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GENERAL INEQUALITIES VIA ISOTONIC SUBADDITIVE FUNCTIONALS

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Abstract. In this manuscript a number of general inequalities for isotonic subadditive functionals on a set of positive mappings are proved and applied. In particular, it is pointed out that these inequalities both unify and generalize some general forms of the Hölder, Popoviciu, Minkowski, Bellman and Power mean inequalities. Also some refinements of some of these results are proved.

1. Introduction

In the monograph [9] a number of classical inequalities like Jensen's, Hölder's, Minkowski's etc. are given in terms of positive isotonic linear functionals. In Article [1] some sharpenings of these inequalities are presented and applied. Here we give an answer to the natural question which arises from the previous results: "Can linearity of functionals be substituted with much weaker conditions?" While in paper [1] we based our investigation on using a functional version of Jensen's and its reversed inequality, now we change the starting point and use some results from [10].

Let S be a nonempty set and P be a set of mappings from the set S to the nonnegative reals R_+. An *Isotonic Subadditive Functional* (ISF) A is a mapping from P into R_+ satisfying the following conditions:

$(A1)$ $A(0) = 0$, $A(af) = aA(f)$, where $a > 0$ and $af, f \in P$ (positive homogenity)

$(A2)$ For $f, g \in P$, $f \leqslant g$ implies $A(f) \leqslant A(g)$ (isotonicity)

$(A3)$ $A(f+g) \leqslant A(f) + A(g)$ for all $f, g, f+g \in P$ (subadditivity).

If we have equality in $(A3)$, then we say that A is *Isotonic Linear* (or additive) *Functional* (ILF). A number of such functionals were presented in [1] (the most important ones are connected to sums and integrals). In Section 2 of this paper we will present a number of isotonic subadditive functionals (ISF) of interest for applications e.g. in the theory of function spaces, interpolation theory, approximation theory and inequalities. In this section we also include some other preliminary results and definitions of importance for our investigation. The main results are presented and proved in Section 3. Finally, some applications are pointed out in Section 4.

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2. Preliminaries

Let us mention the following result related to Jensen's inequality given by J. Pečarić and D. Veljan in [10] :

THEOREM A. *Let A be a functional with properties $(A1)-(A3)$ that is, A is ISF, and let $h(t_1,\dots,t_n)$ be a real-valued function of n variables, which is defined and continuous for $t_i \geqslant 0$, $i=1,\dots,n$. Suppose that h satisfies the following conditions:*

i) if $t_i > 0$ $(i=1,\dots,n)$, then $h(t_1,\dots,t_n) > 0$,

ii) if $\lambda > 0$, then $h(\lambda t_1,\dots,\lambda t_n) = \lambda h(t_1,\dots,t_n)$,

iii) h is a concave function.

Then the inequality

$$A(h(f_1,\dots,f_n)) \leqslant h(A(f_1),\dots,A(f_n)), \quad f_1,\dots,f_n \in P \tag{1}$$

holds.

REMARK 1. Note that h satisfies the conditions of Theorem A if and only if there exists a nonnegative concave continuous function g of $n-1$ variables such that

$$h(t_1,\dots t_n) = t_1 g(\frac{t_2}{t_1},\dots \frac{t_n}{t_1}) \tag{2}$$

if $t_1 > 0$.

REMARK 2. In article [10] it was also shown that Theorem A implies the following result by L. Maligranda (see [5]):

For every non-vanishing function $\varphi : R_+^2 \to R$, which is concave and continuous in each variable and positive homogeneous of degree one and every x,y from a Banach lattice X such that $\varphi(|x|,|y|) \in X$, the following holds:

$$||\varphi(|x|,|y|)||_X \leqslant \varphi(||x||_X,||y||_X).$$

Next we introduce the function $G_{r,s}$, which is crucial for our later investigations.

DEFINITION 1. Let f_i, $i=1,2,\dots,m-1$, be positive functions on $(0,\infty)$ and let $x_i > 0$, $i=1,\dots,m$. For $r \leqslant s$, $r,s \in \{1,2,\dots,m-1\}$, we denote

$$G_{r,s}(x_r,x_{r+1},\dots,x_{s+1}) = x_r f_r\left(\frac{x_{r+1}}{x_r} f_{r+1}\left(\frac{x_{r+2}}{x_{r+1}} \dots f_s\left(\frac{x_{s+1}}{x_s}\right)\right)\right)$$

and

$$G_{s+1,s}(x) = x.$$

If any of the $x_i = 0$, then we define that $G_{r,s}(x_r,x_{r+1},\dots,x_{s+1}) = 0$.

We will finish this Section by pointing out some examples of isotonic subadditive functionals (ISF), which indicate the usefulness of our results proved in Section 3 (e.g., see the applications in Section 4).

EXAMPLE 1. Let A be ILF or ISF. Then we can generate a scale of ISFs A_p, $p \geqslant 1$, by using the well-known technique called *convexification* in the following way

$$A_p(f) = A(f^p)^{1/p}.$$

For a simple proof of this fact see [8].

