Mathematical
Inequalities
& Applications
Volume 10, Number 2 (2007), 251–259

WEIGHTED INTEGRAL INEQUALITIES OF POINCARÉ TYPE

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(*communicated by H. M. Srivastava*)

Abstract. In this paper, we give some weighted integral inequalities which generalize the well-known Poincaré inequality. Our results can be used to generate other integral inequalities.

1. Introduction

The multidimensional integral inequality in [1, 4]

$$\lambda_0 \int_\Omega u^2 \,\mathrm{d}x \leqslant \int_\Omega |\nabla u|^2 \,\mathrm{d}x, \tag{1.1}$$

where Ω is a bounded region in $\mathbb{R}^2$ or $\mathbb{R}^3$, $u \in C^1(\Omega)$, $u = 0$ on $\partial\Omega$ and λ_0 is the smallest eigenvalue of the problem

$$\begin{cases} \Delta u + \lambda u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

is known as the Poincaré's inequality. Many generalized results have recently been found, such as the following theorem in [5].

THEOREM 1.1. *For any $f_\alpha \in C_0^1(\Omega)$ and any real numbers $p_\alpha \geqslant 2$ satisfying: $\sum_\alpha \frac{1}{p_\alpha} = 1$. Then*

$$\int_\Omega \prod_\alpha |f_\alpha| \,\mathrm{d}x \leqslant \frac{1}{n} \sum_\alpha \frac{1}{p_\alpha} \left(\frac{M}{2}\right)^{p_\alpha} \int_\Omega |\nabla f_\alpha|^{p_\alpha} \,\mathrm{d}x. \tag{1.3}$$

It is the purpose of this paper to obtain some weighted inequalities of (1.1).

Throughout this paper we always assume that $\Omega = \prod_{i=1}^n [a_i, b_i] \subset \mathbb{R}^n$ is a field rectangular region, $m \geqslant 2$ and $n \geqslant 2$ is any fixed integers and $C_0^1(\Omega)$ is the collection of all real-valued continuously differentiable functions on Ω which vanish on the boundary $\partial\Omega$ of Ω. The $n-$dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is denoted by $|E|$. We call w a weight if $w \in L^1_{loc}(\mathbb{R}^n)$ and $w > 0\, a.e.$.

Mathematics subject classification (2000): 26D10, 26D15, 39B72.
Key words and phrases: differential forms, A-harmonic tensors, Weighted integral inequalities.

2. Main results

DEFINITION 2.1. We say that the weight $w(x) > 0$ satisfies the $A_r^\lambda-$ condition in Ω, $r > 1$ and $0 < \lambda < \infty$, or that w is an A_r^λ – weight in Ω, write $w \in A_r^\lambda(\Omega)$, if

$$\sup_B \left(\frac{1}{|B|}\int_B w\,\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{\frac{1}{r-1}}\mathrm{d}x\right)^{\lambda(r-1)} < \infty \tag{2.1}$$

for any ball or any cube $B \subset \mathbb{R}^n$.

The following generalized Hölder's inequality will be used repeatedly.

LEMMA 2.1. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$, if f and g are measurable functions on $\mathbb{R}^n$, then*

$$\|f\,g\|_{s,\Omega} \leqslant \|f\|_{\alpha,\Omega}\,\|g\|_{\beta,\Omega} \tag{2.2}$$

for any $\Omega \subset \mathbb{R}^n$.

We also need the following lemmas:

LEMMA 2.2. [6] *If $w \in A_r^1$, $r > 1$, then there exist constants $\gamma > 1$ and C independent of w, such that*

$$\|w\|_{\gamma,Q} \leqslant C|Q|^{(1-\gamma)/\gamma}\|w\|_{1,Q} \tag{2.3}$$

for any ball or any cube $Q \subset \mathbb{R}^n$.

