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SOME INTERPOLATIONS AND REFINEMENTS OF HADAMARD'S INEQUALITY FOR r–CONVEX FUNCTIONS IN CARNOT GROUPS

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Abstract. Some nontrivial increasing functions will be interpolated in the Hadamard's inequality for r-convex function in Carnot groups. The methods are more natural and allow us to extend the condition $\Gamma^2(G)\cap C^w_{H,r}(G)$ to the condition $C(G)\cap C^w_{H,r}(G)$.

1. Introduction

The classical Hadamard's inequality for convex functions states that if $f : [a,b] \to R$ is convex, then

$$\frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}.$$

The generalizations of the Hadamard's inequality to the integral power mean of a positive convex function on an interval $[a,b]$, and to that of a positive r-convex function on an interval $[a,b]$ are obtained by Pearce and Pečarić and other (see [3, 6, 7, 9, 12]). The definition of r-convexity naturally complements the concept of r-concavity, in which the inequality is reversed (see [11]) and which plays an important role in statistics.

Recently, Danielli et al. [1] and Lu et al. [4] have investigated some interesting properties and notions of convex function in Carnot group. Quite recently, Mingbao Sun and Xiaoping Yang [10] have introduced the concept of r-convex in Carnot groups, and derive from their results in the Abelian case $G=R$ more extensive results than the main results in [3, 6, 7, 9].

In what follows, the definition of Carnot groups is introduced, for more details the reader is referred to the paper [2, 5]. A Carnot groups G is a stratified, nilpotent Lie group of step r, with Lie algebra $\mathscr{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$. This means that $[V_1, V_j] = V_{j+1}$ for $j = 1, 2, \ldots, r-1$, whereas $[V_1, V_j] = \{0\}$. For points g, g_0 in G, we denote by $L_{g_0}(g) = g_0 g$ the left-translations on G by an element $g_0 \in G$. Let $m = \dim V_1$, and $X_1, X_2, \ldots, X_m$ be a fixed orthornormal basis of the first layer V_1.

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We continue to denote by $X_1, X_2, \dots, X_m$ the corresponding system of left-invariant vector fields on G. The exponential map $\exp : \mathscr{G} \to G$ is an analytic diffeomorphism, which allows us to define analytic maps $\xi_i : G \to V_i$ for $i = 1, 2, \dots, r$, by letting $g = \exp(\xi_1(g) + \cdots + \xi_r(g))$ for $g \in G$. The mapping $\xi : G \to \mathscr{G}$ is defined by $\xi(g) = \xi_1(g) + \cdots + \xi_r(g)$ is the inverse of the exponential mapping. The stratification of the Lie algebra allows us to define a natural family of nonisotropic dilations $\Delta_\lambda : \mathscr{G} \to \mathscr{G}$ as follows:

$$\Delta_\lambda \xi(g) = \lambda \xi_1(g) + \lambda^2 \xi_2(g) + \cdots + \lambda^r \xi_r(g).$$

Therefore the exponential map induces a group of dilations on G via the formula

$$\delta_\lambda(g) = \exp \circ \Delta_\lambda \circ \exp^{-1}(g), \qquad g \in G.$$

For a given function $f : G \to R$, the action of X_j on f is specified by the equation

$$X_j f(g) = \lim_{t \to 0} \frac{f(g \exp(tX_j)) - f(g)}{t} = \frac{d}{dt} f(g \exp(tX_j)) \Big|_{t=0}.$$

For a given open set $\Omega \in G$, the classes $\Gamma^1(\Omega)$ (respectively, $\Gamma^2(\Omega)$) represent the collection of all functions $u : \Omega \to R$ such that the derivatives $X_\alpha u$ (respectively, $X_\alpha X_{\alpha'} u$), $\alpha, \alpha' = 1, \dots, m$ exist and are continuous functions in Ω. We denote by dg the bi-invariant Haar measure on G obtained by pushing forward Lebesgue on $\mathscr{G}$ via the exponential map. Given a point, the horizontal plane through g_0 as the m-dimensional embedded submanifold of G is given by

$$H_{g_0} = L_{g_0}(\exp(V_1 \times \{0\})),$$

where 0 denotes the $(N-m)$-dimensional zero vector in $\mathscr{G}$, with $N = \dim V_1 + \cdots + \dim V_r$. For a function $u : G \to R$ with $u \in \Gamma^2(G)$, the symmetrized horizontal Hessian of u at $g \in G$ is the $m \times m$ matrix

$$(X^2 u)^*(g) = \left((X^2 u)^*_{i,j}(g) \right)_{i,j=1}^{m}$$

defined by

$$(X^2 u)^*(g) \overset{def}{=} \frac{1}{2} \{ (X_i X_j u(g) + X_i X_j u(g) \}.$$

