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A SOLUTION OF A PROBLEM OF OPPEHEIM

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Abstract. In this paper, the open problem proposed by Oppeheim [A. Oppeheim, E1277, The American Mathematical Monthly, **64**, (1957) p. 504] is discussed carefully. At the same time, the Shafer, Fink and Malesevic inequalities are deduced from the solution of Oppeheim's problem.

1. Introduction

Mitrinovic [1, p. 247] gives us the following result of R. E. Shafer:

THEOREM 1. *If* $x > 0$, *then*

$$\arcsin x > \frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} > \frac{3x}{2+\sqrt{1-x^2}}. \tag{1}$$

Fink [2] obtains a upper bound for inverse sine, and gives the following theorem:

THEOREM 2. *If* $0 \leqslant x \leqslant 1$, *then*

$$\frac{3x}{2+\sqrt{1-x^2}} \leqslant \arcsin x \leqslant \frac{\pi x}{2+\sqrt{1-x^2}}. \tag{2}$$

Furthermore, 3 *and* π *are the best constants in* (2).

In fact, we can improve the upper bound of inverse sine and obtain (see [3])

THEOREM 3. *If* $0 \leqslant x \leqslant 1$, *then*

$$\frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} \leqslant \arcsin x \leqslant \frac{\pi(\sqrt{2}+\frac{1}{2})(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}}. \tag{3}$$

Furthermore, 6 *and* $\pi(\sqrt{2}+\frac{1}{2})$ *are the best constants in* (3).

Malesevic [4] gives a new upper bound for inverse sine, and obtains the following result.

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THEOREM 4. *If* $0 \leqslant x \leqslant 1$, *then*

$$\arcsin x \leqslant \frac{\frac{\pi}{\pi-2}x}{\frac{2}{\pi-2}+\sqrt{1-x^2}}. \tag{4}$$

We can obtain a lower bound for inverse sine, and obtain

THEOREM 5. *If* $0 \leqslant x \leqslant 1$, *then*

$$\frac{\pi(\pi-2)(4-\pi)}{2}\frac{x}{1+\frac{\pi-2}{2}\sqrt{1-x^2}} \leqslant \arcsin x \leqslant \frac{\pi}{2}\frac{x}{1+\frac{\pi-2}{2}\sqrt{1-x^2}}. \tag{5}$$

Furthermore, $\pi/2$ *is the best constant in* (5).

In view of the inequalities

$$\frac{\pi(\sqrt{2}+\frac{1}{2})(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} \leqslant \frac{\pi x}{2+\sqrt{1-x^2}}$$

and

$$\frac{3x}{2+\sqrt{1-x^2}} \leqslant \frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}}$$

for $x \in [0,1]$, combining (2) and (3) gives

THEOREM 6. *If* $0 \leqslant x \leqslant 1$, *then*

$$\frac{3x}{2+\sqrt{1-x^2}} \leqslant \frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} \leqslant \arcsin x$$

$$\leqslant \frac{\pi(\sqrt{2}+\frac{1}{2})(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} \leqslant \frac{\pi x}{2+\sqrt{1-x^2}}. \tag{6}$$

Furthermore, 3 *and* π, 6 *and* $\pi(\sqrt{2}+\frac{1}{2})$ *are the best constants in* (6).

The above discussion brings our attention to a poroblem of Oppeheim. In 1957, Oppeheim [5] asked the following question.

Problem. For each $p > 0$ there is a greatest q and a least r such that

$$q\frac{\sin x}{1+p\cos x} \leqslant x \leqslant r\frac{\sin x}{1+p\cos x}$$

for $0 \leqslant x \leqslant \pi/2$. Determine q and r as functions of p.

In 1958, Carve [6] gave a solution to Oppeheim's problem above using a moving curves method. Unfortunately, the solution given by Carver is incomplete because no obvious expression relating to the greatest value q for values of p in the interval $[1/2, \pi/2-1]$ and $[\pi/2-1, 2/\pi]$ is given.

In the present paper, we study Oppeheim's problem in a concise manner and obtain the following result.

