Mathematical
Inequalities
& Applications
Volume 10, Number 2 (2007), 299–310

OPTIMAL BOUNDS FOR LINEAR FUNCTIONALS ON MONOTONE FUNCTIONS

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(*communicated by I. Olkin*)

Abstract. We determine optimal bounds on linear functionals over the space of square integrable functions on a finite interval, restricted to the nondecreasing elements of the subspace orthogonal to constants. We discuss conditions of bounds attainability and present exemplary applications.

1. Introduction and motivation

It is well known (see, e.g., Dunford and Schwarz [2, Theorem 1, p. 286]) that every continuous linear functional on the space $L^2(a,b)$ of square integrable functions over a (finite) interval $[a,b]$ has the form

$$T_h(g) = (g,h) = \int_a^b g(x)h(x)dx, \quad g \in L^2(a,b), \tag{1.1}$$

where h is an arbitrarily fixed element of $L^2(a,b)$. The norm of the functional amounts to

$$\begin{aligned} ||T_h|| &= \sup_{0\neq g\in L^2(a,b)} \frac{T_h(g)}{||g||} = \sup_{||g||=1} T_h(g) \\ &= ||h|| = \left(\int_a^b h^2(x)dx\right)^{1/2}. \end{aligned} \tag{1.2}$$

In the paper, we improve evaluation (1.2) under the restriction that functions g belong to the family

$$\mathcal{C} = \{g \in L^2(a,b) : \ g \nearrow, \ g \perp 1, \ ||g|| = 1\} \tag{1.3}$$

of nondecreasing, orthogonal to constants elements of $L^2(a,b)$ with the unit norm. Precisely, we try to establish

$$|||T_h||| = \sup_{g\in\mathcal{C}} T_h(g) = \sup_{g\in\mathcal{C}} \int_a^b g(x)h(x)dx \tag{1.4}$$

with arbitrary $h \in L^2(a,b)$. To avoid trivialities, we assume that $h \neq 0$.

Mathematics subject classification (2000): 26D15, 46E30.
Key words and phrases: L^2 -space, linear functional, monotone function, optimal bound.

Evaluations (1.4) have probabilistic interpretations. Consider, e.g., a sequence of independent identically distributed random variables $X_1, X_2, \ldots$ with a distribution function F, finite mean

$$\mu = \mathbb{E}X_1 = \int_0^1 F^{-1}(x)dx = (F^{-1}, 1) \tag{1.5}$$

and variance

$$\sigma^2 = \mathbb{V}\mathrm{ar}X_1 = \int_0^1 (F^{-1}(x) - \mu)^2 dx = ||F^{-1} - \mu||^2. \tag{1.6}$$

Let $M_n = \max\{X_1, \ldots, X_n\}$ and $m_n = \min\{X_1, \ldots, X_n\}$. It can be shown that

$$\mathbb{E}\frac{M_n - \mu}{\sigma} = \int_0^1 \frac{F^{-1}(x) - \mu}{||F^{-1} - \mu||} f_{n:n}(x)dx, \tag{1.7}$$

$$\mathbb{E}\frac{m_n - \mu}{\sigma} = \int_0^1 \frac{F^{-1}(x) - \mu}{||F^{-1} - \mu||} f_{1:n}(x)dx, \tag{1.8}$$

where

$$f_{n:n}(x) = nx^{n-1}, \tag{1.9}$$

$$f_{1:n}(x) = n(1-x)^{n-1}, \tag{1.10}$$

are the density functions of the maximum and minimum, respectively, of independent random variables uniformly distributed on $(0,1)$ (see, e.g., David and Nagaraja [1, p. 34]). Note that (1.7) and (1.8) are linear functionals on $L^2(0,1)$, represented by functions (1.9) and (1.10), respectively, whose arguments belong to (1.3) with $(a,b) = (0,1)$. Accordingly, evaluations (1.4) amount to the sharp bounds on the expectations of deviations of the sample extremes M_n and m_n from the population mean (1.5) in the standard deviation units σ (cf. (1.6)).