In particular, by using any of the ILFs pointed out in [1] and Example 1 we obtain a scale of ISFs as shown in the following examples:

EXAMPLE 2. Let $p \geqslant 1$. Then the functionals

$$\Sigma_p : \quad \Sigma_p(a) := ||a||_{\ell_p} = (\sum_{i=1}^{\infty} a_i^p)^{1/p},$$

where $a = \{a_i\}_0^\infty$, $a_i \geqslant 0$, or more generally

$$S_p : \quad S_p(f) := ||f||_{L_p} = (\int_\Omega f^p d\mu)^{1/p},$$

where (Ω, μ) is a measure space, are all ISFs.

Example 2 is just a special case of the next example of great interest in the theory of function spaces:

EXAMPLE 3. Let M^+ be the cone of μ-measurable functions on R whose values lie in $[0,\infty]$. A mapping $\rho : M^+ \to [0,\infty]$ is called a *Banach function norm* if for all f, g, f_n, $(n \in Z_+)$ in M^+, for all constants $\alpha \geqslant 0$ and for all μ-measurable subsets E of R, it yields that:

$(B1)$ $\rho(f) = 0$ if and only if $f = 0$ μ a.e.; $\rho(\alpha f) = \alpha\rho(f)$,

$(B2)$ $0 \leqslant g \leqslant f$ μ a.e. implies $\rho(g) \leqslant \rho(f)$,

$(B3)$ $\rho(f+g) \leqslant \rho(f) + \rho(g)$,

and, moreover,

$(B4)$ $0 \leqslant f_n \uparrow f$ μ a.e. implies $\rho(f_n) \uparrow \rho(f)$; $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$ and $\int_E f\, d\mu \leqslant C_E \rho(f)$.

According to $(B1)-(B3)$ we see that ρ is a subadditive functional.

REMARK 3. For each Banach function norm ρ, the collection X of all functions f for which $\rho(|f|) < \infty$ is called a Banach function space X with the norm defined by

$$\|f\|_X = \rho(|f|).$$

In particular, when $\rho = \rho_p$ defined by

$$\rho_p(f) = \begin{cases} (\int_\Omega f^p d\mu)^{1/p}, & 1 \leqslant p < \infty \\ \text{ess sup}_\Omega f, & p = \infty \end{cases}$$

we obtain the Lebesgue spaces L^p (see Example 2) also for the limit case $p = \infty$. In fact, several of the most well-established function spaces (e.g. the Lorentz $L_{p,q}$ spaces, the Orlicz spaces, the Hardy-Littlewood $L\log L$ spaces etc.) are defined just via some suitable Banach function norm ρ. For more details see e.g. [3].

Finally we give a fundamental example connected to abstract interpolation theory (see e.g. [3]).

EXAMPLE 4. Let A_0 and A_1 denote two Banach function spaces. For each $t > 0$ and all $a \in A_0 + A_1$ the famous (Peetre) *K-functional* K defined by

$$K(a) = K(t,a;A_0,A_1) := \inf_{a=a_0+a_1, a_0 \in A_0, a_1 \in A_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1})$$

is in fact an ISF. Also the corresponding (Peetre) *J-functional* J, defined by

$$J(a) = J(t,a;A_0,A_1) := \sup_{a\in A_0\cap A_1} (\|a\|_{A_0}, t\|a\|_{A_1})$$

for each $t > 0$ and all $a \in A_0 \cap A_1$, is obviously an ISF.

All modern abstract real interpolation theory is based on these two functionals (see e.g. the books [3] and [4] and the references given there). Hence, our results in this paper directly imply new inequalities and embeddings in this connection.

3. The main results

The following concept is used in [1] and formulated in [2] as shown here:

DEFINITION 2. We say that a set of convex or concave functions $f_i, i = 1,\dots,m-1$, satisfies *Monotonicity Condition* (MC) if all $k = 1,\dots,m-2$ and all pairs (f_k, f_{k+1}) satisfy the following:

(i) When both functions f_k and f_{k+1} are either convex or concave, then f_k is increasing.

(ii) When either f_k is convex and f_{k+1} is concave, or f_k is concave and f_{k+1} is convex, then f_k is decreasing.

Now, we state the following general results for the function $G_{r,s}$.

THEOREM 1. *Let* $\{f_i : i = 1,\dots,m-1\}$, *be a set of positive functions on* $(0,\infty)$ *with the MC property.*

$a)$ *Let* p *and* q *be positive real numbers. If* f_r *is a concave function, then for* $\underline{a} = (a_r,\dots,a_{s+1})$ *and* $\underline{b} = (b_r,\dots,b_{s+1})$ *we have*

$$pG_{r,s}(\underline{a}) + qG_{r,s}(\underline{b}) \leqslant G_{r,s}(p\underline{a} + q\underline{b}) \tag{3}$$

for any s, $s \in \{r,\dots,m-1\}$.

If f_r *is a convex function, then the reversed inequality holds.*

$b)$ *If* f_r *is a concave function, then the function* $G_{r,s}$ *is a superadditive and concave function. If* f_r *is a convex function, then the function* $G_{r,s}$ *is a subadditive and convex function.*

$c)$ *The function* $G_{r,s}$ *is a positively homogeneous function of degree one.*

Proof. The first part of the theorem is proved by using the principle of mathematical induction on the number of variables of the function $G_{r,s}$ and also by using the discrete Jensen's inequality for convex and concave functions.

