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INEQUALITIES INVOLVING TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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(*communicated by J. Pečarić*)

Abstract. We introduce three different proofs of Priceòs inequality and some new trigonometric and hyperbolic inequalities.

1. Introduction

Although the main objective of Price [2] was not to produce new inequalities, it took him a great deal of efforts to produce his inequality (1) (See below) . We were interested in finding a simpler proof of (1) and we were surprised that we found three different proofs, all of which are simpler and shorter than that of Price. The last proof is strikingly simple. We will introduce all of the three proofs in the order they were discovered. We believe the three proofs are interesting and they involve interesting techniques that may be useful to authors in the future.

Let $a \neq b \geqslant 0$, θ be real numbers, and $n \geqslant 1$ an integer throughout this article.

Equation (1) gives Price's inequality [2]:

$$\frac{a^{2n}+b^{2n}-2a^n b^n \cos(n\theta)}{a^2+b^2-2ab\cos\theta} \leqslant \left(\frac{a^n-b^n}{a-b}\right)^2. \tag{1}$$

Since $-1 \leqslant \cos\theta, \cos n\theta \leqslant 1$, it is easy to see that

$$\left(\frac{a^n-b^n}{a+b}\right)^2 \leqslant \frac{a^{2n}+b^{2n}-2a^n b^n \cos(n\theta)}{a^2+b^2-2ab\cos\theta} \leqslant \left(\frac{a^n+b^n}{a-b}\right)^2.$$

So the inequality (1) is nontrivial. It is also interesting to note that if we replace cosine by sine in (1), the inequality does not hold. For example, let $a=2, b=1, n=2$, $\theta=\pi/2$. Then we have

$$\frac{a^{2n}+b^{2n}-2a^n b^n \sin(n\theta)}{a^2+b^2-2ab\sin\theta} = 17 > 9 = \left(\frac{a^n-b^n}{a-b}\right)^2.$$

The first proof to (1) is by applying elementary inequalities. Some of these inequalities are probably not new, but we will include some new proofs of these inequalities. The

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inequality (3) below was used to prove the Bieberbach conjecture when a_n is real in 1931 (see Hille [1], page 355). The second proof is given by using a complex hyperbolic functions. As a consequence we will introduce new inequalities involving elementary trigonometric and hyperbolic functions. We are pleasantly surprised that these inequalities are new.

2. The first proof of Price's inequality

We start this section by establishing the inequality:

$$\sin(n\theta)\cdot\sin\theta \leqslant 2n(1-\cos\theta). \tag{2}$$

Note that (2) is trivially true if $\cos\theta < 0$. We will use induction. Suppose $n = 1$. Then

$$\text{RHS} - \text{LHS} = 2(1-\cos\theta) - \sin^2\theta = 2(1-\cos\theta) - (1-\cos^2\theta) = (1-\cos\theta)^2 \geqslant 0.$$

Hence, (2) holds when $n = 1$. Suppose n is a positive integer such that (2) is valid. Then

$$\begin{aligned}
\sin[(n+1)\theta]\cdot\sin\theta &= [\sin(n\theta)\cdot\sin\theta]\cos\theta + \cos(n\theta)\cdot\sin^2\theta \\
&\leqslant 2n(1-\cos\theta)\cos\theta + \cos(n\theta)\left(1-\cos^2\theta\right), \\
&\qquad \text{by the inductive hypothesis} \\
&= (1-\cos\theta)[2n\cos\theta + \cos(n\theta)\cdot(1+\cos\theta)] \\
&\leqslant (1-\cos\theta)(2n+2), \quad \text{since} \cos\theta \leqslant 1 \\
&= 2(n+1)(1-\cos\theta).
\end{aligned}$$

This establishes (2).

