Mathematical
Inequalities
& Applications
Volume 10, Number 2 (2007), 327–334

CERTAIN PROPERTIES ARISING FROM SOME INEQUALITIES CONCERNING SUBCLASSES OF MULTIVALENTLY ANALYTIC FUNCTIONS

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(*communicated by S. Owa*)

Abstract. In this investigation we prove several theorems involving a general class of multivalently analytic functions in the open unit disk. The applications of the main results are also considered for multivalently starlike functions and multivalenly convex functions.

1. Introduction, notations and definitions

Let $\mathcal{T}(p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots\,, \quad (a_k \in \mathbf{C};\ p \in \mathbf{N} = \{1,2,3,...\}), \tag{1.1}$$

which are *analytic* and *multivalent* in the open unit disk $\mathbf{U} = \{z:\ z \in \mathbf{C} \text{ and } |z| < 1\}$, where $\mathbf{C}$ denotes the set of complex numbers. A function $f(z)$ belonging to $\mathcal{T}(p)$ is said to be *multivalently starlike of order* α in $\mathbf{U}$ if it satisfies the inequality:

$$\Re e\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbf{U}; 0 \leqslant \alpha < p; p \in \mathbf{N}), \tag{1.2}$$

and, a function $f(z) \in \mathcal{T}(p)$ is said to be *multivalently convex of order* α in $\mathbf{U}$ if it satisfies the inequality:

$$\Re e\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in \mathbf{U}; 0 \leqslant \alpha < p; p \in \mathbf{N}). \tag{1.3}$$

By virtue of the definitions (1.2) and (1.3), it is easily seen that a function $f(z)$ is multivalently convex of order α in $\mathbf{U}$ if and only if $zf'(z)/p$ is multivalently starlike of order in α in $\mathbf{U}$, where $f(z) \in \mathcal{T}_n(p)$ and $0 \leqslant \alpha < p$, and $p \in \mathbf{N}$. For their details,

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Mathematics subject classification (2000): 30C45, 30A10.

Key words and phrases: unit disk, multivalently analytic functions, multivalently starlikeness, multivalently convexity, Jack's lemma, rational type functions.

Paper MIA-10-29

one may refer to [1] and [2] (see also [12]). Further, a function $f(z) \in \mathcal{T}(p)$ is said to be in the subclass $\Omega_w(p;\alpha,\lambda)$ if it satisfies the inequality:

$$\Re e\left\{\left(\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right)^w\right\} > \alpha, \tag{1.4}$$

where $z \in \mathbf{U},\ w \in \mathbf{C}^* := \mathbf{C} - \{0\},\ 0 \leqslant \lambda \leqslant 1,\ 0 \leqslant \alpha < p,$ and $p \in \mathbf{N}$.

Here and throughout this paper, the value of the complex expressions like

$$\left(\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right)^w \quad (f(z) \in \mathcal{T}(p); w \in \mathbf{C}^*; 0 \leqslant \lambda \leqslant 1),$$

is taken to be as its principal value.

We mention below some of the subclasses of functions $\Omega_w(p;\alpha,\lambda)$ (defined above). Indeed, we have:

$$\mathcal{V}_w(p;\alpha) \equiv \Omega_w(p;\alpha,0) \ \ (w \in \mathbf{C}^*; 0 \leqslant \alpha < p; p \in \mathbf{N}), \tag{1.5}$$

$$\mathcal{W}_w(p;\alpha) \equiv \Omega_w(p;\alpha,1) \ \ (w \in \mathbf{C}^*; 0 \leqslant \alpha < p; p \in \mathbf{N}), \tag{1.6}$$

$$\mathcal{U}_w(\alpha,\lambda) \equiv \Omega_w(1;\alpha,\lambda) \ \ (w \in \mathbf{C}^*; 0 \leqslant \alpha < 1), \tag{1.7}$$

$$\mathcal{TSK}_\lambda^\delta(p;\alpha) \equiv \Omega_\delta(p;\alpha,\lambda) \ \ (\delta \in \mathbf{R}^*; 0 \leqslant \alpha < p; p \in \mathbf{N}) \ (see[3]), \tag{1.8}$$

$$\mathcal{TS}^\delta(p;\alpha) \equiv \mathcal{TSK}_0^\delta(p;\alpha) \quad (\delta \in \mathbf{R}^*; 0 \leqslant \alpha < p; p \in \mathbf{N}), \tag{1.9}$$

$$\mathcal{TK}^\delta(p;\alpha) \equiv \mathcal{TSK}^\delta(p;\alpha) \quad (\delta \in \mathbf{R}^*; 0 \leqslant \alpha < p; p \in \mathbf{N}), \tag{1.10}$$

$$\mathcal{T}_\lambda(p;\alpha) \equiv \mathcal{TSK}_\lambda^1(p;\alpha) \quad (0 \leqslant \lambda \leqslant 1; 0 \leqslant \alpha < p; p \in \mathbf{N}) \ \ (see[6]), \tag{1.11}$$

where $\mathbf{R}$ denotes the set of real numbers and $\mathbf{R}^* := \mathbf{R} - \{0\}$.