Let us denote $\underline{a}_2 = (a_r, a_{r+1})$ and $\underline{b}_2 = (b_r, b_{r+1})$. For $p,q > 0$ and f_r concave we have

$$\begin{aligned} G_{r,r}(p\underline{a}_2 + q\underline{b}_2) &= (pa_r + qb_r)f_r\left(\frac{pa_{r+1} + qb_{r+1}}{pa_r + qb_r}\right) \\ &\geqslant pa_r f_r\left(\frac{a_{r+1}}{a_r}\right) + qb_r f_r\left(\frac{b_{r+1}}{b_r}\right) \\ &= pG_{r,r}(\underline{a}_2) + qG_{r,r}(\underline{b}_2). \end{aligned}$$

If f_r is a convex function, then we have the reversed inequality.

The assumption of the induction: Let us suppose that every function $G_{r,r+k-2}$, $r, r+k-2 \in \{1,2,\dots,m-1\}$, with k variables satisfies the statement of the first part of Theorem 1.

Let us now consider the function $G_{r,r+k-1}$ of $k+1$ variables. First we consider the case when f_r is a concave function. By the definition we have

$$G_{r,r+k-1}(a_r,\dots,a_{r+k}) = a_r f_r\left(\frac{G_{r+1,r+k-1}(a_{r+1},\dots,a_{r+k})}{a_r}\right).$$

$G_{r+1,r+k-1}$ has k variables and the assumption of induction holds for it. If $G_{r+1,r+k-1}$ satisfies inequality (3), then f_{r+1} is concave and by the MC property f_r is an increasing function.

Dividing the assumption of the induction by pa_r+qb_r and using the monotonicity of the function f_r we have

$$f_r\left(\frac{G_{r+1,r+k-1}(p\underline{a}_k + q\underline{b}_k)}{pa_r+qb_r}\right) \geqslant f_r\left(\frac{pG_{r+1,r+k-1}(\underline{a}_k) + qG_{r+1,r+k-1}(\underline{b}_k)}{pa_r+qb_r}\right), \tag{4}$$

where $\underline{a}_k = (a_{r+1},\dots,a_{r+k})$ and $\underline{b}_k = (b_{r+1},\dots,b_{r+k})$.

Now, multiplying (4) by pa_r+qb_r and using Jensen's inequality for the concave function f_r we find that

$$\begin{aligned}
G_{r,r+k-1}(p\underline{a}_{k+1} + q\underline{b}_{k+1}) &= (pa_r+qb_r) f_r\left(\frac{G_{r+1,r+k-1}(p\underline{a}_k + q\underline{b}_k)}{pa_r+qb_r}\right)\\
&\geqslant pa_r f_r\left(\frac{G_{r+1,r+k-1}(a_{r+1},\dots,a_{r+k})}{a_r}\right)\\
&\quad + qb_r f_r\left(\frac{G_{r+1,r+k-1}(b_{r+1},\dots,b_{r+k})}{b_r}\right)\\
&= pG_{r,r+k-1}(\underline{a}_{k+1}) + qG_{r,r+k-1}(\underline{b}_{k+1}),
\end{aligned}$$

where $\underline{a}_{k+1} = (a_r,\dots,a_{r+k})$ and $\underline{b}_{k+1} = (b_r,\dots,b_{r+k})$, which establishes the statement for the function $G_{r,r+k-1}$ of $k+1$ variables.

Moreover, if $G_{r+1,r+k-1}$ satisfies the inequality reversed to (3), then f_{r+1} is convex and by the MC property f_r is a decreasing function. Hence, the inequality (4) is valid also in this case, and after using the Jensen's inequality for the concave function f_r we obtain that inequality (3) again is valid for the function $G_{r,r+k-1}$ of $k+1$ variables. Thus, by the induction axiom, $a)$ is proved for the case when f_r is concave.

If f_r is a convex function, then similar arguments lead us to the reversed inequality (3).

Obviously, the statement in $b)$ is just a special case of that in $a)$ and the proof of the statement in $c)$ i.e. that

$$G_{r,s}(\lambda a_r,\dots,\lambda a_{s+1}) = \lambda G_{r,s}(a_r,\dots,a_{s+1})$$

for each $\lambda > 0$, is just an easy consequence of the definition of $G_{r,s}$. The proof is complete. □

The following consequence of Theorem A is useful in the sequel but also of independent interest:

THEOREM 2. *Let A be an ISF. Let $a_i \in P$, $i = 1, \dots, m$, be positive functions on a set S, $A(a_i) > 0$, $i = 1, \dots, m$, and let the set $\{f_i : (0,\infty) \to (0,\infty), i = 1, \dots, m-1\}$ have the MC property.*

If, for some $r \in \{1, 2, \dots, m-1\}$, the function f_r is concave, then

$$A(G_{r,s}(a_r, \dots, a_s)) \leqslant G_{r,s}(A(a_r), \dots, A(a_s)) \tag{5}$$

holds for any $s \in \{r, \dots, m-1\}$.

