstract. We give a proof of AG inequality in noncommutative linearly ordered rings.

0. Introduction

AG inequality states that

$$\frac{x_1 + \dots + x_n}{n} \geqslant \sqrt[n]{x_1 \cdot \dots \cdot x_n} \tag{1}$$

where $x_1, \dots, x_n$ are positive real numbers. The equality in (1) holds if and only if $x_1 = \dots = x_n$.

The inequality (1) has the following algebraic form:

$$(x_1 + \dots + x_n)^n \geqslant n^n x_1 \cdot \dots \cdot x_n \tag{2}$$

In this note we consider a noncommutative version of (2). We also give some examples.

Let A be an associative but not necessarily commutative ring. The (partial) orderings on A are in one-one correspondence with positive cones in A. Recall that we say that a subset P of A is a cone if:

(i) $P + P \subseteq P$
(ii) $PP \subseteq P$
(iii) $P \cap (-P) = \{0\}$

(see, for example, [3, p. 105]). The ordering corresponding to P is defined by $a \geqslant b$ if $a - b \in P$. We write simple (A, P) for A ordered by P.

If P satisfies the additional condition

(iv) $A = P \cup (-P) \cup \{0\}$

then we say that A is linearly ordered.

Further, if

(v) $a < b$ and $c > 0$ imply $ac < ab$ and $ca < cb$

then we say that A is strictly ordered.

It is easy to see that a strictly linearly ordered ring is a ring without zero divisors.

Mathematics subject classification (2000): 06F25, 26D05.
Key words and phrases: noncommutative linearly ordered ring, AG inequality, quasi-sum of squares.

Paper MIA-10-01

One of possible extension of AG inequality on a noncommutative partially ordered ring A is the following:

$$n!(t_1+...+t_n)^n \geqslant n^n \sum_{\sigma\in S_n} t_{\sigma(1)}\cdot ...\cdot t_{\sigma(n)} \tag{3}$$

for any positive $t_1,...,t_n \in A$ (here S_n denotes the group of permutation of the set $\{1,...,n\}$). The equality holds if and only if $t_1 = ... = t_n$.

The inequality (3) does not hold on noncommutative partially ordered rings generally. The following example shows that it fails even in the case of matrices.

EXAMPLE 1. Let A be the ring of quadratic matrices of fixed order over an ordered field and let P be the set of nonzero matrices with nonnegative entries. Then (3) fails even for $n=2$.

The existence of non-positive squares in Example 1 is the main reason why AG inequality fails. Another difficulty is the existence of nilpotent elements. These difficulties may be avoided by considering linearly ordered rings. As far as the author can determine, known proofs of the classical inequality (2) do not work in this more general situation.

Historically, the first example of a noncommutative linearly ordered (division) ring was the ring of Hilbert's twisted Laurent series from 1899. Using this ring Hilbert obtained a geometry that satisfies the Desargues' Theorem but not the Pappus' Theorem ([5], see also [1, Ch. I and II]). In the following example we demonstrate the validity of AG inequality (3) in the Hilbert ring.

EXAMPLE 2. Let $F=\mathbb{R}[[t]]$ be the ring of formal power series over $\mathbb{R}$ with ultrametric ordering (i.e. with positive cone consisting of formal power series having positive first nonzero coefficients). Then there is exactly one (ordered) $\mathbb{R}$-automorphism ϕ of F satisfying $\phi(t)=2t$ (then $\phi^m(t^n)=2^{mn}t^n$). Let A be the ring of formal power series over F with standard addition and with multiplication defined by $(\sum a_jx^j)(\sum b_jx^j) = \sum c_jx^j$ where $c_k := \sum_{i+j=k}\phi^j(a_i)b_j$. Then A is linearly ordered noncommutative ring with positive cone consisting of formal power series having positive first nonzero coefficients. We claim that AG inequality holds in A.

We have the following rule for multiplication of monomials in A.

$$a_1x^{r_1}\cdot ...\cdot a_nx^{r_n} = \phi^{r_2+...+r_n}(a_1)\cdot ...\cdot \phi^{r_n}(a_{n-1})a_nx^{r_1+...+r_n}$$

For $f\in A$ we put $\mathrm{ord}(f)=r$ if $f(x)=a_rx^r+a_{r+1}x^{r+1}+...$ with $a_r\neq 0$. It is easy to see that $\mathrm{ord}(fg)=\mathrm{ord}(f)+\mathrm{ord}(g)$, $\mathrm{ord}(f+g)\geqslant \min\{\mathrm{ord}(f),\mathrm{ord}(g)\}$ and $\mathrm{ord}(f+g)=\min\{\mathrm{ord}(f),\mathrm{ord}(g)\}$ if f,g are positive. Also if f,g are positive and $\mathrm{ord}(f)<\mathrm{ord}(g)$ then $f>g$.

