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SOME SCALES OF EQUIVALENT WEIGHT CHARACTERIZATIONS OF HARDY'S INEQUALITY: THE CASE $q < p$

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(*communicated by S. Saitoh*)

Abstract. We consider the weighted Hardy inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{1/q} \leqslant C\left(\int_0^\infty f^p(x)v(x)dx\right)^{1/p}$$

for the case $0 < q < p < \infty$, $p > 1$. The weights $u(x)$ and $v(x)$ for which this inequality holds for all $f(x) \geqslant 0$ may be characterized by the Mazya-Rosin or by the Persson-Stepanov conditions. In this paper, we show that these conditions are not unique and can be supplemented by some continuous scales of conditions and we prove their equivalence. The results for the dual operator which do not follow by duality when $0 < q < 1$ are also given.

1. Introduction

The Hardy inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{1/q} \leqslant C\left(\int_0^\infty f^p(x)v(x)dx\right)^{1/p}, \tag{1}$$

has attracted a lot of interest from the early discoveries of G. H. Hardy even before 1920 and the interest has even increased during the last years (see e.g. the books [2], [6], the recent review work [3] and the references given there).

Let us mention first that for the case $1 < p \leqslant q < \infty$ a necessary and sufficient condition on the weights $u(x) \geqslant 0$ and $v(x) \geqslant 0$ for (1) to hold for all $f(x) \geqslant 0$ is either well-known Muckenhoupt condition

$$\mathbb{A}_M := \sup_{0<t<\infty} U^{1/q}(t)V^{1/p'}(t) < \infty \tag{2}$$

or the following two alternatives which can be found in [7, Theorem 1] (see also [2, Theorem 1.1]):

$$\mathbb{A}_{PS}^{(1)} := \sup_{0<t<\infty} \left(\int_0^t uV^q\right)^{1/q} V^{-1/p}(t) < \infty, \tag{3}$$

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or

$$\mathbb{A}_{PS}^{(2)} := \sup_{0<t<\infty} \left(\int_t^\infty U^{p'} dV \right)^{1/p'} U^{-1/q'}(t) < \infty, \tag{4}$$

where $p' = p/(p-1), p \neq 1$ and for simplicity we suppose, that

$$0 < U(x) := \int_x^\infty u(t)dt < \infty, \;\; 0 < V(x) := \int_0^x v^{1-p'}(t)dt < \infty \text{ for all } x > 0. \tag{5}$$

Moreover, it was recently discovered that these conditions are not unique and can be replaced by the scales of conditions [1] depending on a continuous parameter $s > 0$ of the form

$$\mathbb{A}_M^{(1)}(s) := \sup_{0<t<\infty} \left(\int_t^\infty u V^{q(\frac{1}{p'}-s)} \right)^{1/q} V^s(t) < \infty, \tag{6}$$

$$\mathbb{A}_M^{(2)}(s) := \sup_{0<t<\infty} U^s(t) \left(\int_0^t U^{p'(\frac{1}{q}-s)} dV \right)^{1/p\prime} < \infty, \tag{7}$$

corresponding to (2) and

$$\mathbb{A}_{PS}^{(1)}(s) := \sup_{0<t<\infty} \left(\int_0^t u V^{q(\frac{1}{p'}+s)} \right)^{1/q} V^{-s}(t) < \infty, \tag{8}$$

or

$$\mathbb{A}_{PS}^{(2)}(s) := \sup_{0<t<\infty} \left(\int_t^\infty U^{p'(\frac{1}{q}+s)} dV \right)^{1/p'} U^{-s}(t) < \infty, \tag{9}$$

relating to (3) and (4), respectively.

The main result of [1] reads:

$$C \approx \mathbb{A}_M^{(1)}(s) \approx \mathbb{A}_M^{(2)}(s) \approx \mathbb{A}_{PS}^{(1)}(s) \approx \mathbb{A}_{PS}^{(2)}(s), \tag{10}$$

where C is the best constant in (1), with the constants of equivalence depending only on p, q and s.

In this paper from the same point of view we study the case $0 < q < p < \infty, p > 1, q \neq 1$. Denote $1/r := 1/q - 1/p$. In this case it is known, see [4], [5], [7] and [8], that (1) holds for some finite constant $C \geqslant 0$, if and only if one of the following quantities is finite: the Mazya-Rozin constant

$$\mathbb{B}_{MR} := \left(\int_0^\infty U^{r/p}(x) V^{r/p'}(x) u(x) dx \right)^{1/r} < \infty, \tag{11}$$

or the Persson-Stepanov constant

$$\mathbb{B}_{PS} := \left(\int_0^\infty \left[\int_0^x u(t) V^q(t) dt \right]^{r/p} u(x) V^{q-r/p}(x)\, dx \right)^{1/r} < \infty. \tag{12}$$

Moreover for the best constant C in (1) it yields that

$$C \approx \mathbb{B}_{MR} \approx \mathbb{B}_{PS}. \tag{13}$$

Our main result in this paper is to prove that similar to the case $1 < p \leqslant q < \infty$ the conditions (11) and (12) above are not unique and can be extended by some scales of conditions. However, there is a substantial difference with the case $1 < p \leqslant q < \infty$, because of no duality exists for $0 < q < 1 < p < \infty$. Therefore, we formulate the results for the Hardy inequality with the dual operator too.