Proof. The properties of the function $G_{r,s}$, which are described in Theorem 1, imply that $G_{r,s}$ satisfies the assumptions of Theorem A. Hence, the inequality (5) is a simple consequence of the mentioned theorem. □

While (5) is a Jensen's type inequality, in the following theorem we consider a reversed inequality:

THEOREM 3. *Let A be an ISF. Let c_i, $i = 1, 2, \dots, m$, be positive real numbers, $a_i \in P$, $i = 1, 2, \dots, m$, be positive functions on S, $A(a_i) > 0$, $c_i - A(a_i) > 0$, $i = 1, 2, \dots, m$.*

Let the set $\{f_i : (0,\infty) \to (0,\infty), i = 1, \dots, m-1\}$ have the MC property. Furthermore, suppose that $G_{r,s}(a_r, \dots, a_{s+1}) \in P$, $G_{r,s}(c_r, \dots, c_{s+1}) - A(G_{r,s}(a_r, \dots, a_{s+1})) > 0$ for $r, s \in \{1, 2, \dots, m-1\}$, $r \leqslant s+1$.

Then, for any $r, s \in \{1, 2, \dots, m-1\}$, $r \leqslant s+1$, for which f_r is concave, the following holds:

$$G_{r,s}(c_r - A(a_r), \dots, c_{s+1} - A(a_{s+1})) \leqslant G_{r,s}(c_r, \dots, c_{s+1}) - A(G_{r,s}(a_r, \dots, a_{s+1})). \tag{6}$$

Proof. Since, by Theorem 1, $G_{r,s}$ is a superadditive function we have

$$\begin{aligned} G_{r,s}(c_r - A(a_r), \dots, c_{s+1} - A(a_{s+1})) &\leqslant G_{r,s}(c_r, \dots, c_{s+1}) - G_{r,s}(A(a_r), \dots, A(a_{s+1})) \\ &\leqslant G_{r,s}(c_r, \dots, c_{s+1}) - A(G_{r,s}(a_r, \dots, a_{s+1})) \end{aligned}$$

where in the last inequality Theorem 2 is used. The proof is complete. □

We shall now prove some more general inequalities, which in particular imply sharpenings of some of our previous inequalities.

THEOREM 4. *Let E and F be functionals on P such that the difference $D = F - E$ is an ISF. Let f_i, $i = 1, \dots, m-1$, be positive functions with the MC property. If f_1 is concave, then for $s = 2, \dots, m$, it yields that*

$$\begin{aligned} D(G_{1,m-1}(a_1, \dots, a_m)) \leqslant\ & G_{1,s-1}(F(a_1), \dots, F(a_{s-1}), F(z)) \\ & - G_{1,s-1}(E(a_1), \dots, E(a_{s-1}), E(z)), \end{aligned} \tag{7}$$

where $z = G_{s,m-1}(a_s, \dots, a_m)$, $a_i \in P, i = 1, \dots, m$.

Proof. Since f_i, $i = 1, \dots, m-1$, satisfy the MC property, using discrete Jensen's inequality several times, we get

$$\begin{aligned} & G_{1,s-1}\left(D(a_1), \dots, D(a_{s-1}), D(z)\right) + G_{1,s-1}\left(E(a_1), \dots, E(a_{s-1}), E(z)\right) \\ &= D(a_1) f_1\left(\frac{G_{2,s-1}\left(D(a_2), \dots, D(z)\right)}{D(a_1)}\right) + E(a_1) f_1\left(\frac{G_{2,s-1}\left(E(a_2), \dots, E(z)\right)}{E(a_1)}\right) \end{aligned}$$

$$\begin{aligned}
&\leqslant F(a_1)f_1\left(\frac{G_{2,s-1}\left(D(a_2),\dots,D(z)\right)+G_{2,s-1}\left(E(a_2),\dots,E(z)\right)}{F(a_1)}\right)\\
&\leqslant F(a_1)f_1\left(\frac{F(a_2)}{F(a_1)}f_2\left(\frac{G_{3,s-1}\left(D(a_3),\dots,D(z)\right)+G_{3,s-1}\left(E(a_3),\dots,E(z)\right)}{F(a_2)}\right)\right) \qquad (8)\\
&\leqslant \dots \leqslant G_{1,s-1}\left(F(a_1),\dots,F(a_{s-1}),F(z)\right).
\end{aligned}$$

Indeed, since f_1 is concave, the first inequality holds. Moreover, if f_2 is convex, then f_1 is decreasing and if f_2 is concave, then f_1 is increasing. In either case the second inequality holds. The same reasoning leads to the last inequality. Now, using Theorem 2 we have

$$\begin{aligned}
&D(G_{1,m-1}(a_1,\dots,a_m))\\
&\quad = D(G_{1,m-1}(a_1,\dots,a_{s-1},z))\\
&\quad \leqslant G_{1,s-1}(D(a_1),\dots,D(a_{s-1}),D(z))\\
&\quad \leqslant G_{1,s-1}(F(a_1),\dots,F(a_{s-1}),F(z)) - G_{1,s-1}(E(a_1),\dots,E(a_{s-1}),E(z)).
\end{aligned}$$

This completes the proof. □

As a consequence of the previous Theorems 2 and 4 we get the following result:

THEOREM 5. *Let D and E be ISFs. Let $a_i \in P$, $i=1,\dots,m$, be positive functions on a set S, $D(a_i)>0$ and $E(a_i)>0$, $i=1,\dots,m$, and let a set $\{f_i:(0,\infty)\to(0,\infty), i=1,\dots,m-1\}$ has the MC property. If f_1 is concave, then*

$$\begin{aligned}
&G_{1,s-1}\left(F(a_1),\dots,F(a_{s-1}),F(z)\right)\\
&\qquad \geqslant D(G_{1,m-1}(a_1,\dots,a_m)) + G_{1,s-1}\left(E(a_1),\dots,E(a_{s-1}),E(z)\right) \qquad (9)\\
&\qquad \geqslant F(G_{1,m-1}(a_1,\dots,a_m)),
\end{aligned}$$

where $F=D+E$.