Let $f_1,f_2,...,f_n\in A$ be positive. Then, if $\mathrm{ord}(f_1),...,\mathrm{ord}(f_n)$ are not mutually equal then $\mathrm{ord}(n!(f_1+...+f_n)^n) < \mathrm{ord}(n^n\sum_{\sigma\in S_n} f_{\sigma(1)}\cdot ...\cdot f_{\sigma(n)})$ (strict inequality), hence $n!(f_1+...+f_n)^n > n^n\sum_{\sigma\in S_n} f_{\sigma(1)}\cdot ...\cdot f_{\sigma(n)}$. Therefore, we may assume that $\mathrm{ord}(f_1)=\mathrm{ord}(f_2)=...=\mathrm{ord}(f_n)=m$, and so

$$f_i(x)=a_i(t)x^m+...,$$

where $a_i(t)$, for $i = 1, ..., n$, are strictly positive elements of $\mathbb{R}[[t]]$. Then

$$\operatorname{ord}(n!(f_1 + ... + f_n)^n) = \operatorname{ord}(n^n \sum_{\sigma \in S_n} f_{\sigma(1)} \cdot ... \cdot f_{\sigma(n)}) = mn,$$

so that our inequality is equivalent to

$$n!(\phi^{(n-1)m}(a_1(t) + ... + a_n(t))\phi^{(n-2)m}(a_1(t) + ... + a_n(t)) \cdot ... \cdot (a_1(t) + ... + a_n(t)) \geqslant n^n \sum_{\sigma \in S_n} (\phi^{(n-1)m}(a_{\sigma(1)}(t))\phi^{(n-2)m}(a_{\sigma(2)}(t)) \cdot ... \cdot a_{\sigma(n)}(t))$$

Since ϕ^k does not change the ord, we see that if $\operatorname{ord}(a_i(t))$ are not mutually equal (in $\mathbb{R}[[t]]$), then we have strict inequality (as above). Therefore we may assume that $\operatorname{ord}(a_1(t)) = \operatorname{ord}(a_2(t)) = ... = \operatorname{ord}(a_n(t)) = M$, and so

$$a_i(t) = A_i t^M + ...,$$

where A_i, for $i = 1, ..., n$, are strictly positive real numbers. Now our inequality is equivalent to

$$n!2^{(n-1)mM}2^{(n-2)mM} \cdot ... \cdot 1(\sum A_i)^n \geqslant n^n n!2^{(n-1)mM}2^{(n-2)mM} \cdot ... \cdot 1 \prod A_i,$$

which is ordinary AG inequality over $\mathbb{R}$. From the course of the proof it is clear that the equality holds if and only if $f_1 = ... = f_n$.

In this paper we prove that noncommutative AG inequality (3) holds in arbitrary strictly linearly ordered rings. The organization of the paper is the following. In Section 1. we sketch an algebraic version of classical AG inequality (as it is presented in [4] and [7]). The advantage of this version is that it admits an extension on the noncommutative case. In Section 2. we describe some connections between the rings of homogenous polynomials in commutative and noncommutative variables and in Section 3. we prove the main result.

1. An algebraic version of classical AG inequality

The classical AG inequality (1) has algebraic form (2). Our proof of noncommutative AG inequality (3) is based on an explicit description of $(x_1 + x_2 + ... + x_n)^n - n^n x_1 x_2 ... x_n$, for which we need the notion of quasi-sum of squares.

DEFINITION 1. ([4, Definition 1.]) Let f be a homogenous symmetric n-degree polynomial in n variables. We say that f is a quasi-sum of squares if

$$f = \sum_{1 \leqslant i < j \leqslant n} f_{i,j}(X_i - X_j)^2$$

where $f_{i,j}$ (for each i, j) is a homogenous polynomial of degree $n-2$ that is a linear combination of monomials with nonnegative coefficients.

