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SOME NEW PROPERTIES FOR THE RESOLVENT OPERATOR

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(*communicated by Th. M. Rassias*)

Abstract. The general mixed quasi variational inequalities (GMVI) are considerably difficult to solve directly, and hence, we often try to resolve this problem via solving some equivalent forms of GMVI. It is well known that GMVI are equivalent to the fixed point problems and resolvent equations. In this paper, we use these alternative equivalent formulations to suggest some important properties for the resolvent operator.

1. Introduction

A useful and important generalization of variational inequalities is the general mixed variational inequality containing a nonlinear term φ. But the applicability of the projection method is limited due to the fact that it is not easy to find the projection except in very special cases. Secondly, the projection method can not be applied to suggest iterative algorithms for solving general mixed variational inequalities involving the nonlinear term φ. This fact has motivated many authors to develop the auxiliary principle technique for solving the mixed variational inequalities. Lions and Stampacchia [4], Glowinski et al. [2] used this technique to study the existence of solution for the mixed variational inequalities.

In recent years some iterative methods have been suggested for special cases of the general mixed quasi variational inequalities. For example, if the bifunction is proper, convex and lower semicontinuous function with respect to the first argument, then one can show that the general mixed quasi variational inequalities are equivalent to the fixed-point problems and the implicit resolvent equations using the resolvent operator technique see [1-13]. M. A. Noor, K. I. Noor and Th. M. Rassias [7] have used the resolvent operator technique to establish the equivalence among generalized set-valued variational inequalities, fixed point problems and the generalized set-valued resolvent equations. In 2000, M. A. Noor and Th. M. Rassias [8] have used this equivalent formulation to suggest and analyze some iterative methods, the convergence of these methods requires that the operator is both strongly monotone and Lipschitz continuous. Secondly, it is very difficult to evaluate the resolvent of the operator except for very simple cases. To overcome this disadvantage, in [12] Noor M. A. and Noor K. I.

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have used these alternative equivalent formulations to suggest and analyze modified resolvent iterative method for general mixed quasi variational inequalities, where the skew-symmetry of the nonlinear bifunction plays a crucial part in the convergence analysis of this method. In this paper, we give some important properties of the resolvent operator.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot,\cdot\rangle$ respectively. Let K be a closed convex set in H and $T : H \to H$ be a nonlinear operator. Let $\varphi(.,.) : H\times H \longrightarrow R\cup\{+\infty\}$ be a continuous bifunction. We consider the problem of finding $u^*\in H$ such that

$$\langle Tu^*, g(v)-g(u^*)\rangle + \varphi(g(v),g(u^*)) - \varphi(g(u^*),g(u^*)) \geqslant 0, \quad \forall v\in H. \tag{2.1}$$

Problem (2.1) is called the general mixed quasi variational inequality .

For $\varphi(v,u^*) = \varphi(v), \forall u^*\in H,$ problem (2.1) reduces to finding $u^*\in H$ such that

$$\langle Tu^*, g(v)-g(u^*)\rangle + \varphi(g(v)) - \varphi(g(u^*)) \geqslant 0, \quad \forall v\in H, \tag{2.2}$$

which is known as the general mixed variational inequality, see Noor [9].

If $\varphi(.,.) = \varphi(.)$ is an indicator function of a closed convex set K in $H,$ then the problem (2.1) is equivalent to finding $u^*\in H$ such that $g(u^*)\in K$ and

$$\langle T(u^*), g(v)-g(u^*)\rangle \geqslant 0, \qquad \forall g(v)\in K. \tag{2.3}$$

Problem (2.3) is called the general variational inequality, which first introduced and studied by Noor [5] in 1988. For the applications, formulation and numerical methods of general variational inequalities (2.3), we refer the reader to the survey [11].

If $g\equiv I,$ then the problem (2.3) is equivalent to finding $u^*\in K$ such that

$$\langle Tu^*, v-u^*\rangle \geqslant 0, \quad \forall v\in K, \tag{2.4}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [13] in 1964.