Next, we use (2) to prove the following:

$$1-\cos(n\theta) \leqslant n^2(1-\cos\theta). \tag{3}$$

We will use induction again. The inequality is trivial when $n = 1$.
Suppose n is a positive integer such that (3) is valid. Then

$$\begin{aligned}
1-\cos[(n+1)\theta] &= 1-\cos(n\theta)\cos\theta + \sin(n\theta)\sin\theta \\
&\leqslant 1-\cos(n\theta)\cos\theta + 2n(1-\cos\theta), \quad \text{by (2)} \\
&= (1-\cos(n\theta))\cos\theta + (1-\cos\theta) + 2n(1-\cos\theta) \\
&\leqslant n^2(1-\cos\theta)\cos\theta + (2n+1)(1-\cos\theta), \\
&\qquad \text{by the inductive hypothesis} \\
&= (1-\cos\theta)\left(n^2\cos\theta + 2n + 1\right) \\
&\leqslant (n+1)^2(1-\cos\theta), \quad \text{since } \cos\theta \leqslant 1.
\end{aligned}$$

This completes the proof of (3).

The next elementary inequality is an application of the arithmetic-geometric mean inequality. Let $x = \frac{a}{b}$. Then

$$\sum_{k=0}^{n} a^k b^{n-k} = b^n \sum_{k=0}^{n} x^k \geqslant b^n (n+1) \left(x^0 x^1 \cdots x^n\right)^{\frac{1}{n+1}}$$
$$= (n+1) b^n \left(x^{\frac{n(n+1)}{2}}\right)^{\frac{1}{n+1}} = (n+1) a^{\frac{n}{2}} b^{\frac{n}{2}}.$$

So we have

$$\sum_{k=0}^{n} a^k b^{n-k} \geqslant (n+1) a^{n/2} b^{n/2}. \tag{4}$$

Now, we are ready to prove (1). If $\theta = 2k\pi$ for some integer k, then (1) is trivial. So we assume that $\theta \neq 2k\pi$ for any integer k. Let us denote $\frac{1-\cos(n\theta)}{1-\cos\theta} = f$. By (3), we have $f \leqslant n^2$. Also we notice that

$$\begin{aligned}
&(a^n - b^n)^2 \left(a^2 + b^2 - 2ab\cos\theta\right) - (a-b)^2 \left(a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)\right) \\
&= (a^n - b^n)^2 (a^2 + b^2 - 2ab\cos\theta) \\
&\qquad - (a^n - b^n)^2 (a^2 + b^2 - 2ab) - 2a^n b^n (a-b)^2 (1 - \cos(n\theta)) \\
&= 2ab(1-\cos\theta)\left[(a^n - b^n)^2 - f\, a^{n-1} b^{n-1} (a-b)^2\right] \\
&\geqslant 2ab(1-\cos\theta)\left[(a^n - b^n)^2 - n^2 a^{n-1} b^{n-1} (a-b)^2\right], \quad \text{by (3)} \\
&= 2ab(1-\cos\theta)(a-b)^2 \left[\left(\sum_{k=0}^{n-1} a^k b^{(n-1)-k}\right)^2 - n^2 a^{n-1} b^{n-1}\right] \\
&\geqslant 2ab(1-\cos\theta)(a-b)^2 \left[\left(n a^{\frac{n-1}{2}} b^{\frac{n-1}{2}}\right)^2 - n^2 a^{n-1} b^{n-1}\right], \quad \text{by (4)} \\
&= 0.
\end{aligned}$$

This proves the Price's Inequality.

3. The second proof of Price's inequality

Our approach to the second proof is to use complex numbers. This can be achieved as a consequence of Theorem 1 below. This theorem enables us to prove (1) as well as to give us other new inequalities. First, note that if $|z| \neq 1$ is a complex number, then

$$\left|\frac{z^n - 1}{z - 1}\right| = \left|\sum_{k=0}^{n-1} z^k\right| \leqslant \sum_{k=0}^{n-1} |z|^k = \frac{|z|^n - 1}{|z| - 1}.$$

So

$$\left|\frac{z^n - 1}{z - 1}\right| \leqslant \frac{|z|^n - 1}{|z| - 1}. \tag{5}$$

THEOREM 1. *Let $z = x + iy$ be a complex number with $x \neq 0$. If n is a positive integer, then we have*

$$\left|\frac{\sinh(nz)}{\sinh z}\right| \leqslant \frac{\sinh(nx)}{\sinh x}. \tag{6}$$

Proof.

$$\text{LHS} = \left|\frac{e^{nz} - e^{-nz}}{e^{z} - e^{-z}}\right| = \left|\frac{e^{-nz}\left(e^{2nz} - 1\right)}{e^{-z}\left(e^{2z} - 1\right)}\right| = \frac{e^{-nx}}{e^{-x}}\left|\frac{e^{2nz} - 1}{e^{2z} - 1}\right|.$$