LEMMA 2.3. [4] *Let $r_i \geqslant 0$ and $s > 0$, then*

$$\left(\sum_i r_i\right)^s \leqslant C(s,n)\sum_i r_i^s, \tag{2.4}$$

where

$$C(s,n) = \begin{cases} n^{s-1}, & \text{if } s > 1, \\ 1, & \text{if } 0 \leqslant s \leqslant 1. \end{cases}$$

LEMMA 2.4. [5] *Let $f_\alpha \in C_0^1(\Omega)$ and q_α be any real positive numbers with $q := \sum\limits_\alpha q_\alpha \geqslant 2$, then*

$$\int_\Omega \prod_\alpha |f_\alpha|^{q_\alpha} \leqslant \frac{1}{n}\left(\frac{M}{2}\right)^q \sum_\alpha \frac{q_\alpha}{q}\int_\Omega |\nabla f_\alpha|^q. \tag{2.5}$$

LEMMA 2.5. [7] *Let $q_\alpha > 0$ and $c_\alpha > 0$, then*

$$\prod_\alpha c_\alpha^{q_\alpha} \leqslant \frac{1}{q}\sum_\alpha q_\alpha c_\alpha^q, \tag{2.6}$$

where $q := \sum\limits_\beta q_\beta$.

THEOREM 2.6. *Let $f_\alpha \in C_0^1(\Omega)$ $\alpha = 1,\dots,m$. There exists a constant $\gamma > 1$, such that if $w \in A_r^1 \cap A_{nk}^{\frac{1}{n}}$ for some $r > 1$ and k with $\frac{\gamma-1}{\gamma} > k > \frac{1}{n}$, then*

$$\left(\frac{1}{|\Omega|}\int_\Omega \prod_\alpha |f_\alpha| w \,\mathrm{d}x\right) \leqslant C\,(mn)^{(1-\gamma)/\gamma}\left(\frac{M}{2}\right)^m \sum_\alpha \left(\frac{1}{|\Omega|}\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{1/n}, \tag{2.7}$$

where C is a constant independent of f_α and $M = \max\{b_i - a_i, i = 1,\dots,n\}$.

Proof. Since $w \in A_r^1$ for some $r > 1$, by using lemma 2.2, there exist constants $\gamma > 1$ and $C_1 > 0$, such that

$$\|w\|_{\gamma,\Omega} \leqslant C_1 |\Omega|^{(1-\gamma)/\gamma} \|w\|_{1,\Omega}. \tag{2.8}$$

Let $s > 1$, and $t = \frac{s\gamma}{\gamma-1}$. Then $1 < s < t$ and $\frac{s}{t} + \frac{t-s}{t} = 1$. By using lemma 2.1 and lemma 2.4, we have

$$\begin{aligned}
\int_\Omega \prod_\alpha |f_\alpha| w\,\mathrm{d}x &\leqslant \left(\int_\Omega \prod_\alpha |f_\alpha|^{\frac{t}{s}}\,\mathrm{d}x\right)^{\frac{s}{t}} \left(\int_\Omega w^{\frac{t}{t-s}}\,\mathrm{d}x\right)^{\frac{t-s}{t}} \\
&\leqslant C_1 |\Omega|^{\frac{1-\gamma}{\gamma}} \|w\|_{1,\Omega} \left(\int_\Omega \prod_\alpha |f_\alpha|^{\frac{t}{s}}\,\mathrm{d}x\right)^{\frac{s}{t}} \\
&\leqslant C_1 |\Omega|^{\frac{1-\gamma}{\gamma}} \|w\|_{1,\Omega} \left(\frac{1}{n}\left(\frac{M}{2}\right)^{\frac{tm}{s}} \sum_\alpha \frac{1}{m}\int_\Omega |\nabla f_\alpha|^{\frac{tm}{s}}\,\mathrm{d}x\right)^{\frac{s}{t}} \\
&= C_1 |\Omega|^{\frac{1-\gamma}{\gamma}} \left(\frac{1}{mn}\right)^{\frac{s}{t}} \left(\frac{M}{2}\right)^m \|w\|_{1,\Omega} \left(\sum_\alpha \int_\Omega |\nabla f_\alpha|^{\frac{tm}{s}}\,\mathrm{d}x\right)^{\frac{s}{t}}.
\end{aligned} \tag{2.9}$$