In particular:

$$\mathcal{S}(p;\alpha) \equiv \mathcal{TS}^{\,1}(p;\alpha),\ \ \mathcal{K}(p;\alpha) \equiv \mathcal{TK}^1(p;\alpha)) \ \ (0 \leqslant \alpha < p; p \in \mathbf{N}),$$
$$\mathcal{S}(\alpha) \equiv \mathcal{S}(1;\alpha),\ \ \mathcal{K}(\alpha) \equiv \mathcal{K}(1;\alpha) \ \ (0 \leqslant \alpha < 1),$$
$$\mathcal{S}(p) \equiv \mathcal{S}(p;0),\ \ \mathcal{K}(p) \equiv \mathcal{K}(p;0),$$

and

$$\mathcal{S} \equiv \mathcal{S}(1;0),\ \ \mathcal{K} \equiv \mathcal{K}(1;0).$$

From the literature, we well know that the important subclasses in the Geometric Function Theory such as the multivalently starlike functions $\mathcal{S}(p;\alpha)$ of order α $(0 \leqslant \alpha < p; p \in \mathbf{N})$ in $\mathbf{U}$, the multivalently convex functions $\mathcal{K}(p;\alpha)$ of order α $(0 \leqslant \alpha < p; p \in \mathbf{N})$ in $\mathbf{U}$, the multivalently starlike functions $\mathcal{S}(p)$ in $\mathbf{U}$, the multivalently convex functions $\mathcal{K}(p)$ in $\mathbf{U}$, the starlike functions $\mathcal{S}(\alpha)$ of order α $(0 \leqslant \alpha < 1)$ in $\mathbf{U}$, the convex functions $\mathcal{K}(\alpha)$ of order α $(0 \leqslant \alpha < 1)$ in $\mathbf{U}$, the starlike functions $\mathcal{S}$ in $\mathbf{U}$, and the convex functions $\mathcal{K}$ in $\mathbf{U}$, are seen to be easily identifiable with the aforementioned classes ([1], [2] and [13]).

By introducing a general subclass $\Omega_w(p;\alpha,\lambda)$, w is a complex number with $w \neq 0$, of functions $f(z) \in \mathcal{T}(p)$ satisfying the inequality (1.4), our motive in this paper is to obtain sufficient conditions for a function to belong to the above subclass. The other results investigated include certain inequalities concerning functions of multivalently starlikeness and multivalently convexity in the open unit disk. Several corollaries are deduced as worthwhile consequences of our main results. We note that some of the results in this investigation are also generalizations of the results of the earlier paper in [3]. Inequalities concerning analytic and multivalent functions were also studied in [4], [6]-[8], and [10]-[12].

2. The main results

Before stating and proving our main results, we need the following assertion which is popularly known as Jack's Lemma (see [9]).

LEMMA 1. *Let* $w(z)$ *be non-constant and analytic function in the open unit disk* $\mathbf{U}$, *such that* $w(0)=0$. *If* $|w(z)|$ *attains its maximum value on the circle* $|z|=r<1$ *at a point* z_0, *then* $z_0 w'(z_0) = c w(z_0)$, *where* $c \geqslant 1$.

THEOREM 1. *Let* $f(z) \in \mathcal{T}(p)$, $w \in \mathbf{C}^*$, $0 \leqslant \alpha < p$, $p \in \mathbf{N}$ *and* $z \in \mathbf{U}$. *Then, if the function* $\mathcal{F}(z)$, *defined by*

$$\mathcal{F}(z) = (1-\lambda)f(z) + \lambda z f'(z) \quad (0 \leqslant \lambda \leqslant 1), \tag{2.1}$$

satisfies any one of the cases of the following inequalities:

$$\Re e \left(\frac{1 + z\left(\frac{\mathcal{F}''(z)}{\mathcal{F}'(z)} - \frac{\mathcal{F}'(z)}{\mathcal{F}(z)}\right)}{1 - p^w \left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{-w}} \right) \begin{cases} < |w|^{-2}\Re e\{w\} & \text{when } \Re e\{w\} > 0 \\ > |w|^{-2}\Re e\{w\} & \text{when } \Re e\{w\} < 0 \end{cases} \tag{2.2}$$