Many other similar functionals are studied in statistics. In particular, it is interesting to evaluate linear combinations $\sum_{i=1}^n c_i X_{i:n}$ of order statistics $X_{i:n}$ with fixed coefficients c_i, $1 \leqslant i \leqslant n$, where $X_{i:n}$ denotes the ith smallest value among $X_1, \ldots, X_n$. In this case we have

$$\mathbb{E}\sum_{i=1}^n c_i \frac{X_{i:n} - \mu}{\sigma} = \int_0^1 \frac{F^{-1}(x) - \mu}{||F^{-1} - \mu||} \sum_{i=1}^n c_i f_{i:n}(x)dx,$$

where

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} x^{i-1}(1-x)^{n-i}, \quad 1 \leqslant i \leqslant n, \tag{1.11}$$

are the densities of ith order statistics in the standard uniform sequence. Analogous evaluations are valid for records. An upper record in a numerical sequence is an element which is greater than all the preceding ones. If the sequence consists of independent identically continuously distributed random variables $X_1, \ldots, X_n, \ldots$, then the respective sequence of records $R_0 = X_1$ (by convention), $R_1, \dots, R_n, \dots$ is infinite, and the respective standardized expectations have the forms

$$\mathbb{E}\frac{R_n - \mu}{\sigma} = \int_0^1 \frac{F^{-1}(x) - \mu}{||F^{-1} - \mu||} f_n(x)dx,$$

with

$$f_n(x) = -\frac{[-\ln(1-x)]^n}{n!}, \quad n \geqslant 0, \tag{1.12}$$

(see David and Nagaraja [1, p. 32]). Further examples of stochastic notions that can be likewise evaluated are presented in Kamps [4] and Rychlik [9].

First observe that for some functions h functional (1.1) restricted to set (1.3) may take on values of a fixed sign. Consequently, in contrast to (1.2), bounds (1.4) for specific h can be negative as well. The methods of determining positive and negative bounds are absolutely different. They are presented in Sections 2 and 3, respectively. Their discrete counterparts for sequences, are described in Rychlik [8], and Goroncy and Rychlik [3] respectively.

We finally note that the sharp lower evaluations of (1.1) over (1.3) can be immediately deduced from the upper ones due to the obvious relation

$$\inf_{g \in \mathcal{C}} T_h(g) = -\sup_{g \in \mathcal{C}} T_{-h}(g).$$

Besides of some exceptional cases

$$\inf_{g \in \mathcal{C}} T_h(g) \neq -\sup_{g \in \mathcal{C}} T_h(g)$$

i.e., lower and upper evaluations over (1.3) are not symmetric about zero, unlike to the general ones.

2. Positive bounds

Since $g \in \mathcal{C}$ integrate to zero, we have

$$T_h(g) = \int_a^b g(x)[h(x) - c]dx \leqslant ||h - c||$$

for arbitrary real c, and

$$\inf_{c \in \mathbb{R}} ||h - c|| = ||h_0||,$$

with

$$h_0(x) = h(x) - \frac{1}{b-a}\int_a^b h(t)dt \tag{2.1}$$

being the projection of h onto the linear subspace of functions orthogonal to constants. The equality in

$$T_h(g) \leqslant ||h_0||, \quad g \in \mathcal{C}, \tag{2.2}$$

holds if

$$g(x) = \frac{h_0(x)}{||h_0||} \in \mathcal{C},$$

i.e., if h itself is nondecreasing. Otherwise inequality (2.2) can be improved by the results of the following two Theorems. A more general version of the first one can be found in Moriguti [7].

THEOREM 1. *Let $\underline{h}_0$ denote the (right-continuous version, say, of) derivative of the greatest convex minorant $\underline{H}_0$ of the antiderivative H_0 of (2.1). Then for arbitrary nondecreasing g for which the integrals below exists, we have*

$$\int_a^b g(x)h_0(x)dx \leqslant \int_a^b g(x)\underline{h}_0(x)dx. \tag{2.3}$$

The equality in (2.3) holds iff g is constant on every interval contained in the set

$$\mathcal{H} = \{a < x < b: \ \underline{H}_0(x) < H_0(x)\}. \tag{2.4}$$

Proof. The continuous function

$$H_0(x) = \int_a^x h_0(t)dt \tag{2.5}$$

has the null values at the interval end-points a and b. The greatest convex minorant $\underline{H}_0$ of H_0, defined as the supremum of all convex functions not greater H_0, is a convex function vanishing at a and b as well. It is continuous and differentiable except for at countably many points at most, where the left and right derivatives exist. Function $\underline{h}_0$ is therefore well defined. Set (2.4) is open and so consists of countably many disjoint open intervals, (a_i, b_i), say, at most. On each interval, $\underline{H}_0$ is linear, and has a particular form

$$\underline{H}_0(x) = H_0(a_i) + \frac{H_0(b_i) - H_0(a_i)}{b_i - a_i}(x - a_i), \quad a_i < x < b_i.$$