Since $\frac{s}{t} < 1$ we have $C\left(\frac{s}{t}, m\right) = 1$. Note that $ns - t = s\left(n - \frac{\gamma}{\gamma-1}\right) > 0$ and $\frac{s}{t} = \frac{1}{n} + \frac{ns-t}{nt}$, by using Hölder's inequality and lemma 2.3, we have

$$\begin{aligned}
\left(\sum_\alpha \int_\Omega \left(|\nabla f_\alpha|^m\right)^{\frac{t}{s}}\,\mathrm{d}x\right)^{\frac{s}{t}} &\leqslant C\left(\frac{s}{t}, m\right) \sum_\alpha \left(\int_\Omega \left(|\nabla f_\alpha|^m\right)^{\frac{t}{s}}\,\mathrm{d}x\right)^{\frac{s}{t}} \\
&= \sum_\alpha \left(\int_\Omega \left(|\nabla f_\alpha|^m\right)^{\frac{t}{s}}\,\mathrm{d}x\right)^{\frac{s}{t}} \\
&= \sum_\alpha \left(\int_\Omega \left(|\nabla f_\alpha|^m w^{\frac{1}{n}} w^{-\frac{1}{n}}\right)^{\frac{t}{s}}\,\mathrm{d}x\right)^{\frac{s}{t}} \\
&\leqslant \sum_\alpha \left(\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}} \left(\int_\Omega \left(\frac{1}{w}\right)^{\frac{t}{ns-t}}\,\mathrm{d}x\right)^{\frac{ns-t}{nt}}.
\end{aligned} \tag{2.10}$$

Since $1 < nk < n(\gamma-1)/\gamma = ns/t$, by using theorem 2.4 in [8], we know that $w \in A_{nk}^{\frac{1}{n}} \subset A_{ns/t}^{1/n}$. Therefore

$$\begin{aligned} &\left(\int_\Omega w\,\mathrm{d}x\right)\left(\int_\Omega \left(\frac{1}{w}\right)^{\frac{t}{ns-t}}\,\mathrm{d}x\right)^{\frac{ns-t}{nt}} \\ &\quad= \left(\frac{1}{|\Omega|}\int_\Omega w\,\mathrm{d}x\right)\left(\frac{1}{|\Omega|}\int_\Omega \left(\frac{1}{w}\right)^{\frac{t}{ns-t}}\,\mathrm{d}x\right)^{\frac{ns-t}{nt}} |\Omega|^{1+\frac{\gamma-1}{\gamma}-\frac{1}{n}} \\ &\quad\leqslant C_2|\Omega|^{1+\frac{\gamma-1}{\gamma}-\frac{1}{n}} \end{aligned} \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9) implies that

$$\int_\Omega \prod_\alpha |f_\alpha| w\,\mathrm{d}x \leqslant C\left(\frac{1}{mn}\right)^{\frac{\gamma-1}{\gamma}}\left(\frac{M}{2}\right)^m |\Omega|^{1-\frac{1}{n}} \sum_\alpha \left(\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}}. \tag{2.12}$$

That is

$$\left(\frac{1}{|\Omega|}\int_\Omega \prod_\alpha |f_\alpha| w\,\mathrm{d}x\right) \leqslant C(mn)^{\frac{1-\gamma}{\gamma}}\left(\frac{M}{2}\right)^m \sum_\alpha \left(\frac{1}{|\Omega|}\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}}. \tag{2.13}$$

The proof of theorem 2.6 is completed. □

We now prove other weighted inequalities.