The paper is organized as follows: In order not to disturb our discussions later on some notations and other preliminaries are collected in Section 2. The discussed main result (see Theorem 1) can be found in Section 3 while the corresponding main result for the dual operator is formulated in Section 4. Finally, Section 5 is reserved for a concluding remark of independent interest.

2. Preliminaries

If no misunderstanding can occur we some times delete the integration variable, e.g. we write $\int_0^x uV^q$ instead of $\int_0^x u(t)V^q(t)dt$. Moreover, for positive constants A and B we write $A \ll B$ ($A \gg B$) whenever there exists a finite constant c independent on weight functions u and v such that $A \leqslant cB$ $(cA \geqslant B)$. If both $A \ll B$ and $B \ll A$ it yields that $A \approx B$ (c.f. e.g. (10) above).

Suppose, $0 < q < p < \infty, p > 1, q \neq 1$. It is known ([8], Remark on p. 93), that under the condition (5) the Mazya-Rozin constant has an equivalent form

$$\mathcal{B}_{MR} := \left(\int_0^\infty U^{r/q} V^{r/q'} dV\right)^{1/r}, \tag{14}$$

and

$$\mathbb{B}_{MR}^r = \frac{q}{p'} \mathcal{B}_{MR}^r. \tag{15}$$

Similarly, a counterpart to the Persson-Stepanov constant is

$$\mathcal{B}_{PS} := \left(\int_0^\infty \left[\int_0^x u(t)V^q(t)dt\right]^{r/q} V^{-r/q} dV(x)\right)^{1/r}. \tag{16}$$

Moreover, (see [7])

$$\mathbb{B}_{PS}^r = \frac{q}{p}\mathcal{B}_{PS}^r \quad \text{if } V(\infty) = \infty \tag{17}$$

and

$$\mathbb{B}_{PS}^r = \frac{q}{r}\left(\int_0^\infty uV^q\right)^{r/q} V^{-r/p}(\infty) + \frac{q}{p}\mathcal{B}_{PS}^r \quad \text{if } 0 < V(\infty) < \infty. \tag{18}$$

For $s > 0$ we define the following functionals:

$$\mathbb{B}_{MR}^{(1)}(s) := \left(\int_0^\infty \left[\int_t^\infty uV^{q(1/p'-s)}\right]^{r/p} V^{q(1/p'-s)+rs}(t)u(t)\,dt\right)^{1/r}, \tag{19}$$

$$\mathcal{B}_{MR}^{(1)}(s) := \left(\int_0^\infty \left[\int_t^\infty uV^{q(1/p'-s)}\right]^{r/q} V^{rs-1}(t)\,dV(t)\right)^{1/r}, \tag{20}$$

$$\mathbb{B}_{PS}^{(1)}(s) := \left(\int_0^\infty \left[\int_0^t uV^{q\left(1/p'+s\right)} \right]^{r/p} u(t)V^{q\left(1/p'+s\right)-sr}(t)\,dt \right)^{1/r}, \tag{21}$$

$$\mathcal{B}_{PS}^{(1)}(s) := \left(\int_0^\infty \left[\int_0^t uV^{q\left(1/p'+s\right)} \right]^{r/q} V^{-rs-1}(t)\,dV(t) \right)^{1/r}. \tag{22}$$

Then, similar to (15), (17) and (18) we have

$$\left[\mathbb{B}_{MR}^{(1)}(s)\right]^r = qs\left[\mathcal{B}_{MR}^{(1)}(s)\right]^r, \tag{23}$$

$$\left[\mathbb{B}_{PS}^{(1)}(s)\right]^r = qs\left[\mathcal{B}_{PS}^{(1)}(s)\right]^r \quad \text{if } V(\infty)=\infty \tag{24}$$

and

$$\left[\mathbb{B}_{PS}^{(1)}(s)\right]^r = \frac{q}{r}\left(\int_0^\infty uV^{q(1/p'+s)}\right)^{r/q} V^{-rs}(\infty) + qs\left[\mathcal{B}_{PS}^{(1)}(s)\right]^r, \tag{25}$$

if $0 < V(\infty) < \infty$. Also note, that

$$\mathbb{B}_{MR}^{(1)}(\frac{1}{p'}) = \mathbb{B}_{MR}, \quad \mathcal{B}_{MR}^{(1)}(\frac{1}{p'}) = \mathcal{B}_{MR} \tag{26}$$

and

$$\mathbb{B}_{PS}^{(1)}(\frac{1}{p}) = \mathbb{B}_{PS}, \quad \mathcal{B}_{PS}^{(1)}(\frac{1}{p}) = \mathcal{B}_{PS}. \tag{27}$$

3. The main result

THEOREM 1. *Let $0 < q < p < \infty,\ 1 < p < \infty$ and $q \neq 1$. Then the Hardy inequality (1) holds for some finite constant $C \geqslant 0$ if and only if any of the constants $\mathbb{B}_{MR}^{(1)}(s)$ or $\mathbb{B}_{PS}^{(1)}(s)$ is finite for some $s > 0$. Moreover, for the best constant C in (1) we have*

$$C \approx \mathbb{B}_{MR}^{(1)}(s) \approx \mathbb{B}_{PS}^{(1)}(s). \tag{28}$$

REMARK 1. In view of the equalities (26) and (27) the equivalence (13) follows from (28).