THEOREM 7. *Let* $0 \leqslant x \leqslant \pi/2$ *and* $p > 0$, *then*

$$q\frac{\sin x}{1+p\cos x} \leqslant x \leqslant r\frac{\sin x}{1+p\cos x} \tag{7}$$

holds in cases:

(a) *When* $0 < p < 1/2$, *we have* $q = p+1, r = \pi/2$;
(b) *When* $1/2 \leqslant p < \pi/2 - 1$, *we have* $q = 4p(1-p^2), r = \pi/2$;
(c) *When* $\pi/2 - 1 \leqslant p < 2/\pi$, *we have* $q = 4p(1-p^2), r = p+1$;
(d) *When* $2/\pi \leqslant p < +\infty$, *we have* $q = \pi/2, r = p+1$.

Furthermore, these paired numbers q *and* r *showed in* (a) *and* (d) *are the best constants in* (7); *the values of* r *showed in* (b) *and* (c) *are the best.*

In last section, we discover the relations between the inequalities above and Oppeheim's problem.

2. A Lemma

LEMMA 1. $([7,8])$ *Let* $f, g : [a,b] \to \mathbb{R}$ *be two continuous functions which are differentiable on* (a,b). *Further, let* $g' \neq 0$ *on* (a,b). *If* f'/g' *is increasing (or decreasing) on* (a,b), *then the functions*

$$\frac{f(x) - f(b)}{g(x) - g(b)}$$

and

$$\frac{f(x) - f(a)}{g(x) - g(a)}$$

are also increasing (or decreasing) on (a,b).

3. Solution to Oppeheim's problem

Let $H(x) = \frac{(1+p\cos x)x}{\sin x}$, $f(x) = (1+p\cos x)x, g(x) = \sin x$, and $x \in (0, \frac{\pi}{2}]$, then we have

$$H(x) = \frac{f(x)}{g(x)} = \frac{f(x) - f(0)}{g(x) - g(0)}$$

and

$$k(x) = \frac{f'(x)}{g'(x)} = \frac{1 + p\cos x - px\sin x}{\cos x}.$$

Now, we try to find the maximum and minimum values of $H(x)$ using Lemma 1 and the monotonicity of $k(x)$.

We compute

$$k'(x) = \frac{u(x)}{\cos^2 x},$$

where, $u(x) = \sin x - px - \frac{1}{2}p\sin 2x$ and $u'(x) = \cos x(1 - 2p\cos x)$ for $x \in (0, \frac{\pi}{2}]$. Let $\cos x = t$, then $t \in [0,1)$ and $u'(x) = v(t) = -2pt(t - \frac{1}{2p})$.

There are three cases to consider.

(1) $0 < p < 1/2$.

Since $0 \leqslant t < 1 < 1/2p, u'(x) = v(t) \geqslant 0$ (with equality if and only if $t = 0$), we deduce that the function $u(x)$ is increasing on $(0, \pi/2]$. At the same time, $u(0) = 0$, so $u(x) > 0$ and $k'(x) > 0$. That is, f'/g' is increasing on $(0, \pi/2]$, and $H(x) = \frac{f(x)}{g(x)} = \frac{f(x)-f(0)}{g(x)-g(0)}$ is increasing on $(0, \pi/2]$ by Lemma 1.

Therefore, $\max_{x\in(0,\pi/2]} H(x) = H(\pi/2) = \pi/2$ and $\min_{x\in(0,\pi/2]} H(x) = H(0^+) = 1 + p$. That is, $q = p + 1$ and $r = \pi/2$ hold in (7).

(2) $2/\pi \leqslant p < +\infty$.

Let $\xi = \arccos(1/2p)$, then we have

(A) When $t \in (1/2p, 1)$ or $x \in (0, \xi)$, $u'(x) = v(t) < 0$, so $u(x)$ is decreasing on $(0, \xi)$. Now, $u(0) = 0$, then $u(x) < 0$ (certainly, $u(\xi) < 0$) and $k'(x) < 0$ for $x \in (0, \xi)$.

(B) When $t \in [0, 1/2p)$ or $x \in (\xi, \pi/2]$, $u'(x) = v(t) > 0$, so $u(x)$ is increasing on $(\xi, \pi/2]$. We note that $u(\xi) < 0$ and $u(\pi/2) = 1 - p\pi/2 \leqslant 0$, so $u(x) \leqslant 0$ holds on $(\xi, \pi/2]$.