In what follows, the definition of r-convexity in Carnot group G and some extended means will be given. Given two point $g, g' \in G$, for $\lambda \in [0,1]$, denote

$$g_\lambda = g \delta_\lambda (g^{-1} g').$$

A function $u : G \to (-\infty, \infty]$ is called weakly H-convex (see [1]) if it is proper, which means that $\{g \in G : u(g) = \infty\} \neq G$, and if for any $g \in G$ one has for every $\lambda \in [0,1]$,

$$u(g_\lambda) \leqslant \lambda u(g') + (1-\lambda) u(g)$$

for every $g' \in H_g$.

Recall the power mean $M_r(x,y;\lambda)$ of order r of positive numbers x,y is defined by

$$M_r(x,y;\lambda)=\begin{cases}(\lambda x^r+(1-\lambda)y^r)^{\frac{1}{r}}, & \text{if } r\neq 0,\\ x^{\lambda}y^{1-\lambda}, & \text{if } r=0.\end{cases}$$

A natural idea of weakly H-r-convexity may be introduced via power means.

A function $u:G\to(0,+\infty]$ is said to be weakly H-r-convex if it is proper, and if for any $g\in G$, one has for any $\lambda\in[0,1]$, $u(g_\lambda)\leqslant M_r(u(g'),u(g);\lambda)$ for every $g'\in H_g$. The class of all weakly H-r-convex functions on G is denoted by $C^w_{H,r}(G)$.

A function $u:G\to(0,+\infty]$ is called strongly H-r-convex if it is proper, and if for any $g,g'\in G$, one has $u(g_\lambda)\leqslant M_r(u(g'),u(g);\lambda)$ for every $\lambda\in[0,1]$. The class of all strongly H-r-convex functions on G is denoted by $C^s_{H,r}(G)$.

The above definition of weakly (strongly) H-r-convexity naturally complements the concept of weakly (strongly) H-r-concavity in which the inequality is reversed. The definition of weakly (strongly) H-r-convexity can be expanded as

$$u(g_\lambda)\leqslant\begin{cases}\Big(\lambda u^r(g')+(1-\lambda)u^r(g)\Big)^{\frac{1}{r}}, & \text{if } r\neq 0,\\ u^{\lambda}(g')u^{(1-\lambda)}(g), & \text{if } r=0.\end{cases}\tag{1.1}$$

In particular, when $r=1$, $C^w_{H,1}(G)(C^s_{H,1}(G))$ is just the class of all weakly (strongly) H-convex functions on G where the requirement that an r-convex function must be positive can clearly be relaxed, and is denoted simply by $C^w_H(G)(C^s_H(G))$ (see [1]). When $r=0$, we have that 0-convex function is called weakly (strongly) H-log-convex function.

It is well known that the extended means $E(r,s;x,y)$ (see [8]) are given by $E(r,s;x,x)=x$ *if* $x=y>0$ and for $x\neq y$ by

$$E(r,s;x,y)=\left[\frac{s}{r}\frac{y^r-x^r}{y^s-x^s}\right]^{\frac{1}{(r-s)}},\quad rs(r-s)\neq 0,$$

$$E(r,0;x,y)=E(0,r;x,y)=\left[\frac{1}{r}\frac{y^r-x^r}{\ln y-\ln x}\right]^{\frac{1}{r}},\quad r\neq 0,$$

$$E(r,r;x,y)=e^{\frac{-1}{r}}\left[\frac{x^{x^r}}{y^{y^r}}\right]^{\frac{1}{(x^r-y^r)}},\quad r\neq 0,$$

$$E(0,0;x,y)=\sqrt{xy}.$$

Clearly, $E(p+1,1;x,y)$ is the extended logarithmic mean $L_p(x,y)$ of two positive numbers x,y, while $E(r+1,r;x,y)$ is also the alternative extended logarithmic mean $F_r(x,y)$ of two positive numbers x, y.