THEOREM 2.7. *Let $f_\alpha \in C_0^1(\Omega)$ $\alpha = 1,\ldots,m$. There exists a constant $\gamma > 1$ such that if $w \in A_r^1 \cap A_{\frac{n}{s}-1}^{\frac{s}{n}}$ for some $r > 1$ and $\gamma = \frac{n}{s} > 2$, then*

$$\left(\frac{1}{|\Omega|}\int_\Omega \prod_\alpha |f_\alpha|^s w\,\mathrm{d}x\right)^{\frac{1}{s}} \leqslant C(mn)^{\frac{s-n}{ns}}\left(\frac{M}{2}\right)^m \sum_\alpha \left(\frac{1}{|\Omega|}\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}}, \tag{2.14}$$

where $M = \max\{b_i - a_i, i = 1,\ldots,n\}$.

Proof. Since $w \in A_r^1$ for some $r > 1$, by using lemma 2.2, there exist constant $\gamma > 1$ and $C_1 > 0$ such that

$$\|w\|_{\gamma,\Omega} \leqslant C_1|\Omega|^{\frac{1-\gamma}{\gamma}}\|w\|_{1,\Omega}. \tag{2.15}$$

Note that $\frac{1}{s} = \frac{1}{n} + \frac{n-s}{ns}$ and $0 \leqslant \frac{n-s}{ns} < 1$, then $C\left(\frac{n-s}{ns}, m\right) = 1$. By using lemma 2.2, 2.3 and 2.4, we have

$$\begin{aligned}
&\left(\int_\Omega \prod_\alpha |f_\alpha|^s w\,\mathrm{d}x\right)^{\frac{1}{s}} \\
&\quad\leqslant \left(\int_\Omega \left(w^{\frac{1}{s}}\right)^n \mathrm{d}x\right)^{\frac{1}{n}} \left(\int_\Omega \prod_\alpha |f_\alpha|^{\frac{ns}{n-s}}\,\mathrm{d}x\right)^{\frac{n-s}{ns}} \\
&\quad\leqslant C_1|\Omega|^{\frac{1-\gamma}{s\gamma}}\|w\|_{1,\Omega}^{1/s}\left(\frac{1}{n}\left(\frac{M}{2}\right)^{\frac{nsm}{n-s}}\sum_\alpha \frac{1}{m}\int_\Omega |\nabla f_\alpha|^{\frac{nsm}{n-s}}\,\mathrm{d}x\right)^{\frac{n-s}{ns}} \qquad (2.16)\\
&\quad= C_1|\Omega|^{\frac{1-\gamma}{s\gamma}}\|w\|_{1,\Omega}^{1/s}(mn)^{\frac{s-n}{ns}}\left(\frac{M}{2}\right)^m\left(\sum_\alpha \int_\Omega |\nabla f_\alpha|^{\frac{nsm}{n-s}}\,\mathrm{d}x\right)^{\frac{n-s}{ns}} \\
&\quad\leqslant C_1|\Omega|^{\frac{1-\gamma}{s\gamma}}\|w\|_{1,\Omega}^{1/s}(mn)^{\frac{s-n}{ns}}\left(\frac{M}{2}\right)^m\sum_\alpha\left(\int_\Omega |\nabla f_\alpha|^{\frac{nsm}{n-s}}\,\mathrm{d}x\right)^{\frac{n-s}{ns}}.
\end{aligned}$$

Since $\frac{n}{s} > 2$, then $n-2s>0$ and $\frac{n-s}{ns} = \frac{1}{n} + \frac{n-2s}{ns}$. By using Hölder inequality, we have

$$\begin{aligned}
\left(\int_\Omega (|\nabla f_\alpha|^m)^{\frac{ns}{n-s}}\,\mathrm{d}x\right)^{\frac{n-s}{ns}} &= \left(\int_\Omega \left(|\nabla f_\alpha|^m w^{\frac{1}{n}} w^{\frac{-1}{n}}\right)^{\frac{ns}{n-s}}\,\mathrm{d}x\right)^{\frac{n-s}{ns}} \\
&\leqslant \left(\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}}\left(\int_\Omega \left(\frac{1}{w}\right)^{\frac{s}{n-2s}}\,\mathrm{d}x\right)^{\frac{n-2s}{ns}}.
\end{aligned} \qquad (2.17)$$