Proof. We mainly consider four cases. First we prove that

$$\mathbb{B}_{MR}^{(1)}(s) \approx \mathbb{B}_{MR}. \tag{29}$$

Case 1 : Let $s \geqslant 1/p'$. If $s = 1/p'$, then $\mathbb{B}_{MR}^{(1)}(1/p') = \mathbb{B}_{MR}$ by (26). Let $s > 1/p'$. Then $V^{q(1/p'-s)}$ is decreasing, so that $\mathbb{B}_{MR}^{(1)}(s) \leqslant \mathbb{B}_{MR}$. Since $\mathbb{B}_{MR} \approx \mathbb{B}_{PS}$ by (13), then for the reverse, we show that $\mathbb{B}_{PS} \ll \mathbb{B}_{MR}^{(1)}(s)$. First we write

$$\int_0^t uV^q = \int_0^t uV^{q(1/p'-s)}V^{q(s+1/p)} = \int_0^t V^{q(s+1/p)}\,d\left(-\int_x^t uV^{q(1/p'-s)}\right)$$

[integration by parts]

$$= q(s+1/p)\int_0^t \left(\int_x^t uV^{q(1/p'-s)}\right) V^{q(s-1/r)}(x)dV(x)$$

$$= q(s+1/p)\int_0^t \left\{\left(\int_x^t uV^{q(1/p'-s)}\right) V^{q(s-1/r)+\varepsilon q/p}(x)\right\}\left\{V^{-\varepsilon q/p}(x)\right\} dV(x)$$

[by Hölder's inequality with $r/q, p/q$ and $\varepsilon \in (0,1)$]

$$\leqslant q(s+1/p)\left(\int_0^t \left(\int_x^t uV^{q(1/p'-s)}\right)^{r/q} V^{rs-1+\varepsilon r/p}(x)dV(x)\right)^{q/r} \times$$

$$\times \left(\int_0^t V^{-\varepsilon}(x)dV(x)\right)^{q/p}$$

$$\leqslant \frac{q(s+1/p)}{(1-\varepsilon)^{q/p}}\left(\int_0^t \left(\int_x^\infty uV^{q(1/p'-s)}\right)^{r/q} V^{rs-1+\varepsilon r/p}(x)dV(x)\right)^{q/r} V^{(1-\varepsilon)q/p}(t).$$

Applying this estimate, we find

$$[\mathcal{B}_{PS}]^r = \int_0^\infty \left(\int_0^t uV^q\right)^{r/q} V^{-r/q}(t)\,dV(t)$$

$$\leqslant \frac{[q(s+1/p)]^{r/q}}{(1-\varepsilon)^{r/p}}\times$$

$$\times \int_0^\infty\int_0^t\left(\int_x^\infty uV^{q(1/p'-s)}\right)^{r/q} V^{rs-1+\varepsilon r/p}(x)dV(x)V^{(1-\varepsilon)r/p}(t)V^{-r/q}(t)\,dV(t)$$

[changing the order of integration]

$$= \frac{[q(s+1/p)]^{r/q}}{(1-\varepsilon)^{r/p}}\times$$

$$\times \int_0^\infty\left(\int_x^\infty uV^{q(1/p'-s)}\right)^{r/q} V^{rs-1+\varepsilon r/p}(x)\left(\int_x^\infty V^{-1-\varepsilon r/p}(t)\,dV(t)\right) dV(x)$$

$$\leqslant \frac{p\,[q(s+1/p)]^{r/q}}{\varepsilon r(1-\varepsilon)^{r/p}}\int_0^\infty \left(\int_x^\infty uV^{q(1/p'-s)}\right)^{r/q} V^{rs-1}(x)\,dV(x)$$

$$= \frac{p\,[q(s+1/p)]^{r/q}}{\varepsilon r(1-\varepsilon)^{r/p}}\left[\mathcal{B}_{MR}^{(1)}(s)\right]^r.$$

Taking $\varepsilon = 1/2$, and using the equivalence of $\mathcal{B}_{MR}^{(1)}(s)$ and $\mathbb{B}_{MR}^{(1)}(s)$ we obtain

$$[\mathcal{B}_{PS}]^r \leqslant \frac{p2^{r/q}\,[q(s+1/p)]^{r/q}}{rqs}\left[\mathbb{B}_{MR}^{(1)}(s)\right]^r$$

and in case $V(\infty) = \infty$ the result follows by (17). If $0 < V(\infty) < \infty$, then to finish this case we estimate directly the first term in (18). Write as before

$$\int_0^\infty uV^q \ll \int_0^\infty \left(\int_x^\infty uV^{q(1/p'-s)}\right) V^{q(s-1/r)}(x)\,dV(x)$$

[by Hölder's inequality]

$$\leqslant \left(\int_0^\infty \left(\int_x^\infty uV^{q(1/p'-s)}\right)^{r/q} V^{rs-1}(x)\,dV(x)\right)^{q/r} V^{q/p}(\infty).$$

Hence,

$$\left(\int_0^\infty uV^q\right)^{r/q} V^{-r/p}(\infty) \leqslant \left[\mathcal{B}_{MR}^{(1)}(s)\right]^r = \frac{1}{qs}\left[\mathbb{B}_{MR}^{(1)}(s)\right]^r$$

and the required upper bound $\mathbb{B}_{PS} \ll \mathbb{B}_{MR}^{(1)}(s)$ follows from (18).

