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HYERS–ULAM STABILITY OF FIRST ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract. In this paper, we prove the Hyers-Ulam stability of first order linear partial differential equations with constant coefficients

$$au_x(x,y) + bu_y(x,y) + cu(x,y) + d = 0,$$

where $a, b \in \mathbb{R}$ and $c, d \in \mathbb{C}$ are constants with $\Re(c) \neq 0$ and $\Re(c)$ denotes the real part of c.

1. Introduction

In 1940, S. M. Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [13]). Among those was the question concerning the stability of homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In the following year, D. H. Hyers affirmatively answered in his paper [3] the question of Ulam for the case where G_1 and G_2 are Banach spaces. Furthermore, the result of Hyers has been generalized by Th. M. Rassias (ref. [11].) Since then, the stability problems of various functional equations have been investigated by many authors (see [2, 4].)

Assume that X is a normed space over a scalar field $\mathbb{K}$ and that I is an open interval. Let $a_0, a_1, \ldots, a_{n-1}$ be fixed elements of $\mathbb{K}$. Assume that for a fixed function $g : I \to X$ and for any n times differentiable function $y : I \to X$ satisfying the inequality

$$\|y^{(n)}(t) + a_{n-1}\, y^{(n-1)}(t) + \cdots + a_1\, y'(t) + a_0\, y(t) + g(t)\| \leqslant \varepsilon$$

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for all $t \in I$ and for a given $\varepsilon > 0$, there exists a function $y_0 : I \to X$ satisfying

$$y_0^{(n)}(t) + a_{n-1}\, y_0^{(n-1)}(t) + \cdots + a_1\, y_0'(t) + a_0 y_0(t) + g(t) = 0$$

and $\|y(t) - y_0(t)\| \leqslant K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression of ε with $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$. Then, we say that the above differential equation has the Hyers-Ulam stability. For more detailed definitions of the Hyers-Ulam stability, we refer the reader to [2, 4].

C. Alsina and R. Ger were the first authors who investigated the Hyers-Ulam stability of differential equations: They proved in [1] that if a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(t) - y(t)| \leqslant \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t) - f_0(t)| \leqslant 3\varepsilon$ for any $t \in I$.

This result of Alsina and Ger has been generalized by S.-E. Takahasi, T. Miura and S. Miyajima. Indeed, one of their results in [12] is the following theorem concerning the Hyers-Ulam stability of the Banach space valued differential equation $y'(t) = \lambda y(t)$ (see also [7, 8]):

THEOREM 1. *Let λ be a complex number with $\Re(\lambda) \neq 0$ and let I be an interval with $\inf\{e^{-\Re(\lambda)t} : t \in I\} = 0$. If a continuously differentiable function $\varphi : I \to \mathbb{C}$ satisfies the inequality*

$$|\varphi'(t) - \lambda \varphi(t)| \leqslant \varepsilon$$

for all $t \in I$ and some $\varepsilon \geqslant 0$, then there exists a unique complex number α such that

$$\left|\varphi(t) - \alpha e^{\lambda t}\right| \leqslant \frac{\varepsilon}{|\Re(\lambda)|}$$

for any $t \in I$.

For more recent results about this subject, we can refer to [5, 6, 9, 10]. To the best of our knowledge, however, no author investigated the Hyers-Ulam stability of partial differential equations so far.

In this paper, we will investigate the Hyers-Ulam stability of the first order linear partial differential equations with constant coefficients of the form

$$au_x(x,y) + bu_y(x,y) + cu(x,y) + d = 0, \tag{1}$$

where $a, b \in \mathbb{R}$ and $c, d \in \mathbb{C}$ are constants with $\Re(c) \neq 0$ and $\Re(c)$ denotes the real part of c.

2. Main results

In the following theorem, we will prove the Hyers-Ulam stability of first order linear partial differential equations whose coefficients are constants.