Proof. The first inequality can be obtained by just rewriting inequality (7). Moreover, according to Theorem 2 we get

$$\begin{aligned}
G_{1,s-1}(E(a_1),\dots,E(a_{s-1}),E(z)) &\geqslant E(G_{1,s-1}(a_1,\dots,a_z))\\
&= (D-F)(G_{1,s-1}(a_1,\dots,a_z))\\
&= D(G_{1,m-1}(a_1,\dots,a_m)) - F(G_{1,m-1}(a_1,\dots,a_m)),
\end{aligned}$$

thus, the second inequality in (9) has been established. □

REMARK 4. When $s=m$ (9) becomes

$$\begin{aligned}
G_{1,m-1}(F(a_1),\dots,F(a_m)) &\geqslant D(G_{1,m-1}(a_1,\dots,a_m)) + G_{1,m-1}(E(a_1),\dots,E(a_m))\\
&\geqslant F(G_{1,m-1}(a_1,\dots,a_m)),
\end{aligned} \qquad (10)$$

which is a sharpening of inequality (5).

THEOREM 6. *Let D and E be ISFs and put $F=D+E$. Moreover, let the functions f_i, $i=1\dots,m-1$, be positive functions with the MC property and let $a_i \in P, i=1,\dots,m$, be positive functions such that $E(a_i)>0$, $D(a_i)>0$, $c_i-E(a_i)>0$, $c_i-F(a_i)>0$, where c_i, $i=1\dots,m$, are positive real numbers.*

If f_1 is concave, then for $s = 2, \dots, m$,

$$\begin{aligned} &G_{1,s-1}(c_1, \dots, c_{s-1}, u) - F(G_{1,s-1}(a_1, \dots, a_{s-1}, z)) \\ &\quad \geqslant G_{1,s-1}(c_1 - E(a_1), \dots, c_{s-1} - E(a_{s-1}), u - E(z)) - D(G_{1,m-1}(a_1, \dots, a_m)) \quad (11) \\ &\quad \geqslant G_{1,s-1}(c_1 - F(a_1), \dots, c_{s-1} - F(a_{s-1}), u - F(z)), \end{aligned}$$

where $z = G_{s,m-1}(a_s, \dots, a_m)$, $u = G_{s,m-1}(c_s, \dots, c_m)$ and all the above-mentioned terms are well-defined.

Proof. The left-hand side inequality is a consequence of Theorem 3. Namely, for the functional E we have

$$\begin{aligned} &G_{1,s-1}(c_1 - E(a_1), \dots, c_{s-1} - E(a_{s-1}), u - E(z)) \\ &\leqslant G_{1,s-1}(c_1, \dots, c_{s-1}, u) - E(G_{1,s-1}(a_1, \dots, a_{s-1}, z)) \\ &= G_{1,s-1}(c_1, \dots, c_{s-1}, u) - F(G_{1,s-1}(a_1, \dots, a_{s-1}, z)) + D(G_{1,s-1}(a_1, \dots, a_{s-1}, z)) \end{aligned}$$

and our claim is proved.

The proof of the right-hand side of inequality (11) is based on Theorems 1 and 2. In fact, using Theorem 2 we obtain

$$\begin{aligned} &G_{1,s-1}(c_1 - F(a_1), \dots, c_{s-1} - F(a_{s-1}), u - F(z)) + D(G_{1,m-1}(a_1, \dots, a_m)) \\ &\quad \leqslant G_{1,s-1}(c_1 - F(a_1), \dots, c_{s-1} - F(a_{s-1}), u - F(z)) + G_{1,m-1}(D(a_1), \dots, D(a_m)) \\ &\quad = G_{1,s-1}(c_1 - F(a_1), \dots, c_{s-1} - F(a_{s-1}), u - F(z)) + G_{1,s-1}(D(a_1), \dots, D(z)). \end{aligned}$$

Moreover, according to Theorem 1,

$$\begin{aligned} &G_{1,s-1}(c_1 - F(a_1), \dots, c_{s-1} - F(a_{s-1}), u - F(z)) + G_{1,s-1}(D(a_1), \dots, D(z)) \\ &\quad \leqslant G_{1,s-1}(c_1 - F(a_1) + D(a_1), \dots, c_{s-1} - F(a_{s-1}) + D(a_{s-1}), u - F(z) + D(z)) \\ &\quad = G_{1,s-1}(c_1 - E(a_1), \dots, c_{s-1} - E(a_{s-1}), u - E(z)). \end{aligned}$$

Now we easily obtain the right-hand side inequality (11) and the proof is complete. □

REMARK 5. When $s = m$ (11) reads:

$$\begin{aligned} &G_{1,m-1}(c_1, \dots, c_m)) - F(G_{1,m-1}(a_1, \dots, a_m)) \\ &\quad \geqslant G_{1,m-1}(c_1 - E(a_1), \dots, c_m - E(a_m)) - D(G_{1,m-1}(a_1, \dots, a_m)) \quad (12) \\ &\quad \geqslant G_{1,m-1}(c_1 - F(a_1), \dots, c_m - F(a_m)), \end{aligned}$$

which obviously is a sharpening of inequality (6).