Let $Z = e^{2z}$. Then by (5), we have

$$\begin{aligned}\text{LHS} = e^{-(n-1)x}\left|\frac{Z^n - 1}{Z - 1}\right| &\leqslant e^{-(n-1)x}\frac{|Z|^n - 1}{|Z| - 1} \\ &\leqslant e^{-(n-1)x}\left(\frac{e^{2nx} - 1}{e^{2x} - 1}\right) = \frac{\sinh(nx)}{\sinh x}.\end{aligned}$$

In order to simplify the second proof of Price's Inequality and the subsequent material, we list some properties of hyperbolic functions. Again, let $z = x + iy$. Then

$$\begin{aligned}|\sinh z|^2 &= (\sinh z)(\sinh \overline{z}) \\ &= \frac{1}{4}\left[\left(e^{z+\overline{z}} + e^{-(z+\overline{z})}\right) - \left(e^{z-\overline{z}} + e^{-(z-\overline{z})}\right)\right] \\ &= \frac{1}{4}\left[\left(e^{2x} + e^{-2x}\right) - \left(e^{i2y} + e^{-i2y}\right)\right] \\ &= \frac{1}{2}\left[\cosh(2x) - \cosh(2iy)\right] \\ &= \frac{1}{2}\left[\cosh(2x) - \cos(2y)\right].\end{aligned} \tag{a}$$

Similarly, we have

$$|\cosh z|^2 = \frac{1}{2}\left[\cosh(2x) + \cos(2y)\right]. \tag{b}$$

Now, we are ready to give a new proof to the Price Inequality. Let LHS be the left hand side of the inequality (1) in the statement of the Price Inequality. Let $c = \frac{a}{b}$ and $z = \frac{\ln c + i\theta}{2}$. Then by (a), we have

$$\begin{aligned}|\sinh nz|^2 &= \frac{\cosh(n\ln c) - \cos(n\theta)}{2} \\ &= \frac{c^n + c^{-n} - 2\cos(n\theta)}{4} \\ &= \frac{1}{4}(ab)^{-n}\left(a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)\right).\end{aligned} \tag{c}$$

So we have

$$\text{LHS} = (ab)^{n-1}\left|\frac{\sinh(nz)}{\sinh z}\right|^2.$$

Using Theorem 1 and (c) and by noting that $\cos 0 = 1$, we have

$$\text{LHS} \leqslant (ab)^{n-1}\left[\frac{\sinh\left(\frac{n\ln c}{2}\right)}{\sinh\left(\frac{\ln c}{2}\right)}\right]^2 = (ab)^{n-1}\frac{(ab)^{-n}\left(a^{2n}+b^{2n}-2a^nb^n\right)}{(ab)^{-1}\left(a^2+b^2-2ab\right)} = \left(\frac{a^n-b^n}{a-b}\right)^2.$$

Next we introduce a short proof of (1).

4. The third proof of Price's inequality

It is not difficult to see that (1) in fact is a simple application of (5). If we square both sides of (5) and replace z by $(b/a)\,e^{i\theta}$ in the resulting inequality we obtain the desired Price's inequality (1). Notice that

$$\begin{aligned} |z^n-1|^2 &= |z^n|^2+1-2Re(z^n) \\ &= |(b/a)^n e^{in}\theta \quad|^2+1-2(b/a)^n\cos(n\theta) \\ &= (b^{2n}/a^{2n})+1-2(b/a)^n\cos(n\theta), \end{aligned}$$

which implies Price's inequality.

5. Similar inequalities

As applications of Theorem 1, we point out the following two consequences:

Let $z = x+iy$ be a complex number. Then

$$\left|\frac{\sin(inz)}{\sin(iz)}\right| \leqslant \frac{\sin(inx)}{\sin(ix)} \text{ if } x \neq 0, \text{ and } \left|\frac{\sin(nz)}{\sin(z)}\right| \leqslant \frac{\sin(ny)}{\sin(y)} \text{ if } y \neq 0. \tag{7}$$

In order to see this, we note that $\sinh z = -i\sin(iz)$. Then we obtain the first inequality from Theorem 1. The second is obtained from the first by replacing z by iz.