or

$$\Im m \left(\frac{1 + z\left(\frac{\mathcal{F}''(z)}{\mathcal{F}'(z)} - \frac{\mathcal{F}'(z)}{\mathcal{F}(z)}\right)}{1 - p^w \left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{-w}} \right) \begin{cases} < |w|^{-2}\Im m\{\overline{w}\} & \text{when } \Im m\{\overline{w}\} > 0 \\ > |w|^{-2}\Im me\{\overline{w}\} & \text{when } \Im m\{\overline{w}\} < 0, \end{cases} \tag{2.3}$$

then $f(z) \in \Omega_w(p;\beta,\lambda)$, *where* $\beta := \Re e\{p^w\} - |(p-\alpha)^w| \geqslant 0$.

Proof. Let $f(z) \in \mathcal{T}(p)$ and the function $\mathcal{F}(z)$ be defined by (2.1). With the help of (1.1) and (2.1), we easily see that

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} = \frac{p + (p+1)c_{p+1}z + \cdots}{1 + c_{p+1}z + \cdots}, \tag{2.4}$$

where

$$c_k = \left\{ \frac{1+\lambda(k-1)}{1+\lambda(p-1)} \right\} a_k \quad (k = p+1, p+2, \ldots),$$

and

$$z \in \mathbf{U}, \;\; 0 \leqslant \lambda \leqslant 1, \;\; 0 \leqslant \alpha < p, \;\; \text{and} \;\; p \in \mathbf{N}.$$

Let us define a function $v(z)$ by

$$\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^w - p^w = (p-\alpha)^w v(z) \quad (z \in \mathbf{U}; w \in \mathbf{C}^*; 0 \leqslant \alpha < p; p \in \mathbf{N}), \tag{2.5}$$

where the values of above complex powers are considered to be as their principal values.

Clearly, the function $v(z)$ defined above is analytic in $\mathbf{U}$ with $v(0) = 0$. Differentiation of (2.5) with respect to the variable z gives us

$$1 + z\left(\frac{\mathcal{F}''(z)}{\mathcal{F}'(z)} - \frac{\mathcal{F}'(z)}{\mathcal{F}(z)}\right) = \frac{zv'(z)(p-\alpha)^w}{w[p^w + (p-\alpha)^w v(z)]} \quad (w \in \mathbf{C}^*). \tag{2.6}$$

Hence, from (2.5) and (2.6), we get

$$\mathcal{H}(z) := \frac{1 + z\left(\frac{\mathcal{F}''(z)}{\mathcal{F}'(z)} - \frac{\mathcal{F}'(z)}{\mathcal{F}(z)}\right)}{1 - p^w\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{-w}} = \frac{1}{w} \cdot \frac{zv'(z)}{v(z)} \quad (z \in \mathbf{U}; w \in \mathbf{C}^*). \tag{2.7}$$

We now claim that $|v(z)| < 1$ in $\mathbf{U}$. For otherwise (by Lemma), there exists a point $z_0 \in \mathbf{U}$ such that $z_0 v'(z_0) = cv(z_0)$, where $|v(z_0)| = 1 \ (c \geqslant 1)$.

Therefore, (2.7) yields

$$\Re e\{\mathcal{H}(z_0)\} = \frac{c}{|w|^2}\,\Re e\{\overline{w}\}\begin{cases} \geqslant \dfrac{1}{|w|^2} \cdot \Re e\{w\} & \text{when } \Re e\{w\} > 0 \\[2ex] \leqslant \dfrac{1}{|w|^2} \cdot \Re e\{w\} & \text{when } \Re e\{w\} < 0, \end{cases} \tag{2.8}$$

and also

$$\Im m\{\mathcal{H}(z_0)\} = \frac{c}{|w|^2}\,\Im m\{\overline{w}\}\begin{cases} \geqslant \dfrac{1}{|w|^2} \cdot \Im m\{\overline{w}\} & \text{when } \Im m\{\overline{w}\} > 0 \\[2ex] \leqslant \dfrac{1}{|w|^2} \cdot \Im m\{\overline{w}\} & \text{when } \Im m\{\overline{w}\} < 0, \end{cases} \tag{2.9}$$

which contradict the assumptions in (2.2) and (2.3), respectively. Therefore, $|v(z)| < 1$ holds true for all $z \in \mathbf{U}$, and we conclude from (2.5) that

$$\left|\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^w - p^w\right| = |(p-\alpha)^w v(z)| < |(p-\alpha)^w| \tag{2.10}$$

which implies that

$$\Re e\left\{\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^w\right\} > \Re e\{p^w\} - |(p-\alpha)^w| \geqslant 0, \tag{2.11}$$

where $z \in \mathbf{U}$, $w \in \mathbf{C}^*$, $0 \leqslant \alpha < p$, and $p \in \mathbf{N}$.

