INEQUALITIES FOR Ψ FUNCTION

Slavko Simić

(*communicated by N. Elezović*)

1. Introduction

As usual, we denote by $\Psi(x)$ the logarithmic derivative of Gamma function, i.e.

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}; \quad x \neq 0, -1, -2, \cdots.$$

The bounds for $\Psi(x)$ are of self-evident importance. In [1, p. 453], J. Sandor gives the following inequality

$$\log(x - \frac{1}{2}) < \Psi(x) < \log(x - \frac{1}{2} + \frac{1}{16x}),$$

where $x > 1$.

Our task here is to improve Sandor's inequality and to extend its validity over the whole real axis. As a consequence we obtain a precise upper bound for $\log \Gamma(x + \frac{1}{2})$.

2. Results

Our main result is contained in the next proposition.

PROPOSITION 1. *Let $\Psi(x)$ be defined as above. Then*
(*i*) *for $x \geqslant 1/2$,*

$$\frac{1}{2} \log(x^2 - x + \frac{1}{3} - \frac{1}{45x^2 + 4}) < \Psi(x) < \frac{1}{2} \log(x^2 - x + \frac{1}{3});$$

(*ii*) *for $0 < x < 1/2$,*

$$-\frac{1}{x} + \frac{1}{2} \log(x^2 + x + \frac{1}{3} - \frac{1}{45(x+1)^2 + 4}) < \Psi(x) < -\frac{1}{x} + \frac{1}{2} \log(x^2 + x + \frac{1}{3});$$

(*iii*) *for $x > 0$, $x \neq 1, 2, \cdots$,*

$$\pi \cot(\pi x) + \frac{1}{2} \log(x^2 + x + \frac{1}{3} - \frac{1}{45(x+1)^2 + 4}) < \Psi(-x) < \pi \cot(\pi x) + \frac{1}{2} \log(x^2 + x + \frac{1}{3}).$$

Mathematics subject classification (2000): 26D10, 33B15.
Key words and phrases: inequalities, psi function, gamma function.

The proof is based on the following precise and useful inequality [3].

LEMMA 1. *For each $x \geqslant 1/2$, we have*

$$\frac{1}{2} \log \left[\frac{(x + 1/2)^2 + 1/12 - 1/(45x^2 + 5)}{(x - 1/2)^2 + 1/12 - 1/(45x^2 + 5)} \right] < \frac{1}{x}$$
$$< \frac{1}{2} \log \left[\frac{(x + 1/2)^2 + 1/12 - 1/(45x^2 + 4)}{(x - 1/2)^2 + 1/12 - 1/(45x^2 + 4)} \right].$$

Proof of this lemma is deduced in a standard way.
Define

$$f(x, a) := \log \left[\frac{(x + 1/2)^2 + 1/12 - 1/(45x^2 + a)}{(x - 1/2)^2 + 1/12 - 1/(45x^2 + a)} \right] - \frac{2}{x};$$

then

$$f_x'(x, a) = \frac{2x^2(7a - 30) + 2(1 - a/3)^2}{x^2[(45x^2 + a)((x - 1/2)^2 + 1/12) - 1][(45x^2 + a)((x + 1/2)^2 + 1/12) - 1]};$$

therefore, for $x \geqslant 1/2$,

$$f_x'(x, 5) > 0; \quad f_x'(x, 4) < 0,$$

i.e. $f(x, 5)$ is monotone increasing and $f(x, 4)$ is monotone decreasing for $x \geqslant 1/2$.
Since $\lim_{x \to \infty} f(x, a) = 0$, the assertion of lemma follows.
A corollary we shall use in the sequel is

LEMMA 2. *For each $m \in [m_1, m_2]$; $m_2 > m_1 \geqslant 1/2$, we have*

$$\frac{1}{2} \log \left[\frac{(m + 1/2)^2 + 1/12 - 1/(45m_2^2 + 5)}{(m - 1/2)^2 + 1/12 - 1/(45m_2^2 + 5)} \right] < \frac{1}{m}$$
$$< \frac{1}{2} \log \left[\frac{(m + 1/2) + 1/12 - 1/(45m_1^2 + 4)}{(m - 1/2)^2 + 1/12 - 1/(45m_1^2 + 4)} \right].$$