We also need the following well known results and concepts.

DEFINITION 2.1. The bifunction $\varphi(.,.)$ is said to be *skew-symmetric,* if,

$$\varphi(u,u)-\varphi(u,v)-\varphi(v,u)+\varphi(v,v)\geqslant 0, \quad \forall u,v\in H. \tag{2.5}$$

DEFINITION 2.2 [1]. Let A be a maximal monotone operator, then the resolvent operator associated with A is defined as

$$J_A(u) = (I+\rho A)^{-1}(u), \quad \forall u\in H,$$

where $\rho>0$ is a constant and I is the identity operator.

REMARK 2.1. It is well known that the subdifferential $\partial\varphi(.,.)$ of a convex , proper and lower-semicontinuous function $\varphi(.,.) : H \times H \longrightarrow R \cup \{+\infty\}$ is a maximal monotone with respect to the first argument, we can define its resolvent by

$$J_{\varphi(u)} = (I + \rho\partial\varphi(.,u))^{-1}) \equiv (I + \rho\partial\varphi(u))^{-1}, \tag{2.6}$$

where $\partial\varphi(u) \equiv \partial\varphi(.,u)$.

The resolvent operator $J_{\varphi(u)}$ defined by (2.6) has the following characterization,

LEMMA 2.1. ([10]) *For a given* $u \in H$, $z \in H$ *satisfies the inequality*

$$\langle u - z, v - u\rangle + \rho\varphi(v,u) - \rho\varphi(u,u) \geqslant 0, \quad \forall v \in H, \tag{2.7}$$

if and only if

$$u = J_{\varphi(u)}[z],$$

where $J_{\varphi(u)}$ *is resolvent operator defined by* (2.6).

It follows from Lemma 2.1 that

$$\langle J_{\varphi(u)}[z]-z, v-J_{\varphi(u)}[z]\rangle + \rho\varphi(v, J_{\varphi(u)}[z]) - \rho\varphi(J_{\varphi(u)}[z], J_{\varphi(u)}[z]) \geqslant 0, \quad \forall u,v,z \in H \tag{2.8}$$

The following result can be proved by using Lemma 2.1.

LEMMA 2.2. u^* *is solution of problem* (2.1) *if and only if* $u^* \in H$ *satisfies the relation:*

$$g(u^*) = J_{\varphi(u^*)}[g(u^*) - \rho T(u^*)], \tag{2.9}$$

where $\rho > 0$.

From Lemma 2.2, it is clear that u is solution of (2.1) if and only if u is a zero point of the function

$$r(u,\rho) := g(u) - J_{\varphi(u)}[g(u) - \rho T(u)].$$

3. The main Theorem

The following lemma shows that $\|r(u,\rho)\|$ is a non-decreasing function, while $\dfrac{\|r(u,\rho)\|}{\rho}$ is a non-increasing one with respect to ρ .

LEMMA 3.1. *Suppose the bifunction* $\varphi(.,.)$ *is skew-symmetric. Then for all* $u \in H$ *and* $\rho' \geqslant \rho > 0$, *it holds that*

$$\|r(u,\rho')\| \geqslant \|r(u,\rho)\| \tag{3.1}$$

and

$$\frac{\|r(u,\rho')\|}{\rho'} \leqslant \frac{\|r(u,\rho)\|}{\rho}. \tag{3.2}$$

Proof. Let $t := \frac{\|r(x,\rho')\|}{\|r(x,\rho)\|}$, we only need to prove that $1 \leqslant t \leqslant \frac{\rho'}{\rho}$. Note that its equivalent expression is