Case $2:\ \ 0 < s < 1/p'$. Now $\mathbb{B}_{MR}^{(1)}(s) \geqslant \mathbb{B}_{MR}$ holds true and it only remains to show the reverse. To this end we note that

$$\int_t^\infty uV^{q(1/p'-s)} = \int_t^\infty V^{q(1/p'-s)}(x)\,d(-U(x))$$

[integration by parts]

$$\leqslant U(t)V^{q(1/p'-s)}(t) + q(1/p'-s)\int_t^\infty U(x)V^{q(1/p'-s)-1}(x)\,dV(x).$$

This implies

$$\left[\mathcal{B}_{MR}^{(1)}(s)\right]^r = \int_0^\infty \left(\int_t^\infty uV^{q(1/p'-s)}\right)^{r/q} V^{rs-1}(t)\,dV(t) \ll I_1 + I_2.$$

We have

$$I_1 := \int_0^\infty U^{r/q}\left[V^{q(1/p'-s)}\right]^{r/q} V^{rs-1}\,dV = \mathcal{B}_{MR}.$$

For the second term we write first for some $\varepsilon > 0$

$$\int_t^\infty UV^{q(1/p'-s)-1}\,dV = \int_t^\infty \left\{UV^{q(1/p'-s)-q/r+\varepsilon q/p}\right\}\left\{V^{-q/p-\varepsilon q/p}\right\}dV$$

[by Hölder's inequality]

$$\leqslant \left(\int_t^\infty U^{r/q}V^{r(1/p'-s)-1+\varepsilon r/p}dV\right)^{q/r}\left(\int_t^\infty V^{-1-\varepsilon}dV\right)^{q/p}$$

$$\leqslant \varepsilon^{-q/p}\left(\int_t^\infty U^{r/q}V^{r(1/p'-s)-1+\varepsilon r/p}dV\right)^{q/r} V^{-\varepsilon q/p}(t).$$

Then

$$\begin{aligned} I_2 :&= \int_0^\infty \left(\int_t^\infty UV^{q(1/p'-s)-1}dV\right)^{r/q} V^{rs-1}(t)\,dV(t) \\ &\leqslant \varepsilon^{-r/p}\int_0^\infty \left(\int_t^\infty U^{r/q}V^{r(1/p'-s)-1+\varepsilon r/p}dV\right) V^{-\varepsilon r/p+rs-1}(t)\,dV(t) \\ &= \varepsilon^{-r/p}\int_0^\infty U^{r/q}(x)V^{r(1/p'-s)-1+\varepsilon r/p}(x)\left(\int_0^x V^{r(s-\varepsilon/p)-1}dV\right)dV(x). \end{aligned}$$

Let ε be so small that $s-\varepsilon/p>0$. Then

$$I_2=\frac{1}{r(s-\varepsilon/p)\varepsilon^{r/p}}\int_0^\infty U^{r/q}V^{r/q'}\,dV\approx\mathcal{B}_{MR}^r.$$

Summarizing the above estimates we conclude

$$\left[\mathbb{B}_{MR}^{(1)}(s)\right]^r=qs\left[\mathcal{B}_{MR}^{(1)}(s)\right]\ll\mathcal{B}_{MR}^r=\frac{p'}{q}\mathbb{B}_{MR}^r$$

and (29) is proved.

Now we show that

$$\mathbb{B}_{PS}\approx\mathbb{B}_{PS}^{(1)}(s). \tag{30}$$

Case 3 : $s\geqslant 1/p$. If $s=1/p$, then the equality is valid by (27). Let $s>1/p$. Then $V^{q(s-1/p)}$ is increasing and $\mathbb{B}_{PS}^{(1)}(s)\leqslant\mathbb{B}_{PS}$. For the reverse we show that $C\ll\mathbb{B}_{PS}^{(1)}(s)$ for the best constant of (1) and the result will follow by (13). Let $\int_0^\infty f^pv<\infty$. Put $\alpha:=s-1/p>0$ and suppose first that $V(\infty)=\infty$. Write

$$\begin{aligned}
I_0:&=\int_0^\infty\left(\int_0^x f\right)^q u(x)\,dx=\int_0^\infty\left(\int_0^x f\right)^q u(x)V^{q+\alpha q}(x)V^{-q-\alpha q}(x)\,dx\\
&=q(1+\alpha)\int_0^\infty\left(\int_0^x f\right)^q u(x)V^{q+\alpha q}(x)\left(\int_x^\infty V^{-q-\alpha q-1}\,dV\right)dx\\
&=q(1+\alpha)\int_0^\infty V^{-q-\alpha q-1}(t)\left(\int_0^t\left(\int_0^x f\right)^q u(x)V^{q+\alpha q}(x)dx\right)dV(t)\\
&\leqslant q(1+\alpha)\int_0^\infty\left\{\left(\int_0^t f\right)^q V^{-q}(t)\right\}\left\{\left(\int_0^t uV^{q+\alpha q}\right)V^{-1-\alpha q}(t)\right\}dV(t)
\end{aligned}$$