Let u be a positive function on G , and $u(g_\lambda)$ be an integrable function on $[0,1]$ with respect to λ for every $g,g'\in G$. Define the two-parameter mean of the function $u(g_\lambda)$ on $[0,1]$ with respect to λ by

$$M_{p,q}(u;g,g')=\begin{cases}\left[\frac{\int_0^1 u^p(g_\lambda)d\lambda}{\int_0^1 u^q(g_\lambda)d\lambda}\right]^{\frac{1}{(p-q)}}, & \text{if } p\neq q,\\ \exp\frac{\int_0^1 u^q(g_\lambda)\ln u(g_\lambda)d\lambda}{\int_0^1 u^q(g_\lambda)d\lambda}, & \text{if } p=q.\end{cases}$$

In particular, when $q = 0$, denote $M_{p,0}(u;g,g') = M_p(u;g,g')$, which is the integral power mean. In [10], Mingbzo Sun and Xiaoping Yang established the following Theorems. These results establish a generalization of the classical Hadamard's inequality for weakly H-r-convex functions in a Carnot group G, which subsumes the relationship between extended mean values and weakly H-r-convex functions on G as special cases. It is easy to see that the following results continue to be valid if in their statements we replace $C^w_{H,r}(G)$ with $C^s_{H,r}(G)$.

THEOREM A. *Let G be a Carnot groups, given $u : G \to (0,+\infty]$, with $u \in C(G) \cap C^w_{H,r}(G)$, and $F : (0,+\infty) \to R$. For any fixed $g \in G$, and for every $g' \in H_g$, if F is increasing on $u[g,g'] = \{u(g_\lambda) : 0 \leqslant \lambda \leqslant 1\}$, then*

$$\int_0^1 F(u(g_\lambda))d\lambda \leqslant \frac{r\int_{u(g)}^{u(g')} x^{r-1}F(x)dx}{u^r(g') - u^r(g)}, \tag{1.2}$$

for $u(g) = u(g')$, the right-hand side of (1.2) is defined by $F(u(g))$, while if F is decreasing, the inequality (1.2) is reversed.

THEOREM B. *Let G be a Carnot groups, given $u : G \to (0,+\infty)$, with $u \in \Gamma^2(G) \cap C^w_{H,r}(G)$, and $F_1, F_2 : (0,+\infty) \to R$. For any fixed $g \in G$, and for every $g' \in H_g$, suppose that $u[g,g'] = \{u(g_\lambda) : 0 \leqslant \lambda \leqslant 1\}$, F_2 a positive integrable function on $u[g,g']$ and the ratio F_1/F_2 is increasing on $u[g,g']$, then*

$$\frac{\int_0^1 F_1(u(g_\lambda))d\lambda}{\int_0^1 F_2(u(g_\lambda))d\lambda} \leqslant \frac{\int_{u(g)}^{u(g')} x^{r-1}F_1(x)dx}{\int_{u(g)}^{u(g')} x^{r-1}F_2(x)dx}, \tag{1.3}$$

for $u(g) = u(g')$, the right-hand side of (1.3) is defined by $F_1(u(g))/F_2(u(g'))$, while if F_1/F_2 is decreasing, the inequality (1.3) is reversed.

Sun and Yang have also proved that Theorem B is valid when the condition $u \in \Gamma^2(G) \cap C^w_{H,r}(G)$ replaces by $u \in C(G) \cap C^w_{H,r}(G)$ in the case of $r = 1$ or 0. Applying the Theorem A and Theorem B, they also obtain the some interesting inequalities that subsume the relationship between extended mean values and weakly H-r-convex functions on Carnot groups.

We note that, Sun and Yang can obtain the inequality (1.2) by taking $F_2 \equiv 1$ in Theorem B, but they prove that only in the case of $u \in \Gamma^2(G) \cap C^w_{H,r}(G)$. Also, the proof of Theorem B is very complicated. The main purpose of this paper is to establish a connection to inequality (1.1) for weakly (strongly) H-r-convexity, and then some nontrivial increasing functions shall be interpolated in the inequalities (1.2), (1.3) and some interesting inequalities on Carnot groups established by Sun and Yang, respectively. The methods are more natural and allow us to extend the condition $\Gamma^2(G) \cap C^w_{H,r}(G)$ to the condition $C(G) \cap C^w_{H,r}(G)$ in Theorem B. In the Section 3, the Abelian case $G = R$, the interpolations, extensions and generalizations of the results given by the authors in [3, 6, 7, 9, 10, 12] will be pointed out.

2. Main results

The following Theorem plays an improtant role in the proof of the next two results.