Note that $w \in A^{\frac{s}{n}}_{\frac{n}{s}-1}$, then

$$\begin{aligned}
&\left(\int_\Omega w\,\mathrm{d}x\right)^{\frac{1}{s}}\left(\int_\Omega \left(\frac{1}{w}\right)^{\frac{s}{n-2s}}\,\mathrm{d}x\right)^{\frac{n-2s}{ns}} \\
&\quad= |\Omega|^{\frac{2}{s}-\frac{2}{n}}\left(\left(\frac{1}{|\Omega|}\int_\Omega w\,\mathrm{d}x\right)\left(\frac{1}{|\Omega|}\int_\Omega\left(\frac{1}{w}\right)^{\frac{s}{n-2s}}\,\mathrm{d}x\right)^{\frac{n-2s}{n}}\right)^{\frac{1}{s}} \qquad (2.18)\\
&\quad\leqslant C_2|\Omega|^{\frac{2}{s}-\frac{2}{n}}.
\end{aligned}$$

Substituting (2.17) and (2.18) into (2.16), we obtain

$$\left(\int_\Omega \prod_\alpha |f_\alpha|^s w\,\mathrm{d}x\right)^{\frac{1}{s}} \leqslant C|\Omega|^{\frac{1}{s}-\frac{1}{n}}(mn)^{\frac{s-n}{ns}}\left(\frac{M}{2}\right)^m\sum_\alpha\left(\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}}. \qquad (2.19)$$

That is

$$\left(\frac{1}{|\Omega|}\int_\Omega \prod_\alpha |f_\alpha|^s w\,\mathrm{d}x\right)^{\frac{1}{s}} \leqslant C(mn)^{\frac{s-n}{ns}}\left(\frac{M}{2}\right)^m\sum_\alpha\left(\frac{1}{|\Omega|}\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}}. \qquad (2.20)$$

We complete the proof of theorem 2.7. $\square$

THEOREM 2.8. *Let* $f_\alpha \in C_0^1(\Omega)$ $\alpha = 1,\dots,m$. *if* $1 < s < \frac{n}{2}$ *and* $w \in A^1_{\frac{n}{s}-1}$, *then there exists a constant* C, *independent of* f_α, *such that*

$$\left(\frac{1}{|\Omega|}\int_\Omega \prod_\alpha |f_\alpha|^s w^{s/n}\,\mathrm{d}x\right)^{\frac{1}{s}} \leqslant C\,(mn)^{\frac{s-n}{ns}}\left(\frac{M}{2}\right)^m \sum_\alpha \left(\frac{1}{|\Omega|}\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}}, \quad (2.21)$$

where $M = \max\{b_i - a_i, i = 1,\dots,n\}$.

Proof. Since $\frac{1}{s} = \frac{1}{n} + \frac{n-s}{ns}$ and $0 < \frac{n-s}{ns} < 1$, then $C\left(\frac{n-s}{ns}, m\right) = 1$. By using lemma 2.1, 2.3 and 2.4, we have