In a word, $k'(x) \leqslant 0$ holds on $(0, \pi/2]$. That is, f'/g' is decreasing on $(0, \pi/2]$, and $H(x) = \frac{f(x)}{g(x)} = \frac{f(x)-f(0)}{g(x)-g(0)}$ is decreasing on $(0, \pi/2]$ by Lemma 1.

Therefore, $\max_{x\in(0,\pi/2]} H(x) = H(0^+) = 1+p$ and $\min_{x\in(0,\pi/2]} H(x) = H(\pi/2) = \pi/2$. That is, $q = \pi/2$ and $r = p + 1$ hold in (7).

(3) $1/2 \leqslant p < 2/\pi$.

At this moment, the function $H(x)$ is not fully monotone on $(0, \pi/2]$.

Let η be only one point of minimum of the function $H(x)$. We can obtain by direct calculation

$$H'(x) = \frac{\sin x + \frac{p}{2}\sin 2x - x\cos x - px}{\sin^2 x}.$$

Since $H'(\eta) = 0$, we have

$$\sin\eta + \frac{p}{2}\sin 2\eta - \eta\cos\eta - p\eta = 0. \tag{8}$$

That is

$$\frac{\eta}{\sin\eta} = \frac{1 + p\cos\eta}{p + \cos\eta}. \tag{9}$$

Substituting (9) into the expression of $H(x)$, we obtain

$$H(\eta) = \frac{(1 + p\cos\eta)\eta}{\sin\eta} = \frac{(1 + p\cos\eta)^2}{p + \cos\eta}. \tag{10}$$

Because the minimum of the function $h(x) = \frac{(1+px)^2}{p+x}$ on $(0,1)$ is $4p(1-p^2)$, we have $H(\eta) \geqslant 4p(1-p^2)$.

Finally, we obtain results in two cases:

(i) If $1 + p < \pi/2$ or $1/2 \leqslant p < \pi/2 - 1$, we have $q = 4p(1-p^2), r = \pi/2$;

(ii) If $1 + p \geqslant \pi/2$ or $\pi/2 - 1 \leqslant p < 2/\pi$, we have $q = 4p(1-p^2), r = 1 + p$.

4. The special cases of Oppeheim's inequality

(i) Let $p = 1/2$, then $q = 3/2, r = \pi/2$. Setting $x = \arcsin t$ or $t = \sin x$, we have $t \in [0,1]$ and (2) holds by Theorem 7.

(ii) Let $p = \pi/2 - 1$, then $q = 4p(1-p^2) = \pi(\pi-2)(4-\pi)/2, r = \pi/2$. Setting $x = \arcsin t$ or $t = \sin x$, we have $t \in [0,1]$ and (5) holds by Theorem 7.

(iii) Let $p = 2/\pi$, then $q = 4p(1-p^2) = 8(\pi^2-4)/\pi^3, r = 2/\pi+1$. Setting $x = \arcsin t$ or $t = \sin x$, we have $t \in [0,1]$ and obtain the following result by Theorem 7.

THEOREM 8. *If* $0 \leqslant t \leqslant 1$, *then*

$$\frac{8(\pi^2-4)}{\pi^3} \frac{\pi t}{\pi + 2\sqrt{1-t^2}} \leqslant \arcsin t \leqslant (\frac{2}{\pi}+1)\frac{\pi t}{\pi + 2\sqrt{1-t^2}}. \tag{11}$$

Furthermore, $2/\pi + 1$ *is the best constants in* (11).

(iv) Let $p = 1 > 2/\pi$, then $q = \pi/2, r = 2$. Setting $x = \arcsin t$ or $t = \sin x$, we have $t \in [0,1]$ and obtain the following result by Theorem 7.

THEOREM 9. *If* $0 \leqslant t \leqslant 1$, *then*

$$\frac{\pi}{2}\frac{t}{1+\sqrt{1-t^2}} \leqslant \arcsin t \leqslant 2\frac{t}{1+\sqrt{1-t^2}}. \tag{12}$$

Furthermore, $\pi/2$ *and* 2 *are the best constants in* (12).

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