$$(t-1)(t-\frac{\rho'}{\rho}) \leqslant 0. \tag{3.3}$$

Using inequality (2.8) we have

$$\begin{aligned}\langle\, g(u)-\rho T(u)-J_{\varphi(u)}[g(u)-\rho T(u)], J_{\varphi(u)}[g(u)-\rho T(u)]-J_{\varphi(u)'}[g(u)-\rho' T(u)]\rangle \\ +\rho\varphi(J_{\varphi(u)'}[g(u)-\rho' T(u)], J_{\varphi(u)}[g(u)-\rho T(u)]) \\ -\rho\varphi(J_{\varphi(u)}[g(u)-\rho T(u)], J_{\varphi(u)}[g(u)-\rho T(u)]) \geqslant 0,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\langle\, g(u)-\rho' T(u)-J_{\varphi(u)'}[g(u)-\rho' T(u)], J_{\varphi(u)'}[g(u)-\rho' T(u)]-J_{\varphi(u)}[g(u)-\rho T(u)]\rangle \\ +\rho'\varphi(J_{\varphi(u)}[g(u)-\rho T(u)], J_{\varphi(u)'}[g(u)-\rho' T(u)]) \\ -\rho'\varphi(J_{\varphi(u)'}[g(u)-\rho' T(u)], J_{\varphi(u)'}[g(u)-\rho' T(u)]) \geqslant 0,\end{aligned} \tag{3.5}$$

where $J_{\varphi(u)'} = (I+\rho'\partial\varphi(u))^{-1}$, from (3.4) and using

$$J_{\varphi(u)}[g(u)-\rho T(u)] - J_{\varphi(u)'}[g(u)-\rho' T(u)] = r(u,\rho') - r(u,\rho),$$

we obtain

$$\begin{aligned}\langle\, r(u,\rho), r(u,\rho')-r(u,\rho)\rangle \;\geqslant\; & \rho\langle\, T(u), r(u,\rho')-r(u,\rho)\rangle \\ & -\rho\varphi(J_{\varphi(u)'}[g(u)-\rho' T(u)], J_{\varphi(u)}[g(u)-\rho T(u)]) \\ & +\rho\varphi(J_{\varphi(u)}[g(u)-\rho T(u)], J_{\varphi(u)}[g(u)-\rho T(u)]).\end{aligned} \tag{3.6}$$

Similarly, we have

$$\begin{aligned}\langle\, r(u,\rho'), r(u,\rho)-r(u,\rho')\rangle \;\geqslant\; & \rho'\langle\, T(u), r(u,\rho)-r(u,\rho')\rangle \\ & -\rho'\varphi(J_{\varphi(u)}[g(u)-\rho T(u)], J_{\varphi(u)'}[g(u)-\rho' T(u)]) \\ & +\rho'\varphi(J_{\varphi(u)'}[g(u)-\rho' T(u)], J_{\varphi(u)'}[g(u)-\rho' T(u)]).\end{aligned} \tag{3.7}$$

Multiplying (3.6) and (3.7) by ρ' and ρ respectively, and then adding them, using (2.5) we get

$$\langle\, \rho' r(u,\rho) - \rho r(u,\rho'), r(u,\rho')-r(u,\rho)\rangle \;\geqslant 0 \tag{3.8}$$

and consequently

$$\rho'\|r(u,\rho)\|^2 + \rho\|r(u,\rho')\|^2 \leqslant (\rho+\rho')\langle\, r(u,\rho), r(u,\rho')\rangle\,. \tag{3.9}$$

From Cauchy-Schwarz inequality, we have

$$\langle\, r(u,\rho), r(u,\rho')\rangle \;\leqslant \|r(u,\rho)\|\cdot\|r(u,\rho')\|.$$

Then

$$\rho'\|r(u,\rho)\|^2 + \rho\|r(u,\rho')\|^2 \leqslant (\rho+\rho')\|r(u,\rho)\|\cdot\|r(u,\rho')\|. \tag{3.10}$$

Dividing (3.10) by $\|r(u,\rho)\|^2$ we obtain

$$\rho' + \rho t^2 \leqslant (\rho + \rho')t$$

and thus (3.3) holds and the lemma is proved. □

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