$$\begin{aligned}
&\left(\int_\Omega \prod_\alpha |f_\alpha|^s w^{s/n}\,\mathrm{d}x\right)^{\frac{1}{s}} \\
&\leqslant \left(\int_\Omega \left(w^{\frac{1}{n}}\right)^n \mathrm{d}x\right)^{\frac{1}{n}} \left(\int_\Omega \prod_\alpha |f_\alpha|^{\frac{ns}{n-s}}\,\mathrm{d}x\right)^{\frac{n-s}{ns}} \\
&\leqslant \left(\int_\Omega w\,\mathrm{d}x\right)^{\frac{1}{n}} \left(\frac{1}{n}\left(\frac{M}{2}\right)^{\frac{nsm}{n-s}} \sum_\alpha \frac{1}{m}\int_\Omega |\nabla f_\alpha|^{\frac{nsm}{n-s}}\,\mathrm{d}x\right)^{\frac{n-s}{ns}} \qquad (2.22)\\
&\leqslant \left(\int_\Omega w\,\mathrm{d}x\right)^{\frac{1}{n}} \left(\frac{1}{mn}\right)^{\frac{n-s}{ns}} \left(\frac{M}{2}\right)^m C\left(\frac{n-s}{ns}, m\right) \sum_\alpha \left(\int_\Omega |\nabla f_\alpha|^{\frac{nsm}{n-s}}\,\mathrm{d}x\right)^{\frac{n-s}{ns}} \\
&= \left(\int_\Omega w\,\mathrm{d}x\right)^{\frac{1}{n}} (mn)^{\frac{s-n}{ns}} \left(\frac{M}{2}\right)^m \sum_\alpha \left(\int_\Omega |\nabla f_\alpha|^{\frac{nsm}{n-s}}\,\mathrm{d}x\right)^{\frac{n-s}{ns}}.
\end{aligned}$$

Note that $1 < s < \frac{n}{2}$, we have $n - 2s > 0$ and $\frac{n-s}{ns} = \frac{1}{n} + \frac{n-2s}{ns}$. By using Hölder inequality, we get

$$\begin{aligned}
\sum_\alpha \left(\int_\Omega |\nabla f_\alpha|^{\frac{nsm}{n-s}}\,\mathrm{d}x\right)^{\frac{n-s}{ns}} &= \sum_\alpha \left(\int_\Omega \left(|\nabla f_\alpha|^m w^{\frac{1}{n}} w^{\frac{-1}{n}}\right)^{\frac{ns}{n-s}}\,\mathrm{d}x\right)^{\frac{n-s}{ns}} \qquad (2.23)\\
&\leqslant \sum_\alpha \left(\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}} \left(\int_\Omega \left(\frac{1}{w}\right)^{\frac{s}{n-2s}}\,\mathrm{d}x\right)^{\frac{n-2s}{ns}}.
\end{aligned}$$

Since $w \in A^1_{\frac{n}{s}-1}$, we have

$$\begin{aligned}
&\left(\int_\Omega w\,\mathrm{d}x\right)^{\frac{1}{n}} \left(\int_\Omega \left(\frac{1}{w}\right)^{\frac{s}{n-2s}}\,\mathrm{d}x\right)^{\frac{n-2s}{ns}} \\
&= \left(\left(\frac{1}{|\Omega|}\int_\Omega w\,\mathrm{d}x\right)\left(\frac{1}{|\Omega|}\int_\Omega \left(\frac{1}{w}\right)^{\frac{s}{n-2s}}\,\mathrm{d}x\right)^{\frac{n-2s}{s}}\right)^{\frac{1}{n}} |\Omega|^{\frac{1}{s}-\frac{1}{n}} \qquad (2.24)\\
&\leqslant C|\Omega|^{\frac{1}{s}-\frac{1}{n}}.
\end{aligned}$$

By (2.22), (2.23) and (2.24), we obtain

$$\left(\int_\Omega \prod_\alpha |f_\alpha|^s w^{s/n}\,\mathrm{d}x\right)^{\frac{1}{s}} \leqslant C|\Omega|^{\frac{1}{s}-\frac{1}{n}}\,(mn)^{\frac{s-n}{ns}}\left(\frac{M}{2}\right)^m \sum_\alpha \left(\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}}. \quad (2.25)$$

That is,

$$\left(\frac{1}{|\Omega|}\int_\Omega \prod_\alpha |f_\alpha|^s w^{s/n}\,\mathrm{d}x\right)^{\frac{1}{s}} \leqslant C\,(mn)^{\frac{s-n}{ns}}\left(\frac{M}{2}\right)^m \sum_\alpha \left(\frac{1}{|\Omega|}\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}}. \quad (2.26)$$