An alternate way to express Theorem 1 using (a) is the following:

Let $x \neq 0$ and y be real numbers. Then

$$\frac{\cosh(2nx)-\cos(2ny)}{\cosh(2x)-\cos(2y)} \leqslant \left(\frac{\sinh(nx)}{\sinh x}\right)^2. \tag{8}$$

The equality holds when $y = k\pi$ for any integer k.

As a result of discovering Theorem 1, we obtain several interesting inequalities not only similar to Theorem 1 but also similar to Price's Inequality.

THEOREM 2. *Let $z = x+iy$ be a complex number with $x \neq k\pi$ for any integer k. If n is a positive integer, then*

$$\left|\frac{\tanh(nz)}{\tanh z}\right| \leqslant |\coth x|. \tag{9}$$

Proof. First, note that

$$\left|\frac{\cosh z}{\cosh(nz)}\right| = \left|\frac{e^{-z}}{e^{-nz}}\right|\left|\frac{e^{2z}+1}{e^{2nz}+1}\right| \leqslant \frac{e^{-x}}{e^{-nx}}\left(\frac{e^{2x}+1}{|e^{2nx}-1|}\right) = \frac{\cosh x}{|\sinh(nx)|}.$$

So by Theorem 1 and above,

$$\text{LHS} = \left|\frac{\sinh(nz)}{\sinh z}\right|\left|\frac{\cosh z}{\cosh(nz)}\right| \leqslant \left(\frac{\sinh(nx)}{\sinh x}\right)\left(\frac{\cosh x}{|\sinh(nx)|}\right) = |\coth x|.$$

Let $z = x + iy$ be a complex number. Then, as in inequality (7), we have

$$\left|\frac{\tan(inz)}{\tan(iz)}\right| \leqslant |\coth x| \quad \text{when } x \neq 0, \quad \text{and} \quad \left|\frac{\tan(nz)}{\tan z}\right| \leqslant |\coth y| \quad \text{when } y \neq 0. \tag{10}$$

In order to obtain an inequality similar to Price's Inequality, we let θ and $a \neq b > 0$ be real numbers. Let $c = \frac{a}{b}$ and $z = \frac{\ln c + i\theta}{2}$. Then as in (c), we have

$$\begin{aligned} |\cosh nz|^2 &= \frac{\cosh(n\ln c) + \cos(n\theta)}{2} \quad \text{by } (b) \\ &= \frac{c^n + c^{-n} + 2\cos(n\theta)}{4} \qquad (d) \\ &= \frac{1}{4}(ab)^{-n}\left(a^{2n} + b^{2n} + 2a^n b^n \cos(n\theta)\right). \end{aligned}$$

so by (c) and (d), we have

$$|\tanh(nz)|^2 = \frac{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)}{a^{2n} + b^{2n} + 2a^n b^n \cos(n\theta)}.$$

Note that $\coth^2((\ln c)/2) = \left(\frac{c+1}{c-1}\right)^2 = \left(\frac{a+b}{a-b}\right)^2$. Using inequality in Theorem 2, we have

$$\frac{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)}{a^{2n} + b^{2n} + 2a^n b^n \cos(n\theta)} \leqslant \left(\frac{a+b}{a-b}\right)^2 \left[\frac{a^2 + b^2 - 2ab\cos\theta}{a^2 + b^2 + 2ab\cos\theta}\right]. \tag{11}$$

If z=x+iy,it is known that $|\sinh x| \leqslant |\cosh z| \leqslant \cosh x$ and $|\sinh x| \leqslant |\sinh z| \leqslant \cosh x$. However, if $z = x + iy$ with $x \neq 0$, then

$$\tanh x \leqslant |\coth z| \leqslant \coth x. \tag{12}$$

In order to see this, we note that

$$|\coth z| = \left|\frac{e^z + e^{-z}}{e^z - e^{-z}}\right| = \left|\frac{e^{2z}+1}{e^{2z}-1}\right| \geqslant \frac{|e^{2z}| - 1}{|e^{2z}| + 1} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x,$$

and

$$|\coth z| = \left|\frac{e^z + e^{-z}}{e^z - e^{-z}}\right| = \left|\frac{e^{2z}+1}{e^{2z}-1}\right| \leqslant \frac{|e^{2z}| + 1}{|e^{2z}| - 1} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \coth x.$$