THEOREM 2.1. *Let G be a Carnot groups and $u : G \to (0,\infty)$ with $u \in C(G) \cap C^{w}_{H,r}(G)$. For $0 \leqslant \lambda \leqslant 1$, for any fixed $g_0 \in G$ and for every $g_1 \in H_{g_0}$ and if $h_\lambda : [0,1] \to (0,\infty)$ is defined by*

$$h_\lambda(t) = \begin{cases} \left[\lambda u^r(g_{t+(1-t)\lambda}) + (1-\lambda)u^r(g_{(1-t)\lambda})\right]^{\frac{1}{r}}, & \text{if } r \neq 0, \\ u^\lambda(g_{t+(1-t)\lambda})u^{(1-\lambda)}(g_{(1-t)\lambda}), & \text{if } r = 0, \end{cases} \tag{2.1}$$

where $g_{(1-t)\lambda} \in G$ and $g_{t+(1-t)\lambda} \in H_{g_{(1-t)\lambda}}$, then

(i) $h_\lambda(t)$ is increasing on $[0,1]$;

(ii) $h_\lambda(0) = u(g_\lambda)$ and $h_\lambda(1) = \begin{cases} (\lambda u^r(g_1) + (1-\lambda)u^r(g_0))^{\frac{1}{r}}, & \text{if } r \neq 0, \\ u^\lambda(g_1)\cdot u^{1-\lambda}(g_0), & \text{if } r = 0. \end{cases}$

Proof. For any $t \in [0,1]$, using r-convexity of u, we obtain that for $g_{(1-t)\lambda} \in G$, $g_{t+(1-t)\lambda} \in H_{g_{(1-t)\lambda}}$, and for every $\alpha \in [0,1]$,

$$u(g_{t\alpha+(1-t)\lambda}) \leqslant \begin{cases} \Big[(t\alpha + (1-t)\lambda)u^r(g_{t+(1-t)\lambda}) \\ \qquad + (1 - t\alpha - (1-t)\lambda)u^r(g_{(1-t)\lambda})\Big]^{\frac{1}{r}}, & \text{if } r \neq 0, \\ u^{(t\alpha+(1-t)\lambda)}(g_{t+(1-t)\lambda})\cdot u^{(1-t\alpha-(1-t)\lambda)}(g_{(1-t)\lambda}), & \text{if } r = 0, \end{cases}$$

where $g_{t\alpha+(1-t)\lambda} = g_{(1-t)\lambda}\delta_{t\alpha+(1-t)\lambda}g^{-1}_{(1-t)\lambda}g_{t+(1-t)\lambda}$.

Let $0 \leqslant x < y \leqslant 1$. Taking $t = y$ and $\alpha = \frac{x+(y-x)\lambda}{y}$ in the above inequality, we have

$$u(g_{x+(1-x)\lambda}) \leqslant \begin{cases} \Big[(x + (1-x)\lambda)u^r(g_{y+(1-y)\lambda}) \\ \qquad + (1 - x - (1-x)\lambda)u^r(g_{(1-y)\lambda})\Big]^{\frac{1}{r}}, & \text{if } r \neq 0, \\ u^{(x+(1-x)\lambda)}(g_{y+(1-y)\lambda})\cdot u^{(1-x-(1-x)\lambda)}(g_{(1-y)\lambda}), & \text{if } r = 0, \end{cases}$$

Similarly, if $t = y$ and $\alpha = \frac{(y-x)\lambda}{y}$, we have

$$u(g_{(1-x)\lambda}) \leqslant \begin{cases} \left[((1-x)\lambda)u^r(g_{y+(1-y)\lambda}) + (1-(1-x)\lambda)u^r(g_{(1-y)\lambda})\right]^{\frac{1}{r}}, & \text{if } r \neq 0, \\ u^{((1-x)\lambda)}(g_{y+(1-y)\lambda})\cdot u^{(1-(1-x)\lambda)}(g_{(1-y)\lambda}), & \text{if } r = 0, \end{cases}$$

Thus, for $r \neq 0$, we have

$$\begin{aligned} h_\lambda(x) &= [\lambda u^r(g_{x+(1-x)\lambda}) + (1-\lambda)u^r(g_{(1-x)\lambda})]^{\frac{1}{r}} \\ &\leqslant \{\lambda[(x+(1-x)\lambda)u^r(g_{y+(1-y)\lambda}) + (1-x-(1-x)\lambda)u^r(g_{(1-y)\lambda})] \\ &\quad + (1-\lambda)[(1-x)\lambda u^r(g_{y+(1-y)\lambda}) + (1-(1-x)\lambda)u^r(g_{(1-y)\lambda})]\}^{\frac{1}{r}} \\ &= [\lambda u^r(g_{y+(1-y)\lambda}) + (1-\lambda)u^r(g_{(1-y)\lambda})]^{\frac{1}{r}} \\ &= h_\lambda(y), \end{aligned}$$