We completed the proof of theorem 2.8. □

COROLLARY 2.9. *Let $f \in C_0^1(\Omega)$. There exists a constant $\gamma > 1$ such that if $w \in A_r^1 \cap A_{nk}^{\frac{1}{n}}$ for some $r > 1$ and k with $\frac{\gamma-1}{\gamma} > k > \frac{1}{n}$, then*

$$\left(\frac{1}{|\Omega|}\int_\Omega |f|^m w\,\mathrm{d}x\right) \leqslant C\,m^{\frac{1}{\gamma}}\,n^{\frac{1-\gamma}{\gamma}}\left(\frac{M}{2}\right)^m \left(\frac{1}{|\Omega|}\int_\Omega |\nabla f|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}}, \quad (2.27)$$

where $M = \max\{b_i - a_i, i = 1, \ldots, n\}$.

Proof. This follows from theorem 2.6, by setting $f_\alpha = f$ for all α. □

COROLLARY 2.10. *Let $f_\alpha \in C_0^1(\Omega)$ $\alpha = 1, \ldots, m$. There exists a constant $\gamma > 1$, such that if $w \in A_r^1 \cap A_{nk}^{\frac{1}{n}}$ for some $r > 1$ and k with $\frac{\gamma-1}{\gamma} > k > \frac{1}{n}$, then*

$$\left(\frac{1}{|\Omega|}\int_\Omega \sum_\beta \prod_{\alpha\neq\beta} |f_\alpha||\nabla f_\beta| w^{\frac{n(m-1)+1}{mn}}\,\mathrm{d}x\right) \leqslant C\,m^{\frac{m-m\gamma-1}{m\gamma}}\,n^{\frac{(1-\gamma)(m-1)}{m\gamma}}\left(\frac{M}{2}\right)^{m-1} \sum_\alpha \left(\frac{1}{|\Omega|}\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{1/n},$$

where $M = \max\{b_i - a_i, i = 1, \ldots, n\}$.

Proof. By using Hölder inequality, we have

$$\begin{aligned}
&\int_\Omega \sum_\beta \prod_{\alpha\neq\beta} |f_\alpha||\nabla f_\beta| w^{\frac{n(m-1)+1}{mn}}\,\mathrm{d}x \\
&= \sum_\beta \int_\Omega \prod_{\alpha\neq\beta}\left(|f_\alpha| w^{\frac{1}{m}}\right)|\nabla f_\beta| w^{\frac{1}{mn}}\,\mathrm{d}x \\
&\leqslant \sum_\beta \left(\prod_{\alpha\neq\beta}\left(\int_\Omega \left(|f_\alpha| w^{\frac{1}{m}}\right)^m \mathrm{d}x\right)^{\frac{1}{m}} \left(\int_\Omega |\nabla f_\beta|^m w^{\frac{1}{n}}\,\mathrm{d}x\right)^{\frac{1}{m}}\right) \\
&= \sum_\beta \left(\prod_{\alpha\neq\beta}\left(\int_\Omega |f_\alpha|^m w\,\mathrm{d}x\right)^{\frac{1}{m}} \left(\int_\Omega |\nabla f_\beta|^m w^{\frac{1}{n}}\,\mathrm{d}x\right)^{\frac{1}{m}}\right).
\end{aligned}$$

By using corollary 2.9 and Hölder inequality, we get

$$\left(\int_\Omega |\nabla f_\beta|^m w^{\frac{1}{n}}\,\mathrm{d}x\right)^{\frac{1}{m}} \leqslant |\Omega|^{\frac{1}{m}\left(1-\frac{1}{n}\right)}\left(\int_\Omega |\nabla f_\beta|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{mn}}, \tag{2.28}$$

and

$$\left(\int_\Omega |f_\alpha|^m w\,\mathrm{d}x\right)^{\frac{1}{m}} \leqslant \left(C_1|\Omega|^{1-\frac{1}{n}}(mn)^{\frac{1-\gamma}{\gamma}}\left(\frac{M}{2}\right)^m m\left(\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}}\right)^{\frac{1}{m}}.$$