4. Applications

The applications pointed out in $4.1 - 4.2$ are known for ILF, but they are new in this much more general setting. The applications in Sections $4.3 - 4.5$ are essentially new also for ILF.

4.1. Hölder's and Popoviciu's type inequalities for ISFs

As a simple consequence of Theorems 2 and 3 we obtain a functional version of the Hölder's and Popoviciu's inequality (see [6]). Namely, let p_j, $j = 1, 2, \dots, m$, $m \geqslant 2$, be positive numbers such that $\sum_{j=1}^{m} \frac{1}{p_j} = 1$ let the numbers q_j, $j = 1, 2, \dots, m-1$, be as follows

$$\frac{1}{q_1} = 1 - \frac{1}{p_1},$$
$$\frac{1}{q_j} = 1 - \frac{q_1 q_2 \dots q_{j-1}}{p_j}, \quad j = 2, \dots, m-1.$$

It can easily be verified that $q_1, \dots, q_{m-1}$ are positive real numbers less then 1. Hence, the functions $f_i(x) = x^{1/q_i}$, $i = 1, 2, \dots, m-1$, are concave increasing functions and if f_i and a_i satisfy the assumptions of Theorem 2, then inequality (5) holds for $r = 1, s = m-1$ and has the following form:

$$A\left(a_1\left(\frac{a_2}{a_1}\left(\frac{a_3}{a_2}\cdots\left(\frac{a_m}{a_{m-1}}\right)^{1/q_{m-1}}\right)^{1/q_2}\right)^{1/q_1}\right)$$
$$\leqslant A(a_1)\left(\frac{A(a_2)}{A(a_1)}\left(\frac{A(a_3)}{A(a_2)}\cdots\left(\frac{A(a_m)}{A(a_{m-1})}\right)^{1/q_{m-1}}\right)^{1/q_2}\right)^{1/q_1},$$

which after some transformations becomes the following generalized form of Hölder's inequality:

$$A\left(\prod_{i=1}^{m} a_i^{1/p_i}\right) \leqslant \prod_{i=1}^{m} A(a_i)^{1/p_i}, \tag{13}$$

where A is an ISF.

Moreover, choosing the same functions f_i and assuming that A is an ISF, the functions a_i and the real numbers c_i, $i = 1, 2, \dots m$, satisfy the assumptions of Theorem 3, we obtain a functional version of Popoviciu's inequality:

$$c_1^{1/p_1} c_2^{1/p_2} \dots c_m^{1/p_m} - A\left(a_1^{1/p_1} a_2^{1/p_2} \dots a_m^{1/p_m}\right) \geqslant \prod_{i=1}^{m} (c_i - A(a_i))^{1/p_i}. \tag{14}$$

4.2. Minkowski's and Bellman's type inequalities for ISFs

Let $p > 1$ be a real number and f be a real function defined by

$$f(x) = (1 + x^{1/p})^p, \quad x > 0.$$

The function f is concave and increasing and we consider the function $G_{1,m-1}$ with $f_1 = \dots = f_{m-1} = f$. Then

$$G_{1,m-1}(x_1, \dots, x_m) = (x_1^{1/p} + \dots + x_m^{1/p})^p.$$

If A and a_i, $i = 1,\ldots,m$, satisfy the assumptions of Theorem 2, then we obtain the following functional version of the Minkowski's inequality:

$$A((a_1^{1/p}+\ldots+a_m^{1/p})^p) \leqslant (A(a_1)^{1/p}+\ldots+A(a_m)^{1/p})^p, \tag{15}$$

where A is an ISF.

If A and c_i, a_i, $i = 1,\ldots,m$, satisfy the assumptions of Theorem 3, then we obtain the following functional version of the Bellman's inequality:

$$\left(\sum_{i=1}^m (c_i - A(a_i))^{1/p}\right)^p \leqslant \left(\sum_{i=1}^m c_i^{1/p}\right)^p - A((\sum_{i=1}^m a_i^{1/p})^p), \tag{16}$$

where A is an ISF.

4.3. Power mean type inequalities for ISFs

Let w_i, $i = 1,\ldots,m$, be positive real numbers and $\sum_{i=1}^m w_i = 1$ and let the functions f_i be

$$f_i(x) = \left(1 + \frac{w_{i+1}}{w_i}x^r\right)^{1/r}, \quad i = 2,\ldots,m-1,$$
$$f_1(x) = (w_1 + w_2 x^r)^{1/r},$$

where $r \leqslant 1$, $r \neq 0$.

The functions f_i, $i = 1,2,\ldots,m-1$, are increasing and concave. The function $G_{1,m-1}$ for those special functions f_i has the form

$$G_{1,m-1}(x_1,\ldots,x_m) = (w_1 x_1^r + \ldots + w_m x_m^r)^{1/r},$$

which is exactly the power mean of order r of a sequence $x = (x_1,\ldots,x_m)$ with weights $w = (w_1,\ldots,w_m)$. Usually, that power mean is denoted by $M_m^{[r]}(x;w)$.