This is a consequence of Lemma 1 and the fact that if $r > s > 0$ and $t_1 \geqslant t_2 > 0$, then

$$\frac{r - t_1}{s - t_1} \geqslant \frac{r - t}{s - t} \geqslant \frac{r - t_2}{s - t_2}.$$

3. Proof

Let

$$\Psi_n(x) := -\gamma - \frac{1}{x} + \sum_{s=1}^{n}\left(\frac{1}{s} - \frac{1}{s+x}\right) = \Psi_n^{(1)}(x) - \Psi_n^{(2)}(x), \tag{1}$$

where γ denotes Euler's constant and

$$\Psi_n^{(1)}(x) := -\gamma + \sum_{s=1}^{n}\frac{1}{s}; \quad \Psi_n^{(2)}(x) := \sum_{s=0}^{n}\frac{1}{s+x}.$$

Applying Lemma 2 with $m_1 := x$; $m_2 := x + n$, we obtain

$$\begin{aligned}
\Psi_n^{(2)}(x) &= \sum_{s=0}^{n}\frac{1}{s+x} < \frac{1}{2}\sum_{s=0}^{n}\log\left[\frac{(s+x+1/2)^2 + 1/12 - 1/(45x^2+4)}{(s+x-1/2)^2 + 1/12 - 1/(45x^2+4)}\right] \\
&= \frac{1}{2}\log\prod_{s=0}^{n}\frac{(s+x+1/2)^2 + 1/12 - 1/(45x^2+4)}{(s+x-1/2)^2 + 1/12 - 1/(45x^2+4)} \\
&= \frac{1}{2}\log\left[\frac{(n+x+1/2)^2 + 1/12 - 1/(45x^2+4)}{(x-1/2)^2 + 1/12 - 1/(45x^2+4)}\right].
\end{aligned} \tag{2}$$

Analogously

$$\Psi_n^{(2)}(x) > \frac{1}{2}\log\left[\frac{(n+x+1/2)^2 + 1/12 - 1/(45(n+x)^2+5)}{(x-1/2)^2 + 1/12 - 1/(45(n+x)^2+5)}\right]. \tag{3}$$

Since $\Psi_n^{(1)}(x) = \log n + o(1)$ $(n \to \infty)$, combining this with (1), (2) and (3) and the fact that $\Psi(x) = \lim_{n\to\infty}\Psi_n(x)$ [2, p. 56], we obtain the proof of assertion (i) from Proposition 1.

Parts (ii) and (iii) follow from (i) and the shift formulae

$$\Psi(x) = -\frac{1}{x} + \Psi(x+1); \quad \Psi(-x) = \pi\cot(\pi x) + \Psi(x+1),$$

respectively [2, p. 57].

As a corollary we obtain the next interesting inequality.

PROPOSITION 2. *For all $x \geqslant 0$ we have*

$$\log\Gamma(x+1/2) \leqslant \frac{1}{2}x\log(x^2+1/12) - x + \frac{\sqrt{3}}{6}\arctan 2\sqrt{3}x + \frac{1}{2}\log\pi.$$

Integrating the right part of (i) over $[1/2, x]$ and shifting $x \to x + 1/2$, we obtain the proof of the above inequality.

REFERENCES

[1] D. S. MITRINOVIC, J. SANDOR, B. CRSTICI, *Handbook of Number Theory*, Kluwer Acad. Publ., Dordrecht 1996.
[2] F. W. J. OLVER, *Introduction to Asymptotics and Special Functions*, Acad. Press, NY 1974.
[3] S. SIMIC, *Best possible bounds and monotonicity of segments of harmonic series*, Mat. Vesnik, **50**, (1998), 5–10.

Mathematical Institute SANU
Kneza Mihaila 35
11000 Belgrade
Serbia
e-mail: ssimic@turing.mi.sanu.ac.yu

Mathematical Inequalities & Applications
www.ele-math.com
mia@ele-math.com