THEOREM 2. *Let $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a function which has continuous partial derivatives with respect to the first and second variable. Moreover, assume that u satisfies the following inequality*

$$|au_x(x,y) + bu_y(x,y) + cu(x,y) + d| \leqslant \varepsilon \tag{2}$$

for all $x, y \in \mathbb{R}$ *and for some* $\varepsilon \geqslant 0$, *where* $a, b \in \mathbb{R}$ *and* $c, d \in \mathbb{C}$ *are constants with* $b \neq 0$, $\Re(c) \neq 0$. *Then, there exists a unique function* $\theta : \mathbb{R} \to \mathbb{C}$ *such that*

$$\left| u(x,y) - \theta\Big(x - \frac{a}{b}\,y\Big)\, e^{-(c/b)y} + \frac{d}{c} \right| \leqslant \frac{\varepsilon}{|\Re(c)|} \tag{3}$$

for all $x, y \in \mathbb{R}$. *Furthermore, if* u *satisfies the auxiliary inequality*

$$|u(x,0) - g(x)| \leqslant \delta \tag{4}$$

for any $x \in \mathbb{R}$ *and for some* $\delta \geqslant 0$, *where* $g : \mathbb{R} \to \mathbb{C}$ *is a given function, then the* θ *satisfies*

$$\left| \theta(x) - g(x) - \frac{d}{c} \right| \leqslant \frac{\varepsilon}{|\Re(c)|} + \delta \tag{5}$$

for each $x \in \mathbb{R}$.

Proof. We introduce new coordinates (ξ, η) by a suitable change of axes:

$$\begin{cases} \xi = x - \dfrac{a}{b}\,y, \\ \eta = \dfrac{1}{b}\,y. \end{cases} \tag{6}$$

Since $u(x,y) = u(\xi + a\eta, b\eta)$ and the last term can be denoted by $v(\xi, \eta)$, it follows from (6) that

$$\begin{aligned} u_x(x,y) &= \frac{\partial \xi}{\partial x}\, v_\xi(\xi,\eta) + \frac{\partial \eta}{\partial x}\, v_\eta(\xi,\eta) = v_\xi(\xi,\eta), \\ u_y(x,y) &= \frac{\partial \xi}{\partial y}\, v_\xi(\xi,\eta) + \frac{\partial \eta}{\partial y}\, v_\eta(\xi,\eta) = -\frac{a}{b}\, v_\xi(\xi,\eta) + \frac{1}{b}\, v_\eta(\xi,\eta). \end{aligned}$$

Hence, we have

$$au_x(x,y) + bu_y(x,y) = v_\eta(\xi,\eta),$$

and if we apply this equality to (2), we get

$$|v_\eta(\xi,\eta) + cv(\xi,\eta) + d| \leqslant \varepsilon$$

for all $\xi, \eta \in \mathbb{R}$, where $v(\xi,\eta) := u(\xi + a\eta, b\eta) = u(x,y)$.

Furthermore, if we set $v(\xi,\eta) = w(\xi,\eta) - (d/c)$ in the above inequality, then we have

$$|w_\eta(\xi,\eta) + cw(\xi,\eta)| \leqslant \varepsilon \tag{7}$$

for any $\xi, \eta \in \mathbb{R}$.

Since $\inf\{e^{\Re(c)\eta} : \eta \in \mathbb{R}\} = 0$, Theorem 1 together with (7) implies that for each fixed $\xi \in \mathbb{R}$, there exists a unique complex number $\theta(\xi)$ such that

$$\left| w(\xi,\eta) - \theta(\xi) e^{-c\eta} \right| \leqslant \frac{\varepsilon}{|\Re(c)|}$$

for all $\eta \in \mathbb{R}$. As $w(\xi,\eta) = u(x,y) + (d/c)$, the last inequality together with (6) yields the inequality (3). The validity of (5) immediately follows from (4) and (3) with $y = 0$. $\square$