A significance of this notion is in the fact that

$$(X_1 + ... + X_n)^n - n^n X_1 \cdot ... \cdot X_n \tag{4}$$

is a quasi-sum of squares, for all natural numbers n (see [4], and [7] where the explicit formulas are done). Note that Hurwitz ([6]) gave an explicit expression for

$$X_1^n + X_2^n + ... + X_n^n - nX_1X_2...X_n$$

as a quasi-sum of squares (see [2], p.87), which may be understood as an purely algebraic proof of AG inequality for positive numbers $x_1^n, ..., x_n^n$, i.e. a proof that works over arbitrary ordered commutative rings. However, the Hurwitz result is insufficient for purely algebraic proof of (2).

Let us adopt the following notation for polynomials in variables $X_1, X_2, ...X_n$:

$$s_0 := 1$$

$$s_k(X_1, X_2, ..., X_n) := \sum_{i_1<i_2<...<i_k} X_{i_1}X_{i_2} \cdot ... \cdot X_{i_k},$$

for $1 \leqslant k \leqslant n$.

$$s_k^{(ij)}(X_1, X_2, ..., X_n) := s_k(X_1, ..., X_{i-1}, X_{i+1}, ..., X_{j-1}, X_{j+1}, ..., X_n),$$

for $1 \leqslant i < j \leqslant n$.

THEOREM 1. ([7, Theorem 1.]) $(X_1 + X_2 + ... + X_n)^n - n^n X_1X_2...X_n$ *is a quasi-sum of squares with rational coefficients, for all natural* $n \geqslant 2$. *More precisely:*

$$(X_1 + X_2 + ... + X_n)^n - n^n X_1X_2...X_n = \sum_{1\leqslant i<j\leqslant n} f_{i,j}(X_i - X_j)^2$$

where $f_{i,j} := \sum_{k=0}^{k=n-2} \frac{n^k}{\binom{n-1}{k+1}} s_1^{n-2-k} s_k^{(ij)}$.

2. Link between commutative and noncommutative relations

In this section we prepare the result of previous section for the case of noncommutative variables. We adopt the following notation and definitions.

$T_1, ..., T_n; T$ denotes noncommutative variables.

$X_1, ..., X_n; X$ denotes commutative variables.

$\mathcal{C}$ denotes the linear map from the ring of homogenous polynomials (over the field of rational numbers) in noncommutative variables $T_1, ..., T_n, ...; T$ to the ring of homogenous polynomials in the commutative variables $X_1, ..., X_n, ...; X$ defined by $\mathcal{C}(T_i) = X_i$, for all i and $\mathcal{C}(T) = X$. Also, we require that $\mathcal{C}$ is multiplicative on the monomials. For example, $\mathcal{C}(T_1T_2^3T_1^4) = \mathcal{C}(T_1)(\mathcal{C}(T_2))^3(\mathcal{C}(T_1))^4 = X_1^5X_2^3$.

DEFINITION 2. We say that a monomial G in noncommutative variables is associated to a monomial H, if $\mathcal{C}(G) = \mathcal{C}(H)$.

For example, if $G := T_4^2T_1T_3$ then the set of all monomials that are associated to G is $\{T_4^2T_1T_3,\ T_4^2T_3T_1,\ T_4T_1T_4T_3,\ T_4T_3T_4T_1,\ T_4T_1T_3T_4,\ T_4T_3T_1T_4,\ T_1T_4^2T_3,\ T_3T_4^2T_1,\ T_1T_4T_3T_4,\ T_3T_4T_1T_4,\ T_1T_3T_4^2,\ T_3T_1T_4^2\}$.

Further, let $\mathcal{N}$ denote the linear map from the ring of homogenous polynomials in the commutative variables $X_1, ..., X_n, ...; X$ to the ring of homogenous polynomials in noncommutative variables $T_1, ..., T_n, ...; T$ defined as follows. Given a monomial in commutative variables $f := X_{i_1}^{a_1} \cdot ... \cdot X_{i_r}^{a_r}$, with $a_i \geqslant 1$ for all i, then $\mathcal{N}(f)$ is defined to be the sum of monomials in noncommutative variables associated to the monomial $T_{i_1}^{a_1} \cdot ... \cdot T_{i_r}^{a_r}$ divided by the number of these monomials (i.e. by $\frac{(a_1+...+a_r)!}{a_1! \cdot ... \cdot a_r!}$). For example, if $f = X_4^2 X_1 X_3$ then

$$\mathcal{N}(f) = \frac{1}{12}(T_4^2 T_1 T_3 + T_4^2 T_3 T_1 + T_4 T_1 T_4 T_3 + T_4 T_3 T_4 T_1 + T_4 T_1 T_3 T_4 + T_4 T_3 T_1 T_4 + T_1 T_4^2 T_3 + T_3 T_4^2 T_1 + T_1 T_4 T_3 T_4 + T_3 T_4 T_1 T_4 + T_1 T_3 T_4^2 + T_3 T_1 T_4^2).$$