Therefore, by using lemma 2.5 we have

$$\begin{aligned}
&\int_\Omega \sum_\beta \prod_{\alpha\neq\beta} |f_\alpha||\nabla f_\beta| w^{\frac{n(m-1)+1}{mn}}\,\mathrm{d}x \\
&\quad\leqslant \sum_\beta C|\Omega|^{\frac{1}{m}\left(1-\frac{1}{n}\right)(m-1)}(mn)^{\frac{(1-\gamma)(m-1)}{m\gamma}}\left(\frac{M}{2}\right)^{m-1} m^{\frac{m-1}{m}}|\Omega|^{\frac{1}{m}\left(1-\frac{1}{n}\right)}\times \\
&\qquad\quad\times \prod_\alpha \left(\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{mn}} \\
&\quad= C|\Omega|^{1-\frac{1}{n}}\left(\frac{M}{2}\right)^{m-1} m^{\frac{m-1}{m\gamma}} n^{\frac{(1-\gamma)(m-1)}{m\gamma}} \prod_\alpha \left(\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{mn}} \\
&\quad\leqslant C|\Omega|^{1-\frac{1}{n}}\left(\frac{M}{2}\right)^{m-1} m^{\frac{m-1}{m\gamma}} n^{\frac{(1-\gamma)(m-1)}{m\gamma}} n \sum_\alpha \frac{1}{mn}\left(\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}} \\
&\quad= C|\Omega|^{1-\frac{1}{n}}\left(\frac{M}{2}\right)^{m-1} m^{\frac{m-m\gamma-1}{m\gamma}} n^{\frac{(1-\gamma)(m-1)}{m\gamma}} \sum_\alpha \left(\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{\frac{1}{n}}.
\end{aligned}$$

That is

$$\begin{aligned}
&\left(\frac{1}{|\Omega|}\int_\Omega \sum_\beta \prod_{\alpha\neq\beta} |f_\alpha||\nabla f_\beta| w^{\frac{n(m-1)+1}{mn}}\,\mathrm{d}x\right) \\
&\qquad\leqslant C\, m^{\frac{m-m\gamma-1}{m\gamma}} n^{\frac{(1-\gamma)(m-1)}{m\gamma}}\left(\frac{M}{2}\right)^{m-1}\sum_\alpha\left(\frac{1}{|\Omega|}\int_\Omega |\nabla f_\alpha|^{mn} w\,\mathrm{d}x\right)^{1/n}.
\end{aligned}$$

The proof of corollary 2.10 is completed. □

REMARK 2.2 Further interesting integral inequalities of the weighted Poincaré-type could be obtained from the results above, For instance, by letting $m = 2$ in the corollary 2.10, we have

$$\begin{aligned}
&\left(\frac{1}{|\Omega|}\int_\Omega \left(|f||\nabla g| + |g||\nabla f|\right) w^{\frac{n+1}{2n}}\,\mathrm{d}x\right) \\
&\qquad\leqslant C\left(\frac{M}{2}\right)(2n)^{\frac{1-\gamma}{\gamma}}\left(\frac{1}{|\Omega|}\int_\Omega \left(|\nabla f|^{2n} + |\nabla g|^{2n}\right) w\,\mathrm{d}x\right)^{\frac{1}{n}}.
\end{aligned}$$

and by taking $f = g$ in the last expression, we get

$$\left(\frac{1}{|\Omega|}\int_{\Omega}|f\nabla f|w^{\frac{n+1}{2n}}\,\mathrm{d}x\right) \leqslant C\left(\frac{M}{2}\right)(2n)^{\frac{1-\gamma}{\gamma}}\left(\frac{1}{|\Omega|}\int_{\Omega}|\nabla f|^{2n}w\,\mathrm{d}x\right)^{\frac{1}{n}}. \tag{2.29}$$

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(Received January 13, 2006)

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