The derivative $\underline{h}_0$ has the constant value $\frac{H_0(b_i)-H_0(a_i)}{b_i-a_i}$ there. On the completion of (2.4), we have $\underline{H}_0(x) = H_0(x)$ and, consequently, $\underline{h}_0(x) = h_0(x)$. Since $\underline{H}_0(x) \leqslant H_0(x)$, $a \leqslant x \leqslant b$, with the equality at the end-points, we have

$$\begin{aligned}\int_a^b g(x)h(x)dx &= \int_a^b g(x)H_0(dx)\\ &\leqslant \int_a^b g(x)\underline{H}_0(dx) = \int_a^b g(x)\underline{h}_0(x)dx\end{aligned}$$

(cf., e.g., Marshall and Olkin [5, Proposition A. 2(ii), p. 444]). For establishing the equality conditions, we consider

$$\begin{aligned}\int_a^b g(x)[\underline{h}_0(x) - h(x)]dx &= \int_{\mathcal{H}} g(x)[\underline{h}_0(x) - h_0(x)]dx\\ &= \sum_i \left[\int_{a_i}^{b_i} g(x)\underline{H}_0(dx) - \int_{a_i}^{b_i} g(x)H_0(dx)\right].\end{aligned}$$

By definition, $\underline{H}_0(x) < H_0(x)$, $a_i < x < b_i$, and the functions are equal at a_i and b_i. Therefore

$$\int_{a_i}^{b_i} g(x)\underline{H}_0(dx) \geqslant \int_{a_i}^{b_i} g(x)H_0(dx)$$

for each i, and the equality holds iff g is constant on (a_i, b_i). This completes the proof. □

It can be shown that $\underline{h}_0$ defined in Theorem 1 is the projection of h_0 onto the convex cone of nondecreasing functions in $L^2(a,b)$ (see, e.g., Rychlik [9, pp. 14–16]). It is also the projection of h onto the cone of nondecreasing functions orthogonal to constant functions. Clearly, $\underline{h}_0 = h_0$ if h_0 (and the original h) is nondecreasing. We also easily check that the operators of projections onto the linear subspace orthogonal to constants and the convex cone of nondecreasing functions are interchangeable, and we have

$$\underline{h}_0(x) = \underline{h}(x) - \frac{1}{b-a}\int_a^b \underline{h}(t)dt.$$

THEOREM 2. *For every $g \in \mathcal{C}$ defined in* (1.3), *yields*

$$\int_a^b g(x)h(x)dx \leqslant \left(\int_a^b \underline{h}_0^2(x)dx\right)^{1/2} = ||\underline{h}_0||. \tag{2.6}$$

Moreover, if the antiderivative (2.5) *of* (2.1) *is negative for some* $a < x < b$, *than the right-hand side of* (2.6) *is positive and the equality is attained there by the unique function*

$$g_0(x) = \frac{\underline{h}_0(x)}{||\underline{h}_0||}. \tag{2.7}$$

Proof. Since $g \in \mathcal{C}$, we obtain

$$\begin{aligned}\int_a^b g(x)h(x)dx &= \int_a^b g(x)h_0(x)dx \\ &\leqslant \int_a^b g(x)\underline{h}_0(x)dx \leqslant ||g||\,||\underline{h}_0||,\end{aligned} \tag{2.8}$$

by Theorem 1 and the Schwarz inequality. By the latter assumption, $\underline{h}_0$ is a nonzero function and (2.7) is well defined. We easily check that this belongs to (1.3). Moreover, this is the only function with the unit norm which provides the equality in the Schwarz inequality of (2.8). Obviously, g_0 and $\underline{h}_0$ are constant on the same intervals (if any), and so are on all the intervals where $\underline{H}_0$ and H_0 differ, in particular. Theorem 1 implies that (2.7) attains the equality in the former inequality of (2.8) as well, which ends the proof. □