and for $r = 0$, we have

$$\begin{aligned} h_\lambda(x) &= u^\lambda(g_{x+(1-x)\lambda}) \cdot u^{(1-\lambda)}(g_{(1-x)\lambda}) \\ &\leqslant \left[u^{(x+(1-x)\lambda)}(g_{y+(1-y)\lambda}) \cdot u^{(1-x-(1-x)\lambda)}(g_{(1-y)\lambda})\right]^\lambda \times \\ &\quad \times \left[u^{(1-x)\lambda}(g_{y+(1-y)\lambda}) \cdot u^{(1-(1-x)\lambda)}(g_{(1-y)\lambda})\right]^{(1-\lambda)} \\ &= u^\lambda(g_{y+(1-y)\lambda}) \cdot u^{(1-\lambda)}(g_{(1-y)\lambda}) \\ &= h_\lambda(y). \end{aligned}$$

This completes the proof of (i). The proofs of (ii) are obvious, we omit it. Thus, the proofs of Theorem 2.1 are complete.

REMARK 2.2. It is valid that $C^w_{H,r}(G)$ replaces by $C^s_{H,r}(G)$ in Theorem 2.1, and if u is weakly (strongly) H-r-concavity then $h_\lambda(t)$ is decreasing decreasing on [0,1]. Also, we note that the $h_\lambda(t)$ in (2.1) is a mapping in connection to inequality (1.1) for weakly (strongly) H-r-convexity.

Applying Theorm 2.1, we obtain the following two Theorems easily.

THEOREM 2.3. *Let G be a Carnot groups, given $u : G \to (0,+\infty)$ with $u \in C(G) \cap H^w_{H,r}(G)$, and $F : (0,+\infty) \to R$. For $0 \leqslant \lambda \leqslant 1$, for any fixed $g_0 \in G$ and for every $g_1 \in H_{g_0}$, and suppose that $Q : [0,1] \to R$ is defined by $Q(t) = \int_0^1 F(h_\lambda(t))d\lambda$, where $h_\lambda(t)$ is defined as in (2.1). If F is increasing on $[h_\lambda(0), h_\lambda(1)]$, then*

(i) $Q(t)$ is increasing on $[0,1]$;

(ii) $Q(0) = \displaystyle\int_0^1 F(u(g_\lambda))d\lambda$, and

$$Q(1) = \begin{cases} \dfrac{r\int_{u(g_0)}^{u(g_1)} x^{r-1}F(x)dx}{u^r(g_1) - u^r(g_0)} & \text{if } r \neq 0, \\ \displaystyle\int_{u(g_0)}^{u(g_1)} \frac{F(x)}{x(\ln u(g_1) - \ln u(g_0))}dx, & \text{if } r = 0, \end{cases}$$

for $u(g_0) = u(g_1)$, $Q(1)$ is defined by $F(u(g_0))$, while if F is decreasing, the $Q(t)$ is decreasing on $[0,1]$.

Proof. By Theorem 2.1, we have $h_\lambda(t)$ is increasing on [0,1], and using F is increasing on $[h_\lambda(0), h_\lambda(1)]$, we obtain the $Q(t)$ is increasing on [0,1]. This completes the proof of (i). Proof of (ii). The first term of (ii) is obvious. To prove the second term of (ii) first we suppose that $u(g_0) = u(g_1)$, by definition on $Q(1)$, we have $Q(1) = F(u(g_0))$, so that we may assume $u(g_0) \neq u(g_1)$. By (ii) in Theorem 2.1, we obtain

$$Q(1) = \int_0^1 F(h_\lambda(1))d\lambda = \begin{cases} \int_0^1 F\Big((\lambda u^r(g_1) + (1-\lambda)u^r(g_0))^{\frac{1}{r}}\Big)d\lambda, & \text{if } r \neq 0, \\ \int_0^1 F\Big(u^\lambda(g_1)\cdot u^{1-\lambda}(g_0)\Big)d\lambda, & \text{if } r = 0. \end{cases}$$

By changing the variable, we have the second term of (ii). This completes the proof of (ii). Thus the proofs of Theorem 2.3 are complete.

For the convenience, we define the following sets.