Further, let ψ_{ij}, $1 \leqslant i,j \leqslant n$ denote the map from the ring of homogenous polynomials in noncommutative variables $T_1, ..., T_n; T$ to the ring of homogenous polynomials in noncommutative variables $T_1, ..., T_n$ defined by $\psi_{ij}(F(T_1, ..., T_n, T) := F(T_1, ..., T_n, T_i - T_j))$. It is easy to see that ψ_{ij} are linear for all i,j.

LEMMA 1. *$\mathcal{C}\mathcal{N}(f) = f$ for all polynomials f in commutative variables.*

Proof. Obvious. □

DEFINITION 3. We say that a homogenous polynomial F in noncommutative variables is complete if it satisfies the following property: if F contains a monomial G with coefficient c then it contains all monomials that are associated to G with the same coefficient c.

LEMMA 2. *A polynomial F is complete if and only if $\mathcal{N}\mathcal{C}(F) = F$.*

Proof. Since $\mathcal{C}$ is linear and constant on any set of mutually associated monomials, we see that $\mathcal{N}\mathcal{C}(F) = F$ for all complete polynomials. Assume, now, that $\mathcal{N}\mathcal{C}(F) = F$, for a polynomial F. Assume that $G = \sum c_k E_k$ is a part of F where $\{E_k\}_{k=1}^{k=m}$ is the set of all monomials that are associated to a fixed monomial that occurs in F. We claim that all c_k are mutually equal. It must be $\mathcal{N}\mathcal{C}(G) = G$, so that $\frac{\sum c_k}{m} \sum E_k = \sum c_k E_k$, hence $c_1 = ... = c_m$.

For example, $F := T_1^2 T_2 + T_1 T_2 T_1 + T_2 T_1^2$ is complete. We see that $\mathcal{C}(F) = 3X_1^2 X_2$, and $\mathcal{N}\mathcal{C}(F) = \frac{3(T_1^2 T_2 + T_1 T_2 T_1 + T_2 T_1^2)}{3} = F$. □

LEMMA 3. *Assume that F, G are complete. Then $\mathcal{C}(F) = \mathcal{C}(G)$ implies $F = G$.*

Proof. Assume $\mathcal{C}(F) = \mathcal{C}(G)$, for complete polynomials F, G. Then $\mathcal{N}\mathcal{C}(F) = \mathcal{N}\mathcal{C}(G)$, hence, by Lemma 2, $F = G$. □

LEMMA 4. *Assume that $f = g \cdot (X_i - X_j)^2$ for a homogenous polynomial g in variables $X_1, ..., X_n$. Then $\mathcal{N}(f) = \psi_{ij}\mathcal{N}(g \cdot X^2)$.*

Proof. Since $\mathcal{N}$ and ψ_{ij} are linear maps we may assume that g is a monomial, say $g := X_{i_1}^{a_1} \cdot ... \cdot X_{i_r}^{a_r}$. By Lemma 1, we have $\mathcal{C}\mathcal{N}(f) = f$. We claim that $\mathcal{C}(\psi_{ij}\mathcal{N}(gX^2)) = f$, too. Let $\{E_k\}_{k=1}^{k=m}$ be the set of all monomials that are associated to $T_{i_1}^{a_1} \cdot ... \cdot T_{i_r}^{a_r} T^2$. Then we have, $\mathcal{C}(\psi_{ij}\mathcal{N}(gX^2)) = \mathcal{C}(\psi_{ij}(\frac{\sum E_k}{m})) = \frac{1}{m}\sum \mathcal{C}(\psi_{ij}(E_k)) = f$ (since $\mathcal{C}(\psi_{ij}(E_k)) = \mathcal{C}(T_{i_1}^{a_1} \cdot ... \cdot T_{i_r}^{a_r}(T_i - T_j)^2) = f$, for all k). Now, using Lemma 3, we see that it is sufficient to prove that $\psi_{ij}\mathcal{N}(g \cdot X^2)$ is complete. Denote by a the multiplicity of X_i in g and by b the multiplicity of X_j in g (note that a, b are nonnegative integers). Then we have three classes of monomials in $\psi_{ij}\mathcal{N}(g \cdot X^2)$: (i) those having X_i with multiplicity $a+2$, (ii) those having X_i with multiplicity $a+1$, and (iii) those having X_i with multiplicity a.