In particular, (1.9), (1.10), (1.11) and (1.12) are density functions on $[0,1]$, and respective modifications (2.1) arise by subtracting one from them. They are nondecreasing in cases (1.9) and (1.12), and so coincide with their projections onto the cone of nondecreasing functions. By Theorem 2, we immediately get the following sharp evaluations of expected maxima and records

$$\mathbb{E}\frac{M_n-\mu}{\sigma} \leqslant ||f_{n:n}-1|| = \frac{n-1}{(2n-1)^{1/2}},$$
$$\mathbb{E}\frac{R_n-\mu}{\sigma} \leqslant ||f_n-1|| = \left[\binom{2n}{n}-1\right]^{1/2}.$$

Functions $f_{i:n}-1$, $2\leqslant i\leqslant n-1$, defined in (1.11) are first increasing, and then decreasing. Respective antiderivatives are first convex decreasing, then convex increasing, and eventually concave increasing. Respective convex minorants first coincide with the original antiderivatives, and are ultimately linear. We can easily conclude that

$$\mathbb{E}\frac{X_{i:n}-\mu}{\sigma} \leqslant ||\underline{f}_{i:n}-1||,$$

where

$$\underline{f}_{i:n}(x) = f_{i:n}(\min\{x,\alpha\})$$

for a unique $0<\alpha<1$ satisfying

$$f_{i:n}(x) = \frac{1-F_{i:n}(x)}{1-x},$$

where

$$F_{i:n}(x) = \sum_{k=i}^{n}\binom{n}{k}x^k(1-x)^{n-k}$$

is the distribution function of (1.11). In the minimum case, the antiderivative $F_{1:n}(x)-x = 1-x-(1-x)^n$ of $f_{1:n}(x)-1 = n(1-x)^{n-1}-1$ is strictly concave on $(0,1)$, and the respective convex minorant and its derivative are zero functions. Therefore Theorem 2 provides a trivial bound

$$\mathbb{E}m_n-\mu = \mathbb{E}(\min\{X_1,\ldots,X_n\}-X_1) \leqslant 0.$$

In the next section we try to improve the inequality analyzing the case of functionals h whose projections $\underline{h}_0$ onto the family of nondecreasing functions orthogonal to constants amount to zero.

3. Nonpositive bounds

Now we focus on evaluations (1.4) on functionals determined by elements h for which bounds (2.6) of Theorem 2 are zero. Accordingly, we assume that the continuous function

$$H_0(x) = \int_a^x h(t)dt - \frac{x-a}{b-a}\int_a^b h(t)dt \tag{3.1}$$

has no negative values in $[a,b]$. The problem is easily solved by means of methods described in the previous section if (3.1) has a zero value at an interior point of the interval.

THEOREM 3. *If* $H_0(x) \geqslant 0$ *for all* $a \leqslant x \leqslant b$ *and* $H_0(\theta) = 0$ *for some* $a < \theta < b$, *then for all* $g \in \mathcal{C}$, *we have*

$$\int_a^b g(x)h(x)dx \leqslant 0, \tag{3.2}$$

and the equality holds for the function satisfying

$$g_\theta(x) = \begin{cases} -\left[\frac{b-\theta}{(\theta-a)(b-a)}\right]^{1/2}, & a < x < \theta, \\ \left[\frac{\theta-a}{(b-\theta)(b-a)}\right]^{1/2}, & \theta < x < b. \end{cases} \tag{3.3}$$

Proof. By assumption, $h_0(x) = 0$, $a \leqslant x \leqslant b$, and (3.2) immediately follows from (2.3). By Theorem 1 again, the equality holds if g is constant on (a, θ) and (θ, b), and a positive jump is admitted at θ. The only such a function that is orthogonal to constants and has the unit norm is presented in (3.3). □

If (3.1) has a single zero in (a, b), then (3.3) is the unique element of (1.3) attains the zero bound in (3.2). Clearly, we can freely choose its values at a, b, and θ. If (3.1) has several zeros, then there exist various stepwise functions in (1.3) with jumps at the zeros of H_0. If it happens that $H_0(x) = 0$ on an interval, then bound (3.2) is attained by functions strictly increasing on the interval.

All the remaining cases are treated in the following theorem.