The following functions are all the real-valued function defined on $[0,\infty)$. Let

$I_1=\{(f,g) \mid f$ is nonnegative and nondecreasing and g is positive and nonincreasing$\}$,
$I_2=\{(f,g) \mid f$ is nonnegative and nonincreasing and g is negative and nonincreasing$\}$,
$I_3=\{(f,g) \mid f$ is nonpositive and nondecreasing and g is positive and nondecreasing$\}$,
$I_4=\{(f,g) \mid f$ is nonpositive and nonincreasing and g is negative and nondecreasing$\}$,
$J_1=\{(f,g) \mid f$ is nonnegative and nonincreasing and g is positive and nondecreasing$\}$,
$J_2=\{(f,g) \mid f$ is nonnegative and nondecreasing and g is negative and nondecreasing$\}$,
$J_3=\{(f,g) \mid f$ is nonpositive and nonincreasing and g is positive and nonincreasing$\}$,
$J_4=\{(f,g) \mid f$ is nonpositive and nondecreasing and g is negative and nonincreasing$\}$,
$D_1 = I_1 \cup I_2 \cup I_3 \cup I_4$ and $D_2 = J_1 \cup J_2 \cup J_3 \cup J_4$.

THEOREM 2.4. *Let G be a Carnot groups, given $u : G \to (0,+\infty)$ with $u \in C(G) \cap H^w_{H,r}(G)$, and $F_1, F_2 : (0,+\infty) \to R$. For $0 \leqslant \lambda \leqslant 1$, for any fixed $g_0 \in G$ and for every $g_1 \in H_{g_0}$, and suppose that $Q : [0,1] \to R$ is defined by $Q(t) = \int_0^1 F_1(h_\lambda(t))d\lambda / \int_0^1 F_2(h_\lambda(t))d\lambda$, where $h_\lambda(t)$ as defined in $(2,1)$, and F_1 and F_2 integrable on $[h_\lambda(0), h_\lambda(1)]$.If $(F_1,F_2) \in D_1$, then*

(i) $Q(t)$ *is increasing on* $[0,1]$, *and*

(ii) $Q(0) = \dfrac{\int_0^1 F_1(u(g_\lambda))d\lambda}{\int_0^1 F_2(u(g_\lambda))d\lambda}$,

$$Q(1) = \frac{\int_{u(g_0)}^{u(g_1)} x^{r-1}F_1(x)dx}{\int_{u(g_0)}^{u(g_1)} x^{r-1}F_2(x)dx},$$

for $u(g_0) = u(g_1)$, the $Q(1)$ is defined by $F_1(u(g_0))/F_2(u(g_0))$, while if $(F_1,F_2) \in D_2$, then $Q(t)$ is decreasing on $[0,1]$.

Proof. Let $0 \leqslant s < t \leqslant 1$ and $0 \leqslant \lambda \leqslant 1$, we have

$$
\begin{aligned}
Q(t) - Q(s) &= \frac{\int_0^1 F_1(h_\lambda(t))d\lambda}{\int_0^1 F_2(h_\lambda(t))d\lambda} - \frac{\int_0^1 F_1(h_\lambda(t))d\lambda}{\int_0^1 F_2(h_\lambda(s))d\lambda} \\
&\quad + \frac{\int_0^1 F_1(h_\lambda(t))d\lambda}{\int_0^1 F_2(h_\lambda(s))d\lambda} - \frac{\int_0^1 F_1(h_\lambda(s))d\lambda}{\int_0^1 F_2(h_\lambda(s))d\lambda} \\
&= \frac{\int_0^1 F_1(h_\lambda(t))d\lambda \cdot \int_0^1 \Big(F_2(h_\lambda(s))d\lambda - F_2(h_\lambda(t))\Big)d\lambda}{\int_0^1 F_2(h_\lambda(t))d\lambda \cdot \int_0^1 F_2(h_\lambda(s))d\lambda} \\
&\quad + \frac{1}{\int_0^1 F_2(h_\lambda(s))d\lambda}\Big(\int_0^1 (F_1(h_\lambda(t)) - F_1(h_\lambda(s)))d\lambda\Big).
\end{aligned}
$$

By Theorem 2.1, we have $h_\lambda(s) \leqslant h_\lambda(t)$ and using $(F_1, F_2) \in D_1$, we obtain $Q(t) - Q(s) \geqslant 0$. This completes the proof of (i).

To prove (ii) first we observe that

$$
Q(0) = \frac{\int_0^1 F_1(u(g_\lambda))d\lambda}{\int_0^1 F_2(u(g_\lambda))d\lambda}.
$$

To prove the second term of (ii) we suppose that $u(g_0) = u(g_1)$, then it is obvious $Q(1) = F_1(u(g_0))/F_2(u(g_1))$, so that we may assume $u(g_0) \neq u(g_1)$. By changing the variable, we have

$$
Q(1) = \frac{\int_0^1 F_1\Big((\lambda u^r(g_1) + (1-\lambda)u^r(g_0))^{\frac{1}{r}}\Big)d\lambda}{\int_0^1 F_2\Big((\lambda u^r(g_1) + (1-\lambda)u^r(g_0))^{\frac{1}{r}}\Big)d\lambda} = \frac{\int_{u(g_0)}^{u(g_1)} x^{r-1}F_1(x)dx}{\int_{u(g_0)}^{u(g_1)} x^{r-1}F_2(x)dx}.
$$

This completes the proof of (ii).