[applying Hölder's inequality with p/q and r/q]

$$\begin{aligned}
&\leqslant q(1+\alpha)\left(\int_0^\infty\left(\int_0^t f\right)^p V^{-p}(t)dV(t)\right)^{q/p}\times\\
&\qquad\times\left(\int_0^\infty\left(\int_0^t uV^{q+\alpha q}\right)^{r/q}V^{-r/q-\alpha r}(t)dV(t)\right)^{q/r}\\
&\leqslant q(1/p'+s)(p')^{q/p}\left(\int_0^\infty f^pv\right)^{q/p}\left[\mathcal{B}_{PS}^{(1)}(s)\right]^q\\
&\leqslant\frac{q(1/p'+s)(p')^{q/p}}{(qs)^{q/r}}\left[\mathbb{B}_{PS}^{(1)}(s)\right]^q\left(\int_0^\infty f^pv\right)^{q/p}.
\end{aligned}$$

Hence,

$$C\leqslant\frac{q^{1/q}(1/p'+s)^{1/q}(p')^{1/p}}{(qs)^{1/r}}\mathbb{B}_{PS}^{(1)}(s).$$

If $0<V(\infty)<\infty$, then we use the fact that

$$V^{-q-\alpha q}(x)=q(1+\alpha)\left[\frac{1}{V^{q+\alpha q}(\infty)}+\int_x^\infty V^{-q-\alpha q-1}\,dV\right]$$

and obtain

$$\int_0^\infty \left(\int_0^x f\right)^q u(x)dx = I_0 + q(1+\alpha)I_1.$$

We have shown that $I_0 \ll \mathbb{B}_{PS}^{(1)}(s)\left(\int_0^\infty f^p v\right)^{1/p}$. For the upper bound of I_1, let $\{x_k\} \subset (0,\infty)$, $k \leqslant N < \infty$ such that

$$\int_0^{x_k} f = 2^k, \quad k \leqslant N$$
$$\int_0^{\infty} f \leqslant 2^{N+1}, \quad x_{N+1} = \infty.$$

Since

$$\int_0^\infty f \leqslant \left(\int_0^\infty f^p v\right)^{1/p} V^{1/p'}(\infty) < \infty,$$

then such $N \in \mathbb{N}$ exists. We have

$$\begin{aligned}
V^{q+\alpha q}(\infty)I_1 &= \int_0^\infty \left(\int_0^x f\right)^q u(x)V^{q+\alpha q}(x)\,dx \\
&= \sum_{k\leqslant N}\int_{x_k}^{x_{k+1}} \left(\int_0^x f\right)^q u(x)V^{q+\alpha q}(x)\,dx \\
&\leqslant \sum_{k\leqslant N} 2^{(k+1)q}\int_{x_k}^{x_{k+1}} uV^{q+\alpha q} \\
&\leqslant 4^q \sum_{k\leqslant N}\left(\int_{x_{k-1}}^{x_k} f^p v\right)^{q/p}\left(\int_{x_{k-1}}^{x_k} dV\right)^{q/p'}\int_{x_k}^{x_{k+1}} uV^{q+\alpha q}
\end{aligned}$$

[by Hölder's inequality with p/q, r/q]

$$\begin{aligned}
&\leqslant 4^q\left(\sum_{k\leqslant N}\int_{x_{k-1}}^{x_k} f^p v\right)^{q/p}\left(\sum_{k\leqslant N}\left(\int_{x_{k-1}}^{x_k} dV\right)^{r/p'}\left(\int_{x_k}^{x_{k+1}} uV^{q+\alpha q}\right)^{r/q}\right)^{q/r} \\
&\leqslant 4^q\left(\int_0^\infty f^p v\right)^{q/p}\left(\sum_{k\leqslant N} V^{r/p'}(x_k)\int_{x_k}^{x_{k+1}}\left(\int_{x_k}^x uV^{q+\alpha q}\right)^{r/p} u(x)V^{q+\alpha q}(x)\,dx\right)^{q/r} \\
&\leqslant 4^q\left(\int_0^\infty f^p v\right)^{q/p}\left(\int_0^\infty\left(\int_0^x uV^{q(1/p'+s)}\right)^{r/p} u(x)V^{q(1/p'+s)-sr}(x)V^{r(1/p'+s)}(x)\,dx\right)^{q/r} \\
&\leqslant 4^q V^{q+\alpha q}(\infty)\left(\int_0^\infty f^p v\right)^{q/p}\left[\mathbb{B}_{PS}^{(1)}(s)\right]^q.
\end{aligned}$$

Thus, $C \ll \mathbb{B}_{PS}^{(1)}(s)$ and by (13) the required equivalence $\mathbb{B}_{PS} \approx \mathbb{B}_{PS}^{(1)}(s)$ for $s \geqslant 1/p$ is proved.