As an application of these inequalities, we have

$$\frac{\tanh nx}{\coth x} \leqslant \left|\frac{\coth nz}{\coth z}\right| \leqslant \frac{\coth nx}{\tanh x}, \tag{13}$$

$$\frac{|\sinh nx|}{\cosh x} \leqslant \frac{|\cosh nz|}{|\cosh z|} \leqslant \frac{\cosh nx}{|\sinh x|}, \tag{14}$$

and

$$\frac{|\sinh nx|}{\cosh x} \leqslant \left|\frac{\cosh nz}{\cosh z}\right| \leqslant \frac{\cosh nx}{|\sinh x|}, \tag{15}$$

and

$$\frac{|\sinh nx|}{\cosh x} \leqslant \left|\frac{\sinh nz}{\sinh z}\right| \leqslant \frac{\cosh nx}{|\sinh x|}. \tag{16}$$

As before, if we let $c = \frac{a}{b}$ and $z = \frac{\ln c + i\theta}{2}$, we have

$$|\coth(nz)|^2 = \frac{a^{2n} + b^{2n} + 2a^n b^n \cos(n\theta)}{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)},$$

$$[\tanh((n\ln c)/2)]^2 = (\frac{a^n - b^n}{a^n + b^n})^2 \; [\coth((n\ln c)/2)]^2 = \left(\frac{a^n + b^n}{a^n - b^n}\right)^2.$$

So (12) yields only the following trivial inequality:

$$\left(\frac{a-b}{a+b}\right)^2 \leqslant \frac{a^2 + b^2 + 2ab\cos\theta}{a^2 + b^2 - 2ab\cos\theta} \leqslant \left(\frac{a+b}{a-b}\right)^2.$$

However, if we combine $\left(\frac{a^n+b^n}{a^n-b^n}\right)^2 \leqslant \left(\frac{a+b}{a-b}\right)^2$ and (13), we have that

$$\left(\frac{a-b}{a+b}\right)^4 \frac{a^2 + b^2 + 2ab\cos\theta}{a^2 + b^2 - 2ab\cos\theta} \leqslant \frac{a^{2n} + b^{2n} + 2a^n b^n \cos(n\theta)}{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)}$$
$$\leqslant \left(\frac{a+b}{a-b}\right)^4 \frac{a^2 + b^2 + 2ab\cos\theta}{a^2 + b^2 - 2ab\cos\theta}$$

So by applying (16), we have the following non-trivial inequality:

$$\left(\frac{a-b}{a+b}\right)^6 \leqslant \frac{a^{2n} + b^{2n} + 2a^n b^n \cos(n\theta)}{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)} \leqslant \left(\frac{a+b}{a-b}\right)^6 \tag{17}$$

for any positive integer n.

Unfortunately, (14) and (15) yield uninteresting inequalities as follow:

$$\left(\frac{a^n - b^n}{a+b}\right)^2 \leqslant \frac{a^{2n} + b^{2n} + 2a^n b^n \cos(n\theta)}{a^{2n} + b^{2n} + 2ab\cos(n\theta)} \leqslant \left(\frac{a^n + b^n}{a-b}\right)^2,$$

and

$$\left(\frac{a^n - b^n}{a+b}\right)^2 \leqslant \frac{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)}{a^{2n} + b^{2n} - 2ab\cos(n\theta)} \leqslant \left(\frac{a^n + b^n}{a-b}\right)^2,$$

respectively.

If we replace the trigonometric function cosine by sine in the Price's Inequality (1), the inequality does not hold as we mentioned at the beginning. But the following corollary is a direct consequence of (1).

COROLLARY 1. *Let θ and $a \neq b > 0$ be real numbers. If n is a positive integer of the form $n = 4m+1$ for some integer m, then we have*

$$\frac{a^{2n}+b^{2n}-2a^n b^n \sin(n\theta)}{a^2+b^2-2ab\sin\theta} \leqslant \left(\frac{a^n-b^n}{a-b}\right)^2. \tag{18}$$

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