In (i) each monomial appears with multiplicity $\frac{\binom{a}{2}}{m}$, in (ii) with multiplicity $\frac{ab}{m}$, and in (iii) with multiplicity $\frac{\binom{b}{2}}{m}$. The Lemma is proved.

3. Proof of the noncommutative AG inequality

Before the proving of main result we describe a natural extension (A', P') of a noncommutative linearly strictly ordered ring (A, P). By the definition

$$A' := \{\frac{a}{m} : \ a \in A, m \in \mathbf{N}\}/ \sim$$

and

$$P' := \{\frac{a}{m} : \ a \in P, m \in \mathbf{N}\}/ \sim,$$

with $\frac{a}{m} \sim \frac{b}{r}$ if $ar = bm$. Then $a \mapsto \frac{a}{1}$ defines an ordered inclusion $A \subseteq A'$, with $P = P' \cap A$.

THEOREM 2. *Let A be a noncommutative linearly strictly ordered ring. Then*

$$n!(t_1 + ... + t_n)^n \geqslant n^n \sum_{\sigma \in S_n} t_{\sigma(1)} \cdot ... \cdot t_{\sigma(n)} \tag{5}$$

for any positive $t_1, ..., t_n \in A$. Further, the equality holds if and only if $t_1 = ... = t_n$.

Proof. By Theorem 1

$$(X_1 + ... + X_n)^n - n^n \cdot X_1 \cdot ... \cdot X_n = \sum_{1 \leqslant i < j \leqslant n} f_{i,j}(X_i - X_j)^2$$

for some homogenous polynomials $f_{i,j}$ of degree $n-2$ with nonnegative rational coefficients. Therefore

$$\mathcal{N}((X_1 + ... + X_n)^n - n^n \cdot X_1 \cdot ... \cdot X_n) = \mathcal{N}(\sum_{1 \leqslant i < j \leqslant n} f_{i,j}(X_i - X_j)^2).$$

By the definition of $\mathcal{N}$ and Lemma 4 we get

$$(T_1 + ... + T_n)^n - \frac{n^n}{n!} \sum_{\sigma \in S_n} \prod_{1 \leqslant i \leqslant n} T_{\sigma(i)} = \sum_{1 \leqslant i < j \leqslant n} \psi_{ij}\mathcal{N}(f_{ij} \cdot X^2).$$

We see that $\psi_{ij}\mathcal{N}(f_{ij} \cdot X^2)$ (for every fixed $i < j$) is a linear combination with positive coefficients of monomials in variables $T_1, ..., T_n$ and $T_i - T_j$, where $T_i - T_j$ appears exactly twice. Therefore, after the substitution $T_i \mapsto t_i$ (that makes sense over A'), we get a relation over A

$$n!(t_1 + \ldots + t_n)^n \geqslant n^n \sum_{\sigma \in S_n} t_{\sigma(1)} \cdot \ldots \cdot t_{\sigma(n)} \tag{6}$$

for any positive $t_1, \ldots, t_n \in A$. It is easy to see that the equality holds if and only if $t_1 = \ldots = t_n$. $\square$

EXAMPLE 3. (a) For $n = 2$ we have

$$(X_1 + X_2)^2 - 2^2 X_1 X_2 = (X_1 - X_2)^2.$$

Therefore,

$$2!(T_1 + T_2)^2 - 2^2(T_1T_2 + T_2T_1) = 2!(T_1 - T_2)^2$$

(b) For $n = 3$ we have

$$(X_1 + X_2 + X_3)^3 - 3^3 \cdot X_1X_2X_3 = \frac{1}{2}\Big((X_1 + X_2 + 7X_3)(X_1 - X_2)^2 + \ldots\Big)$$

Therefore

$$3!(T_1+T_2+T_3)^3 - 3^3(T_1T_2T_3 + T_1T_3T_2 + T_2T_1T_3 + T_2T_3T_1 + T_3T_1T_2 + T_3T_2T_1)$$
$$= 3!\frac{1}{2}\Bigg(\Big((T_1+T_2+7T_3)(T_1-T_2)^2 + (T_1-T_2)(T_1 + T_2+7T_3)(T_1-T_2)$$
$$+ (T_1-T_2)^2(T_1+T_2+7T_3)\Big) + \ldots\Bigg)$$

Acknowledgment. The author thanks Professor P. S. Bullen for several very useful comments.

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(Received July 11, 2006)

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