Case 4 : Let $0 < s < 1/p$. Now we have $\mathbb{B}_{PS}^{(1)}(s) \geqslant \mathbb{B}_{PS}$. For the reverse we show that $\mathbb{B}_{PS}^{(1)}(s) \ll \mathcal{B}_{MR} \approx \mathbb{B}_{MR} \approx \mathbb{B}_{PS}$. Write

$$\int_0^t uV^{q(1/p'+s)} = \int_0^t V^{q(1/p'+s)}(x)\, d\left(-\int_x^t u\right)$$

[integration by parts]

$$= q(1/p'+s)\int_0^t \left(\int_x^t u\right) V^{q(1/p'+s)-1}(x) dV(x)$$
$$= q(1/p'+s)\int_0^t \left\{\left(\int_x^t u\right) V^{q(1/p'+s)-1+\varepsilon q/p}(x)\right\}\left\{V^{-\varepsilon q/p}(x)\right\} dV(x)$$

for some $\varepsilon \in (0,1)$ which we determine later. Now Hölder's inequality with the parameters r/q and p/q gives that

$$\ll \left(\int_0^t \left(\int_x^t u\right)^{r/q} V^{r(1/p'+s-1/q+\varepsilon/p)}(x) dV(x)\right)^{q/r} \left(\int_0^t V^{-\varepsilon} dV\right)^{q/p}$$
$$\ll \left(\int_0^t U^{r/q} V^{r(1/p'+s-1/q+\varepsilon/p)} dV\right)^{q/r} V^{(1-\varepsilon)q/p}(t).$$

Using this estimate, we obtain

$$\mathcal{B}_{PS}^{(1)}(s) \ll \int_0^\infty U^{r/q}(t) V^{r(1/p'+s-1/q+\varepsilon/p)}(t)\left(\int_t^\infty V^{-1-r(s-(1-\varepsilon)/p)} dV\right) dV(t).$$

If $0 < s < 1/p$, then there exists $\varepsilon \in (0,1)$ such that $s-(1-\varepsilon)/p > 0$ so we have

$$\mathcal{B}_{PS}^{(1)}(s) \ll \int_0^\infty U^{r/q} V^{r/q'}\, dV = \mathcal{B}_{MR}$$

and if $V(\infty) = \infty$, then the result follows by (13), (15) and (24). If $0 < V(\infty) < \infty$, then as before we find

$$\int_0^\infty uV^{q(1/p'+s)} = q(1/p'+s)\int_0^\infty UV^{q(1/p'+s)-1} dV$$
$$= q(1/p'+s)\int_0^\infty \left\{UV^{q/q'}\right\}\left\{V^{qs+q/r-1}\right\} dV$$
$$\ll \left(\int_0^\infty U^{r/q} V^{r/q'} dV\right)^{q/r} \left(\int_0^\infty V^{ps-1} dV\right)^{q/p}.$$

Hence,

$$V^{-s}(\infty)\left(\int_0^\infty uV^{q(1/p'+s)}\right)^{1/q} \ll \mathcal{B}_{MR}.$$

Thus, $\mathbb{B}_{PS}^{(1)}(s) \ll \mathcal{B}_{MR}$ because of (25) and the equivalence (30) follows. Now (13), (29) and (30) prove (28). $\square$

4. The main result for the dual operator

It was mentioned in the introduction, that in the case $0 < q < 1 < p < \infty$ there is no duality for the Hardy inequality (1). However, the related inequality with the dual operator

$$\left(\int_0^\infty \left(\int_x^\infty g\right)^q u(x)\,dx\right)^{1/q} \leqslant C\left(\int_0^\infty g^p v\right)^{1/p} \tag{31}$$

can be treated by the same method as (1). In the case $1 < q < p < \infty$ it brings alternatives to (28). To formulate the results we need to complement our notation. Suppose for simplicity, that

$$0 < U_*(x) := \int_0^x u < \infty, \quad 0 < V_*(x) := \int_x^\infty v^{1-p'} < \infty,\ x > 0,$$

$$\mathbb{B}^*_{MR} := \left(\int_0^\infty U_*^{r/p} V_*^{r/p'} u\right)^{1/r},$$

$$\mathcal{B}^*_{MR} := \left(\int_0^\infty U_*^{r/q} V_*^{r/q'}\, d(-V_*)\right)^{1/r},$$

$$\mathbb{B}^*_{PS} := \left(\int_0^\infty \left[\int_x^\infty u V_*^q\right]^{r/p} u(x) V_*^{q-r/p}(x)\,dx\right)^{1/r}$$

and

$$\mathcal{B}^*_{PS} := \left(\int_0^\infty \left[\int_x^\infty u V_*^q\right]^{r/q} V_*^{-r/q}(x)\, d\left(-V_*(x)\right)\right)^{1/r},$$

with (15), (17) and (18) valid with replacement of $\mathbb{B}$ by $\mathbb{B}^*$ and $\mathcal{B}$ by $\mathcal{B}^*$, respectively.