REMARK 2.5. It is obvious that Theorem 2.3 and Theorem 2.4 comtinue to be valid if in their statements we replace $C^w_{H,r}(G)$ with $C^s_{H,r}(G)$. Similarly, under the assumption of Theorem 2.3 and Theorem 2.4, respectively, if u is weakly (strongly) H-r-concave, we can derive that $Q(t)$ is decreasing on $[0,1]$ in the case of F be increasing and $(F_1, F_2) \in D_1$, respectively, while $Q(t)$ is increasing on $[0,1]$ is valid in the case of F be decreasing and $(F_1, F_2) \in D_2$, respectively.

REMARK 2.6. In Theorem 2.3, the nontrivial incresing function $Q(t)$ is interpolated in the inequality (1.2). In Theorem 2.4, the condition $u \in \Gamma^2(G) \cap H^w_{H,r}(G)$ in Theorem B is extended to $u \in C(G) \cap H^w_{H,r}(G)$, and the nontrivial increasing function $Q(t)$ is interpolated in the inequality (1.3). Obviously, we can obtain the Theorem 2.3 by taking $F_2 \equiv 1$ in Theorem 2.4, but we only give the proof in the case of $F_1 \in I_1 \cup I_3$.

REMARK 2.7. Using the Theorem 2.3 and Theorem 2.4, we have the following interpolations of the interesting inequalities given by Sum and Yan in $[10]$ that subsume the relationship between extended mean value and weakly H-r-convex functions on

Carnot groups. Also, we note that the inequality (2.4) and (2.5) in [10] only hold in the case of $p-q>0$ and $p>0$, respectively.

If take $F_1(x)=x^p$, $F_2(x)=x^q$ for $0<x<\infty$ and for any real numbers p,q with $p>0>q$ in Theorem 2.4, u as in Theorem 2.4, we have $(F_1,F_2)\in D_1$ and then we can derive the following inequality:

$$M_{p,q}(u;g_0,g_1)=Q(0)\leqslant Q(t)\leqslant Q(1)=E(p+r,q+r,u(g_1),u(g_0)) \tag{2.2}$$

where $Q(t)=\int_0^1 h_\lambda^p(t)d\lambda/\int_0^1 h_\lambda^q(t)d\lambda$, $h_\lambda(t)$ is defined as in (2.1) and $Q(t)$ is increasing on $[0,1]$. This proves that the $Q(t)$ in (2.2) is interpolated in inequality (2.4) in [10]. Also, we note that the condition u in (2.2) is weaker than u in (2.4) in [10]. Similarly, taking $F(x)=x^p$ for $0<x<\infty$ in Theorem 2.3 and for any real number $p>0$, we have F is increasing and then we can derive the following inequality:

$$M_p(u;g_0,g_1)=Q(0)\leqslant Q(t)\leqslant Q(1)=E(p+r,r,u(g_1),u(g_0)) \tag{2.3}$$

where $Q(t)=\int_0^1 h_\lambda^p(t)d\lambda$, $h_\lambda(t)$ is defined as in (2.1), and the $Q(t)$ is increasing on $[0,1]$. This prove that $Q(t)$ in (2.3) is interpolated in inequality (2.5) in [10].

Further, taking $r=1$ and $p=1$ in (2.3), respectively, we have

$$M_p(u;g_0,g_1)=Q(0)\leqslant Q(t)\leqslant Q(1)=L_p(u(g_1),u(g_0)) \tag{2.4}$$

where $Q(t)=\int_0^1[\lambda u(g_{t+(1-t)\lambda})+(1-\lambda)u(g_{(1-t)\lambda})]^p d\lambda$, and

$$\int_0^1 u(g_\lambda)d\lambda=Q(0)\leqslant Q(t)\leqslant Q(1)=F_r(u(g_1),u(g_0)) \tag{2.5}$$

where $Q(t)=\int_0^1 h_\lambda(t)dt$, $h_\lambda(t)$ is defined as in (2.1), respectively, The $Q(t)$ in (2.4) and the $Q(t)$ in (2.5) are interpolated in the inequality $M_p(u;g_0,g_1)\leqslant L_p(u(g_1),u(g_0))$ and $\int_0^1 u(g_\lambda)d\lambda\leqslant F_r(u(g_1),u(g_0))$ given in [10], respectively.