THEOREM 4. *If* $H_0(x) > 0$ *for all* $a < x < b$, *then for every* $g \in \mathcal{C}$ *holds*

$$\int_a^b g(x)h(x)dx \leqslant -\inf_{a<x<b} H_0(x) \left[\frac{b-a}{(x-a)(b-x)}\right]^{1/2}. \tag{3.4}$$

If the infimum of the right-hand side of (3.4) *is attained at some* $a < \theta < b$, *then the equality holds there for*

$$g_\theta(x) = \begin{cases} -\frac{b-\theta}{(b-a)H_0(\theta)}, & a < x < \theta, \\ \frac{\theta-a}{(b-a)H_0(\theta)}, & \theta < x < b. \end{cases} \tag{3.5}$$

If the infimum is attained in the limit as $x \searrow a$ *(* $x \nearrow b$*, respectively), then the equality in* (3.4) *is attained in limit for sequences of functions* (3.5) *with* $\theta \searrow a$ *(* $\theta \nearrow b$*, respectively).*

Proof. Recalling Theorem 1 again yields

$$T_h(g) = \int_a^b g(x)h(x)dx \leqslant 0, \quad g \in \mathcal{C}. \tag{3.6}$$

By assumption, the integral amounts to zero if $g(x) =$ const. on $\mathcal{H} = (a, b)$. However, no element of (1.3) is constant, because $\int_a^b g(x)dx = 0$ implies $g(x) = 0$ and contradicts $||g|| = 1$. Therefore we have the strict inequality in (3.6). Suppose that for a given $g \in \mathcal{C}$

$$T_h(g) = \int_a^b g(x)h(x)dx = -t,$$

for some positive number t. Function $g(x)/t$ is orthogonal to constants, satisfies $T_h(g/t) = -1$ and $||g/t|| = 1/t$. Obviously, both the negative and reciprocal of t increase as t itself decreases. Therefore the original problem of maximizing functional (1.1) over the set (1.3) is dual to that of maximizing the norm functional over the convex set

$$\mathcal{D} = \{g \in L^2(a,b) : \; g \nearrow, \; T_1(g) = 0, \; T_h(g) = -1\}. \tag{3.7}$$

We focus on the latter one. We first note that the norm functional is convex, and so

$$||\alpha g_1 + (1-\alpha)g_2|| \leqslant \alpha||g_1|| + (1-\alpha)||g_2|| \leqslant \max\{||g_1||, ||g_2||\}$$

for every $0 < \alpha < 1$ and arbitrary functions g_1, g_2. For the $L^2(a,b)$ norm, the former inequality is equivalent to the Minkowski inequality, in which the equality is attained iff either $g_1 = 0$ or $g_2 = 0$ or $g_2 = \beta g_1$ for some positive β (see, Mitrinović [6, Theorem 29.6, p. 57]). This implies that for arbitrary distinct $g_1, g_2 \in \mathcal{D}$ and $0 < \alpha < 1$ we have

$$||\alpha g_1 + (1-\alpha)g_2|| < \max\{||g_1||, ||g_2||\},$$

which allows us to confine ourselves to the problem of maximizing the norm over the subset of extreme points of (3.7).

We claim that the subset consists of two-valued functions only. As we noticed, constant functions do not belong to (3.7). A two-valued function with a positive jump at $a < \theta < b$ is an element of the set if its values $\alpha < 0 < \beta$ satisfy the equations

$$\alpha(\theta - a) + \beta(b - \theta) = 0,$$
$$\alpha H_0(\theta) - \beta H_0(\theta) = -1.$$

They have the forms (3.5), dependent on the jump points θ. We easily check that (3.5) are extreme in (3.7). Indeed, any proper convex combination of nondecreasing functions is constant on (a, θ) and (θ, b) if both the components are constant on the respective intervals. The only element of (3.7) with constant values there has the form (3.5).