For $s > 0$ we now define a new portion of functionals

$$\mathbb{B}^{*(1)}_{MR}(s) := \left(\int_0^\infty \left[\int_t^\infty u V_*^{q\left(1/p'-s\right)}\right]^{r/p} V_*^{q(1/p'-s)+rs}(t) u(t)\,dt\right)^{1/r},$$

$$\mathcal{B}^{*(1)}_{MR}(s) := \left(\int_0^\infty \left[\int_0^t u V_*^{q\left(1/p'-s\right)}\right]^{r/q} V_*^{rs-1}(t)\, d\left(-V_*(t)\right)\right)^{1/r},$$

$$\mathbb{B}^{*(1)}_{PS}(s) := \left(\int_0^\infty \left[\int_t^\infty u V_*^{q\left(1/p'+s\right)}\right]^{r/p} u(t) V_*^{q(1/p'-s)+rs}(t)\,dt\right)^{1/r}$$

and

$$\mathcal{B}^{*(1)}_{PS}(s) := \left(\int_0^\infty \left[\int_t^\infty u V_*^{q\left(1/p'+s\right)}\right]^{r/p} V_*^{-rs-1}(t)\, d\left(-V_*(t)\right)\right)^{1/r}$$

with (23) and (24) valid with $\mathbb{B}^*$ and $\mathcal{B}^*$ instead of $\mathbb{B}$ and $\mathcal{B}$ and (25), (26) and (27) are also true with natural modification for $V_*(0) = \infty$ and $0 < V_*(0) < \infty$. Thus we arrive at the following dual version of Theorem 1:

THEOREM 2. *Let $0 < q < p < \infty$, $1 < p < \infty$ and $q \neq 1$. Then the Hardy inequality (31) holds for some finite constant $C \geqslant 0$ if and only if any of the constants*

$\mathbb{B}^{*(1)}_{MR}(s)$ *or* $\mathbb{B}^{*(1)}_{PS}(s)$ *is finite for some* $s>0$. *Moreover, for the best constant* C *in* (31) *we have*

$$C \approx \mathbb{B}^{*(1)}_{MR}(s) \approx \mathbb{B}^{*(1)}_{PS}(s). \tag{32}$$

In the case $1<q<p<\infty$ the Hardy inequality (1) is equivalent to the following

$$\left(\int_0^\infty \left(\int_x^\infty g\right)^{p'} dV(x)\right)^{1/p'} \leqslant C\left(\int_0^\infty g^{q'} u^{1-q'} dx\right)^{1/q'} \tag{33}$$

and is characterized by Theorem 2. If we compose related constants for (33), we obtain the following new characterizations for the initial Hardy inequality (1):

COROLLARY 3. *Let* $1<q<p<\infty$. *Then the inequality* (1) *is characterized by*

$$C \approx \mathbb{B}^{(2)}_{MR}(s) \approx \mathbb{B}^{(2)}_{PS}(s),$$

where

$$\mathbb{B}^{(2)}_{MR}(s) := \left(\int_0^\infty \left[\int_0^t U^{p'(1/q-s)} dV\right]^{r/p'} U^{rs-1}(t)u(t)\,dt\right)^{1/r}$$

and

$$\mathbb{B}^{(2)}_{PS}(s) := \left(\int_0^\infty \left[\int_t^\infty U^{q\left(1/p'+s\right)} dV\right]^{r/p} U^{q\left(1/p'+s\right)-rs}(t)\,dV(t)\right)^{1/r}.$$

Observe that

$$\left[\mathbb{B}^{(2)}_{MR}(s)\right]^r = \frac{1}{sp'}\left[\mathcal{B}^{(2)}_{MR}(s)\right]^r,$$

where

$$\mathcal{B}^{(2)}_{MR}(s) := \left(\int_0^\infty \left[\int_0^t U^{p'(1/q-s)} dV\right]^{r/q'} U^{p'(1/q-s)+rs}(t)\,dV(t)\right)^{1/r}$$

and

$$\left[\mathbb{B}^{(2)}_{PS}(s)\right]^r = qs\left[\mathcal{B}^{(2)}_{PS}(s)\right]^r, \quad U(0)=\infty,$$

$$\left[\mathbb{B}^{(2)}_{PS}(s)\right]^r = \frac{q}{r}\left(\int_0^\infty U^{q(1/p'+s)}\,dV\right)^{r/q} U^{-rs}(0) + qs\left[\mathcal{B}^{(2)}_{PS}(s)\right]^r, \quad 0<U(0)<\infty,$$

where

$$\mathcal{B}^{(2)}_{PS}(s) := \left(\int_0^\infty \left[\int_t^\infty U^{q\left(1/p'+s\right)} dV\right]^{r/q} U^{-rs-1}(t)u(t)\,dt\right)^{1/r}.$$

Analogously, using Theorem 1 for the inequality

$$\left(\int_0^\infty \left(\int_0^x f\right)^{p'} dV_*(x)\right)^{1/p'} \leqslant C\left(\int_0^\infty f^{q'} u^{1-q'} dx\right)^{1/q'},$$

dual to (31), we obtain the following four additional characterizations of (31).