It is easy to see that the above results continuous to be valid if in their statements we replace $C^w_{H,r}(G)$ with $C^s_{H,r}(G)$.

3. Applications

In the Abelian case, when $G=(R,+)$ and $u=f$ is a positive function on $(-\infty,+\infty)$ in Theorem 2.3 and Theorem 2.4, respectively, we have the following two Corollaries.

COROLLARY 3.1. *Let f be a positive function on $(-\infty,+\infty)$, F be a real-valued function on $(0,+\infty)$, and $Q:[0,1]\to R$ be defined by*

$$Q(t)=\int_0^1 F(k_\lambda(t))d\lambda \tag{3.1}$$

where

$$k_\lambda(t) = \begin{cases} [\lambda f^r((t+\lambda-t\lambda)b+(1-t-\lambda+t\lambda)a) \\ \qquad +(1-\lambda)f^r((\lambda-t\lambda)b+(1-\lambda+t\lambda)a)]^{\frac{1}{r}}, & \text{if } r \neq 0, \\ f^\lambda((t+\lambda-t\lambda)b+(1-t-\lambda+t\lambda)a)\times \\ \qquad \times f^{1-\lambda}((\lambda-t\lambda)b+(1-\lambda+t\lambda)a), & \text{if } r = 0. \end{cases} \tag{3.2}$$

When F is increasing on $[k_\lambda(0), k_\lambda(1)]$, if f is r-convex, then

(i) $Q(t)$ is increasing on $[0,1]$;

(ii) $Q(0) = \dfrac{1}{b-a}\displaystyle\int_a^b F(f(x))dx,$

$$Q(1) = \begin{cases} \dfrac{r\int_{f(a)}^{f(b)} x^{r-1}F(x)dx}{f^r(b)-f^r(a)}, & \text{if } r \neq 0, \\ \displaystyle\int_{f(a)}^{f(b)} \frac{F(x)}{x(\ln f(b)-\ln(a))}dx, & \text{if } r = 0, \end{cases}$$

for $f(a) = f(b)$, the $Q(1)$ is defined by $F(f(a))$, while if f is r-concave, $Q(t)$ is decreasing. When F is decreasing, if f is r-convex, $Q(t)$ is decreasing, while if f is r-concave, $Q(t)$ is increasing.

COROLLARY 3.2. *Let f be a positive function on $(-\infty,+\infty)$, F_1, F_2 real-valued functions on $(0,+\infty)$, and $Q:[0,1]\to R$ be defined by*

$$Q(t) = \frac{\int_0^1 F_1(k_\lambda(t))d\lambda}{\int_0^1 F_2(k_\lambda(t))d\lambda} \tag{3.3}$$

where $k_\lambda(t)$ defined as in (3.2), and F_1, F_2 be integrable function on $[k_\lambda(0), k_\lambda(1)]$. When $(F_1,F_2)\in D_1$, D_1 as defined in Section 2, if f is r-convex, than

(i) $Q(t)$ is increasing on $[0,1]$, and

(ii) $Q(0) = \dfrac{\int_a^b F_1(f(x))dx}{\int_a^b F_2(f(x))dx},$

$$Q(1) = \frac{\int_{f(a)}^{f(b)} x^{r-1}F_1(x)dx}{\int_{f(a)}^{f(b)} x^{r-1}F_2(x)dx},$$

for $f(a) = f(b)$, the $Q(1)$ is defined by $F_1(f(a))/F_2(f(a))$, while if f is r-concave, the $Q(t)$ is decreasing on $[0,1]$. When $(F_1,F_2)\in D_2$, if f is r-convex, the $Q(t)$ is decreasing on $[0,1]$, while if f is r-concave, $Q(t)$ is increasing on $[0,1]$.

REMARK 3.3. The increasing function $Q(t)$ in Corollary 3.1 and $Q(t)$ in Corollary 3.2 are interpolated in the inequality (5.3) and (5.1) in [10], respectively. Also, we note that the restrictive conditions of the function f in Corollary 3.2 are weaker than the ones in Corollary 5.1 in [10].

REMARK 3.4. We note that the inequality (5.4) and (5.5) in [10] only holds in the case of $p-q>0$ and $p>0$, respectively. If we suitably choose the F in Corollary 3.1 and F_1, F_2 in Corollary 3.2, respectively, we can obtain the interpolations, extentions and generalizations of the results given in [3, 6, 7, 9]. We leave the details for the reader. Specially, if we take $F(x)=x^p$ in Corollary 3.1 and change the variable to (3.1), then Corollary 3.1 reduce to the Theorem given by Yang and the author in [12].

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