Now we verify that any function from (3.7) with n values ($n \geqslant 3$) is a convex combination of (3.5). Suppose that $g \in \mathcal{D}$ has $n-1$ jumps at $a < \theta_1 < \ldots < \theta_{n-1} < b$, say. Obviously, the linear subspace of n-valued elements of $L^2(a,b)$ orthogonal to constants has dimension $n-1$. Functions g_{θ_i}, $i = 1, \ldots, n-1$, form the basis of the subspace. Indeed, representations

$$g_{\theta_i}(x) = \sum_{j \neq i} \alpha_{ji} g_{\theta_j}(x), \quad i = 1, \ldots n-1,$$

are not possible for any real coefficients α_{ij}, because the left-hand side has a jump at θ_i, and the right-hand side does not. It follows that

$$g(x) = \sum_{i=1}^{n-1} \alpha_i g_{\theta_i}(x) \tag{3.8}$$

for some real α_i, $i = 1, \dots n-1$. Since each g_{θ_i} has the jump of size $1/H_0(\theta_i) > 0$, and (3.8) has the jump $\alpha_i/H_0(\theta_i)$ there, which is nonnegative if $\alpha_i \geqslant 0$. Since

$$-1 = T_h(g) = \sum_{i=1}^{n-1} \alpha_i T_h(g_{\theta_i}) = -\sum_{i=1}^{n-1} \alpha_i,$$

we conclude that (3.8) is a convex combination.

The next step of proof consists in showing that every $g \in \mathcal{D}$ is the limit of piecewise constant elements of $\mathcal{D}$. Our reasoning is standard. We represent

$$g = g^+ - g^- = \max\{g, 0\} - \min\{-g, 0\}$$

as the difference of two nonnegative functions. The former is nondecreasing and the latter is nonincreasing. Then for every positive integer k we partition $[a,b]$ into 2^k intervals $[\theta_{i-1,k}, \theta_{i,k}] = \left[a + \frac{i-1}{2^k}(b-a), a + \frac{i}{2^k}(b-a)\right]$, $i = 1, \dots, 2^k$, and construct stepwise interpolations

$$g_k^+(x) = \sum_{i=1}^{2^k} g^+(\theta_{i-1,k}) \mathbf{1}_{[\theta_{i-1,k},\theta_{i,k}]}(x), \tag{3.9}$$

$$g_k^-(x) = \sum_{i=1}^{2^k} g^-(\theta_{i,k}) \mathbf{1}_{[\theta_{i-1,k},\theta_{i,k}]}(x) \tag{3.10}$$

of g^+ and g^-, respectively. Sequences (3.9) and (3.10) are nondecreasing and converge pointwise to g^+ and g^-, respectively. By the monotone convergence theorem, $||g_k^+ - g^+|| \to 0$ and $||g_k^- - g^-|| \to 0$ so that $g_k = g_k^+ - g_k^-$ tend to g in the norm. This also implies that

$$\int_a^b g_k(x)dx = T_1(g_k) \to T_1(g) = 0,$$

$$\int_a^b g_k(x)h(x)dx = T_h(g_k) \to T_h(g) = -1,$$

$$\int_a^b g_k(x)g(x)dx = T_g(g_k) \to T_g(g) = ||g||^2.$$

Therefore for

$$\tilde{g}_k = \frac{g_k - T_1(g_k)}{|T_h(g_k)|} \in \mathcal{D},$$

we have

$$\begin{aligned} ||g - \tilde{g}_k||^2 &= \left|\left| g - \frac{g_k}{|T_h(g_k)|} + \frac{T_1(g_k)}{|T_h(g_k)|} \right|\right|^2 \\ &= ||g||^2 + \frac{||g_k||^2}{T_h^2(g_k)} + \frac{T_1^2(g_k)}{T_h^2(g_k)}(b-a) \\ &\qquad - 2\frac{T_g(g_k)}{|T_h(g_k)|} - 2\frac{T_1^2(g_k)}{T_h^2(g_k)} \longrightarrow 0 \quad \text{as } k \to \infty. \end{aligned}$$

Summing up, we proved that each element of (3.7) can be approximated by convex combinations of two-valued functions (3.5), and the procedure results in decreasing the norm. Therefore we can restrict ourselves to examining the norms

$$\begin{aligned} ||g_\theta|| &= \left[\int_a^\theta \frac{(b-\theta)^2}{(b-a)^2H_0^2(\theta)}dx + \int_\theta^b \frac{(\theta-a)^2}{(b-a)^2H_0^2(\theta)}dx\right]^{1/2} \\ &= \left[\frac{(\theta-a)(b-\theta)}{b-a}\right]^{1/2}\frac{1}{H_0(\theta)}, \quad a<\theta<b. \end{aligned}$$