As a consequence of Theorems 2 and 3 we obtain the following inequalities:

$$A(M_m^{[r]}(a_1,\ldots,a_m;w)) \leqslant M_m^{[r]}(A(a_1),\ldots,A(a_m);w), \tag{17}$$

and

$$\begin{aligned} &M_m^{[r]}(c_1 - A(a_1),\ldots,c_m - A(a_m);w) \\ &\qquad \leqslant M_m^{[r]}(c_1,\ldots,c_m;w) - A(M_m^{[r]}(a_1,\ldots,a_m;w)), \end{aligned} \tag{18}$$

where A, a_i, c_i, $i = 1,\ldots,m$, satisfy the assumptions of Theorem 2 and 3.

REMARK 6. If $A = A_p$ as in Example 1, then the inequalities (17) and (18) have the following forms:

$$A_p(M_m^{[r]}(a_1,\ldots,a_m;w)) \leqslant M_m^{[r]}(A_p(a_1),\ldots,A_p(a_m);w), \tag{19}$$

respectively

$$\begin{aligned} &M_m^{[r]}(c_1 - A_p(a_1),\ldots,c_m - A_p(a_m);w) \\ &\qquad \leqslant M_m^{[r]}(c_1,\ldots,c_m;w) - A_p(M_m^{[r]}(a_1,\ldots,a_m;w)). \end{aligned} \tag{20}$$

These inequalities are mixed-mean type inequalities. The inequality (19) can be found in [10], while (20) is a new result.

REMARK 7. If A is generated by a functional norm $||.||$ as in Example 3 - Remark 3, then the inequalities (17) and (18) have the following forms:

$$||M_m^{[r]}(a_1,\ldots,a_m;w)|| \leqslant M_m^{[r]}(||a_1||,\ldots,||a_m||;w), \tag{21}$$

and

$$\begin{aligned} &M_m^{[r]}(c_1-||a_1||,\ldots,c_m-||a_m||);w) \\ &\qquad \leqslant M_m^{[r]}(c_1,\ldots,c_m;w)-||M_m^{[r]}(a_1,\ldots,a_m;w)||. \end{aligned} \tag{22}$$

Both of these inequalities are new.

4.4. More refinements of Hölder's, Minkowski's, Bellman's and reversed power means inequalities for ISFs

If the functionals D,E,F and the functions a_i, $i=1,\ldots,m$, satisfy the assumptions of Theorem 4, then for positive real numbers p_i, $\sum_{i=1}^m \frac{1}{p_i}=1$, the following sharpening of a functional version of Hölder's inequality (13) holds:

$$\prod_{i=1}^m F(a_i)^{1/p_i} \geqslant D\left(\prod_{i=1}^m a_i^{1/p_i}\right)+\prod_{i=1}^m E(a_i)^{1/p_i} \geqslant F\left(\prod_{i=1}^m a_i^{1/p_i}\right).$$

Also, for $p>1$ the following sharpening of a functional version of Minkowski's inequality (15) holds:

$$\left(\sum_{i=1}^m F(a_i)^{1/p}\right)^p \geqslant D\left(\sum_{i=1}^m a_i^{1/p}\right)^p+\left(\sum_{i=1}^m E(a_i)^{1/p}\right)^p \geqslant F\left(\sum_{i=1}^m a_i^{1/p}\right)^p.$$

Furthermore, if $r<1$, then the following sharpening of a functional version of the power mean inequality (17) of order r with weights $w=(w_1,\ldots,w_m)$ holds:

$$M_m^{[r]}(F(a);w) \geqslant D(M_m^{[r]}(a;w))+M_m^{[r]}(E(a);w) \geqslant F(M_m^{[r]}(a;w)),$$

where $F(a)=(F(a_1),\ldots,F(a_m))$ etc.

If the functionals D,E,F, the numbers c_i, the functions a_i, $i=1,\ldots,m$ satisfy the assumptions of Theorem 5, then for positive real numbers p_i, $\sum_{i=1}^m \frac{1}{p_i}=1$, the following sharpening of a functional version of Popoviciu's inequality (14) holds:

$$\begin{aligned} \prod_{i=1}^m c_i^{1/p_i}-F\left(\prod_{i=1}^m a_i^{1/p_i}\right) &\geqslant \prod_{i=1}^m (c_i-E(a_i))^{1/p_i}-D\left(\prod_{i=1}^m a_i^{1/p_i}\right) \\ &\geqslant \prod_{i=1}^m (c_i-F(a_i))^{1/p_i}. \end{aligned}$$

Moreover, for $p>1$ the following sharpening of a functional version of Bellman's inequality (16) holds:

$$\begin{aligned} (\sum_{i=1}^m c_i^{1/p})^p-F\left(\sum_{i=1}^m a_i^{1/p}\right)^p &\geqslant \left(\sum_{i=1}^m (c_i-E(a_i))^{1/p}\right)^p-D\left(\sum_{i=1}^m a_i^{1/p}\right)^p \\ &\geqslant \left(\sum_{i=1}^m (c_i-F(a_i))^{1/p}\right)^p. \end{aligned}$$

Finally, if $r < 1$, then the following sharpening of a functional version of the reversed inequality for power mean (18) of order r with weights $w = (w_1, \dots, w_m)$ holds:

$$M_m^{[r]}(c;w) - F(M_m^{[r]}(a;w)) \geqslant M_m^{[r]}(c - E(a);w) - D(M_m^{[r]}(a;w)) \geqslant M_m^{[r]}(c - F(a);w),$$

where $F(a) = (F(a_1), \dots, F(a_m))$ etc.