COROLLARY 4. *Let* $1 < q < p < \infty$. *Then the inequality (31) is characterized by*

$$C \approx \mathbb{B}_{MR}^{*(2)}(s) \approx \mathbb{B}_{PS}^{*(2)}(s),$$

where

$$\mathbb{B}_{MR}^{*(2)}(s) := \left(\int_0^\infty \left[\int_t^\infty U_*^{p'(1/q-s)} dV_* \right]^{r/p'} U_*^{rs-1}(t) u(t)\, dt \right)^{1/r}$$

and

$$\mathbb{B}_{PS}^{*(2)}(s) := \left(\int_0^\infty \left[\int_0^t U_*^{q(1/p'+s)} dV_* \right]^{r/p} U_*^{q(1/p'+s)-rs}(t)\, dV_*(t) \right)^{1/r}$$

with the usual supplemented counterparts:

$$\mathcal{B}_{MR}^{*(2)}(s) := \left(\int_0^\infty \left[\int_t^\infty U_*^{p'(1/q-s)} dV_* \right]^{r/p'} U_*^{p'(1/q-s)+rs}(t) dV_*(t) \right)^{1/r},$$

$$\left[\mathbb{B}_{MR}^{*(2)}(s)\right]^r = \frac{1}{sp'} \left[\mathcal{B}_{MR}^{*(2)}(s)\right]^r$$

and

$$\mathcal{B}_{PS}^{*(2)}(s) := \left(\int_0^\infty \left[\int_0^t U_*^{q(1/p'+s)} dV_* \right]^{r/q} U_*^{-rs-1}(t) u(t)\, dt \right)^{1/r},$$

$$\left[\mathbb{B}_{PS}^{*(2)}(s)\right]^r = qs \left[\mathcal{B}_{PS}^{*(2)}(s)\right]^r \quad \textit{for } U_*(\infty) = \infty,$$

$$\left[\mathbb{B}_{PS}^{(2)}(s)\right]^r = \frac{q}{r} \left(\int_0^\infty U_*^{q(1/p'+s)}\, dV_* \right)^{r/q} U_*^{-rs}(\infty) + qs \left[\mathcal{B}_{PS}^{*(2)}(s)\right]^r,$$

for $0 < U_*(\infty) < \infty$.

Thus, the Hardy inequalities (1) and (31) have eight alternative scale characterizations each with four, roughly speaking, in common.

5. Concluding remark

Theorems 1 and 2 assert that a Hardy inequality can be characterized by the finiteness of some integral scale conditions. We remark, that if we start with an integral condition in Mazya-Rozin or Persson-Stepanov form, then it uniquely corresponds to a Hardy inequality and, consequently, is equivalent to the scales of conditions.

More precisely, let $u(x) \geqslant 0$ and $v(x) \geqslant 0$ be weight functions, and

$$0 < U(t) := \int_t^\infty u < \infty, \quad 0 < V(t) := \int_0^t v < \infty,$$

for all $t > 0$. Suppose that given $\alpha > 0$ and $0 < \beta \neq 1$

$$\mathbb{B}_{\alpha,\beta} := \left(\int_0^\infty U^\alpha V^\beta u \right)^{1/(\alpha+\beta)} < \infty. \tag{34}$$

Then [8] implies that Hardy's inequality

$$\left(\int_0^\infty \left(\int_0^x f\,v\right)^q u(x)\,dx\right)^{1/q} \leqslant C\left(\int_0^\infty f^p v\,dx\right)^{1/p}$$

holds for all $f(x) \geqslant 0$ with $p = 1 + \beta/\alpha$, $q = (\alpha+\beta)/(\alpha+1)$ and $C \approx \mathbb{B}_{\alpha,\beta}$. It follows from (13) and Theorem 1, that (34) is equivalent to continuous scales of integral conditions like (19)-(22). We omit the details.

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REFERENCES

[1] A. GOGATISHVILI, A. KUFNER, L. E. PERSSON AND A. WEDESTIG, Real Anal. Exchange, **29**, (2003/04), 867–880.

[2] A. KUFNER, L. E. PERSSON, *Weighted Inequalities of Hardy Type*, World Scientific, New Jersey/London/Singapore/Hong Kong, 2003.

[3] A. KUFNER, L. MALIGRANDA AND L. E. PERSSON, *The Hardy inequality-about it's history and related results*, accepted book manuscript, Department of Mathematics, Luleå University of Technology, 2006 (150 pages).

[4] V. MAZJA, *Einbettungssätze für Sobolewsche Räume*, Teil 1(Imbedding Theorems for Sobolev Spaces. Part 1), Teubner-Texte zur Mathematik, Leipzig, 1979.

[5] V. MAZJA, *Sobolev Spaces*, Russian Edition: Leningrad University, Leningrad 1985; English trans.: Springer Verlag, Berlin-Heidelberg, 1985.

[6] B. OPIC, A. KUFNER, *Hardy-Type Inequalities*, Longman, Harlow, 1990.

[7] L. E. PERSSON, V. D. STEPANOV, *Weighted integral inequalities with the geometric mean operator*, J. Inequal. Appl., **7**, (2002), 727–746 (an abbreviated version can be found in Dokl. Acad. Nauk., **377**, (2001), 439–440).

[8] G. SINNAMON, V. D. STEPANOV, *The weighted Hardy inequality: new proofs and the case $p = 1$*, J. London Math. Soc., **54**, (1996), 89–101.

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