Coming back to the original problem, we obtain

$$|||T_h||| = -\frac{1}{\sup_{a<\theta<b}||g_\theta||} = -\inf_{a<\theta<b}\left[\frac{b-a}{(\theta-a)(b-\theta)}\right]^{1/2} H_0(\theta). \tag{3.11}$$

We finally note that the function in (3.11) is positive continuous on (a,b). If the infimum is attained at some $a<\theta_0<b$, then $|||T_h||| = T_h(g_{\theta_0})$. If the infimum is attained in limit for $\theta \searrow a$ ($\theta \nearrow b$), then

$$|||T_h||| = \lim_{\theta\searrow a(\theta\nearrow b)} T_h(g_\theta). \qquad \square \tag{3.12}$$

If the function in (3.4) has a minimum, then the respective bound is strictly negative. If the minimum is unique, then it is attained by a unique two-valued function (3.7). If the function in (3.4) has more global minima, the functional is minimized by respective functions (3.7). By the Minkowski theorem, no other functions attain the minimum. The sequences of functions for which the functional attains the limiting value do not need to consist of two-valued functions. If (3.12) holds, then $|||T_h|||$ may be either negative or equal to zero. The null value occurs in the example of sharp bounds on the sample minimum

$$\mathbb{E}\frac{m_n-\mu}{\sigma} \leqslant -\inf_{0<\theta<1}\frac{1-\theta-(1-\theta)^n}{[\theta(1-\theta)]^{1/2}} = 0$$

(cf. (1.8) and (1.10)). This is rather a common case. In order to get a strictly negative bound, we need

$$\begin{aligned} \frac{1}{H_0(\theta)} &= \mathcal{O}\left((\theta-a)^{-1/2}\right), \quad \text{as } \theta\searrow a, \\ \frac{1}{H_0(\theta)} &= \mathcal{O}\left((b-\theta)^{-1/2}\right), \quad \text{as } \theta\nearrow b. \end{aligned}$$

This holds if

$$\begin{aligned} \frac{1}{h_0(\theta)} &= \mathcal{O}\left((\theta-a)^{1/2}\right), \quad \text{as } \theta\searrow a, \\ \frac{1}{h_0(\theta)} &= \mathcal{O}\left((b-\theta)^{1/2}\right), \quad \text{as } \theta\nearrow b, \end{aligned}$$

which means that the centered function (2.1) has to tend to infinity at the domain borders sufficiently fast.

We finally note that similar results can be derived for the Hilbert spaces $L^2((a,b),$ $W(dx))$ with non-uniform measures generated by nondecreasing functions W on (a,b). Since the weighted versions of the Schwarz and Minkowski inequalities are well-known, the only problem is to establish a counterpart of Theorem 1. The idea is sketched below. Suppose that $W(a)=0$ and $W(b)<\infty$, which implies that constant functions belong to the spaces. If $h\in L^2((a,b),W(dx))$, then

$$\int_a^b h(x)W(dx)=\int_0^{W(b)} h(W^{-1}(x))dx$$

is finite, where

$$W^{-1}(x)=\sup\{y:\ W(y)\leqslant x\},\quad 0<x<W(b).$$

Function

$$H_W(x)=\int_0^x h(W^{-1}(t))dt,\quad 0<x<W(b),$$

has a greatest convex minorant $\underline{H}_W$, with the derivative $\underline{h}_W$. Put $\underline{h}(x)=\underline{h}_W(W(x))$, $a<x<b$. Then for all nondecreasing $g\in L^2((a,b),W(dx))$ yields

$$\begin{aligned}\int_a^b g(x)h(x)W(dx) &= \int_0^{W(b)} g(W^{-1}(x))h(W^{-1}(x))dx\\ &\leqslant \int_0^{W(b)} g(W^{-1}(x))\underline{h}_W(x))dx\\ &= \int_a^b g(x)\underline{h}(x)W(dx).\end{aligned}$$

Observe that $\underline{h}$ is the projection of h onto the convex cone of nondecreasing functions in $L^2((a,b),W(dx))$.

Acknowledgment. The research was supported by the Polish Ministry of Education and Science under the Grant no. 1 P03A 015 30.

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(Received September 16, 2006)

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