EXAMPLE 1. Let $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a given function which has continuous partial derivatives with respect to the first and second variable. If u satisfies

$$|u_x(x,y) - 2u_y(x,y) + 4u(x,y) - 3| \leqslant \varepsilon \tag{8}$$

for all $x, y \in \mathbb{R}$ and for some $\varepsilon \geqslant 0$, then Theorem 2 implies that there exists a unique function $\theta : \mathbb{R} \to \mathbb{C}$ such that

$$\left| u(x,y) - \theta\left(x + \frac{1}{2}y\right)e^{2y} - \frac{3}{4} \right| \leqslant \frac{\varepsilon}{4}$$

for all $x, y \in \mathbb{R}$. Moreover, if u satisfies the auxiliary inequality

$$|u(x,0) - \sin x| \leqslant \delta$$

for all $x \in \mathbb{R}$ and for some $\delta \geqslant 0$, then θ satisfies

$$\left| \theta(x) - \sin x + \frac{3}{4} \right| \leqslant \frac{\varepsilon}{4} + \delta$$

for all $x \in \mathbb{R}$.

Indeed, it is easy to show that the real function

$$\left[\sin\left(x + \frac{1}{2}y\right) - \frac{3}{4} \right] e^{2y} + \frac{3}{4}$$

is the solution of the following linear partial differential equation

$$v_x(x,y) - 2v_y(x,y) + 4v(x,y) - 3 = 0 \tag{9}$$

with the auxiliary condition

$$v(x,0) = \sin x.$$

If $a \neq 0$ and the following inequality is given as the auxiliary condition,

$$|u(0,y) - g(y)| \leqslant \delta \tag{10}$$

for all $y \in \mathbb{R}$ and some $\delta \geqslant 0$, where $g : \mathbb{R} \to \mathbb{C}$ is a given function, we may introduce another coordinates (ξ,η) such as

$$\begin{cases} \xi = \dfrac{1}{a}x, \\ \eta = -\dfrac{b}{a}x + y \end{cases}$$

and we can easily prove the following theorem by a similar way as in the proof of Theorem 2, which permits us to omit the proof.

THEOREM 3. *Let* $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ *be a function which has continuous partial derivatives with respect to the first and second variable. Moreover, assume that* u *satisfies the inequality* (2) *for all* $x, y \in \mathbb{R}$ *and for some* $\varepsilon \geqslant 0$*, where* $a, b \in \mathbb{R}$ *and* $c, d \in \mathbb{C}$ *are constants with* $a \neq 0$*,* $\Re(c) \neq 0$*. Then, there exists a unique function* $\theta : \mathbb{R} \to \mathbb{C}$ *such that*

$$\left| u(x,y) - \theta\left(y - \frac{b}{a}x\right) e^{-(c/a)x} + \frac{d}{c} \right| \leqslant \frac{\varepsilon}{|\Re(c)|}$$

for all $x, y \in \mathbb{R}$*. Furthermore, if* u *satisfies the auxiliary condition* (10) *for all* $y \in \mathbb{R}$ *and for some* $\delta \geqslant 0$*, where* $g : \mathbb{R} \to \mathbb{C}$ *is a given function, then the* θ *satisfies the inequality* (5) *for every* $x \in \mathbb{R}$*.*

EXAMPLE 2. Let $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a given function which has continuous partial derivatives with respect to the first and second variable. If u satisfies the inequality (8) for all $x, y \in \mathbb{R}$ and for some $\varepsilon \geqslant 0$, then Theorem 3 implies that there exists a unique function $\theta : \mathbb{R} \to \mathbb{C}$ such that

$$\left| u(x,y) - \theta(2x + y)e^{-4x} - \frac{3}{4} \right| \leqslant \frac{\varepsilon}{4}$$

for all $x, y \in \mathbb{R}$. Moreover, if u satisfies the auxiliary inequality

$$|u(0,y) - \sin y| \leqslant \delta$$

for all $y \in \mathbb{R}$ and for some $\delta \geqslant 0$, then θ satisfies

$$\left| \theta(y) - \sin y + \frac{3}{4} \right| \leqslant \frac{\varepsilon}{4} + \delta$$

for all $y \in \mathbb{R}$.

As we can easily verify, the real function

$$\left[\sin(2x + y) - \frac{3}{4} \right] e^{-4x} + \frac{3}{4}$$

is the solution of the linear partial differential equation (9) with the auxiliary condition, $v(0,y) = \sin y$.

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