4.5. Some inequalities connected to Example 3

Let X be a Banach function space defined as in Example 3. Then the "convexified" space X^p, $p \geqslant 1$, is defined as the space of functions such that $|f|^p \in X$ equipped with the norm

$$||f||_{X^p} := ||\,|f|^p\,||_X^{1/p}.$$

Our results in this paper give a number of both new and well-known inequalities for such spaces. Here we just mention the following inequalities, which follow from our Sections $4.1 - 4.2$ and by the obvious substitutions:

Hölder's inequality: If $\sum_{i=1}^m \frac{1}{p_i} = 1$, $p_i > 0$, then

$$||\prod_{i=1}^m f_i||_X \leqslant \prod_{i=1}^m ||f_i||_{X^{p_i}}.$$

Minkowski's inequality: If $p \geqslant 1$, then

$$||\sum_{i=1}^m f_i||_{X^p} \leqslant \sum_{i=1}^m ||f_i||_{X^p}.$$

Popoviciu's inequality: If $\sum_{i=1}^m \frac{1}{p_i} = 1$, $p_i > 0$ and $0 < ||f_i||_{X^{p_i}} < d_i$, $i = 1, \dots, m$, then

$$\prod_{i=1}^m d_i - ||\prod_{i=1}^m f_i||_X \geqslant \prod_{i=1}^m \left(d_i^{p_i} - ||f_i||_{X^{p_i}}^{p_i}\right)^{1/p_i}.$$

Bellman's inequality: If $p > 1$ and $0 < ||f_i||_{X^p} < d_i$, $i = 1, \dots, m$, then

$$\left(\sum_{i=1}^m (d_i^p - ||f_i||_{X^p})^{1/p}\right)^p \leqslant \left(\sum_{i=1}^m d_i\right)^p - ||\sum_{i=1}^m f_i||_{X^p}^p.$$

REMARK 8. The first two inequalities are well-known but the other two seem to be new in this form.

4.6. Some inequalities connected to the Peetre K-functional

Let $\Omega = (\Omega, \Sigma, \mu)$ be a measure space and let f be a measurable function. Consider the K-functional defined in our Example 4. Our results can be used to obtain a number of completely new inequalities connected to this functional. We finish this paper by just pointing out some easy examples of such inequalities.

Let A_0 and A_1 be two function spaces on Ω and put $K(t,f) = K(t,f;A_0,A_1)$, $t > 0$. According to (13) we obtain that

$$K(t,\prod_{i=1}^{m} f_i) \leqslant \prod_{i=1}^{m} (K(t,f_i^{p_i}))^{1/p_i}, \tag{23}$$

where $p_i > 0$ and $\sum_{i=1}^{m} \frac{1}{p_i} = 1$.

REMARK 9. It is well-known that

$$K(t,f;L_1,L_\infty) = \int_0^t f^*(s)ds,$$

when f^* is the usual nonincreasing rearrangement of f (see e.g. [3]). For this case (23) just reads

$$\int_0^t (\prod_{i=1}^{m} f_i)^* ds \leqslant \prod_{i=1}^{m} (\int_0^t (f_i^*)^{p_i} ds)^{1/p_i}.$$

Of course, this inequality follows directly from Hölder's inequality.

If we instead of (13) use (15) we obtain that

$$\left(K(t,(\sum_{i=1}^{m} f_i)^p)\right)^{1/p} \leqslant \sum_{i=1}^{m} (K(t,f_i^p))^{1/p},$$

which just informs us that $K(t,f)$, for each $t > 0$, can be used as a norm on the space $A_0 + A_1$.

Moreover, the inequalities (14) and (16) imply that if $\sum_{i=1}^{m} \frac{1}{p_i} = 1$, $p_i > 0$ and $0 < K(t,|f_i|^{p_i}) < d_i$, $i = 1,2,\dots,m$, then

$$\prod_{i=1}^{m} d_i - K(t,\prod_{i=1}^{m} f_i) \geqslant \prod_{i=1}^{m} (d_i^{p_i} - K(t,|f_i|^{p_i}))^{1/p_i}$$

and if $p > 1$ and $0 < K(t,|f_i|^p) < d_i$, $i = 1,2,\dots,m$, then

$$\left(\sum_{i=1}^{m} (d_i^p - K(t,|f_i|^p))^{1/p}\right)^p \leqslant \left(\sum_{i=1}^{m} d_i\right)^p - K(t,|\sum_{i=1}^{m} f_i|^p).$$

REMARK 10. Analogous inequalities can be stated for the Peetre J-functional. Since most of these inequalities are only known in special cases in interpolation theory we strongly believe that our discovery can be useful in this area. We aim to return to these questions in a forthcoming paper.

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