Hence $f(z) \in \Omega_w(p; \beta, \lambda)$, where $\beta = \Re e\{p^w\} - |(p-\alpha)^w|$.

REMARK 1. If we choose the complex parameter w as $w := \delta \in \mathbf{R}^*$ in the Theorem 1, we arrive at the result which was earlier given by Irmak and Raina in [3].

THEOREM 2. *Let* $z \in \mathbf{U};\ \delta \in \mathbf{R}^*;\ n,m,p \in \mathbf{N};\ q-l = n-m \in \mathbf{N};\ l \in \mathbf{N}_0;$ *and* $f(z) \in \mathcal{T}(n);\ g(z) \in \mathcal{T}(m)$. *If*

$$\Re e\left\{\frac{z\mathcal{D}^{1+l}\left(\frac{f(z)}{g(z)}\right)}{\mathcal{D}^{l}\left(\frac{f(z)}{g(z)}\right)}\right\}\begin{cases} < q-l+\dfrac{1}{2\delta} & \text{when } \delta > 0 \\ > q-l+\dfrac{1}{2\delta} & \text{when } \delta < 0, \end{cases} \tag{2.12}$$

then

$$\Re e\left\{\left[\frac{(q-l)!}{q!}\cdot z^{l-q}\cdot \mathcal{D}^{l}\left(\frac{f(z)}{g(z)}\right)\right]^{\delta}\right\} > 0, \tag{2.13}$$

where

$$\mathcal{D}^{l}\{h(z)\} = \frac{s!}{(s-l)!}z^{s-l} + \sum_{k=s+1}^{\infty}\frac{k!}{(k-l)!}a_k z^{k-l}, \qquad (l \leqslant s; l \in \mathbf{N}_0 := \mathbf{N}\cup\{0\}) \tag{2.14}$$

when

$$h(z) = z^s + a_{s+1}z^{s+1} + a_{s+2}z^{s+2} + \cdots \in \mathcal{T}(s) \quad (s \in \mathbf{N}). \tag{2.15}$$

Proof. Let $f(z) \in \mathcal{T}(n)$ and $g(z) \in \mathcal{T}(m)$ with, of course, $n-m \in \mathbf{N}$. Since

$$\frac{f(z)}{g(z)} = z^q + c_1 z^{1+q} + c_2 z^{2+q} + \cdots \in \mathcal{T}(q) \quad (q = n-m \in \mathbf{N}),$$

and using the definition of the operator $\mathcal{D}^l\{\cdot\}$ given by (2.14), we define a function $w(z)$ by

$$\left[\frac{(q-l)!}{q!}\cdot z^{l-q}\cdot \mathcal{D}^{l}\left(\frac{f(z)}{g(z)}\right)\right]^{\delta} = 1 + w(z) \quad (z \in \mathbf{U}; \delta \in \mathbf{R}^*), \tag{2.16}$$

where the value of the above power is taken to be its principal value. Obviously, the function $w(z)$ is analytic in $\mathbf{U}$ and $w(0) = 0$.

Differentiating (2.16), we find

$$l - q + \frac{z\mathcal{D}^{1+l}\left(\frac{f(z)}{g(z)}\right)}{\mathcal{D}^{l}\left(\frac{f(z)}{g(z)}\right)} = \frac{1}{\delta}\cdot\frac{zw'(z)}{1+w(z)}. \tag{2.17}$$

If we next suppose that there exists a point $z_0 \in \mathbf{U}$ such that $z_0 w'(z_0) = cw(z_0)$, where $|w(z_0)| = 1\ \ (c \geqslant 1)$, i.e., $w(z_0) = e^{i\theta}\ \ (\theta \in [0,2\pi) - \{\pi\})$, then (2.17) gives

$$\begin{aligned}\Re e\left\{\left.\frac{z\mathcal{D}^{1+l}\left(\frac{f(z)}{g(z)}\right)}{\mathcal{D}^{l}\left(\frac{f(z)}{g(z)}\right)}\right|_{z=z_0}\right\} &= q-l+\frac{1}{\delta}\,\Re e\left(\frac{z_0 w'(z_0)}{1+w(z_0)}\right) \\ &= q-l+\frac{c}{\delta}\,\Re e\left(\frac{e^{i\theta}}{1+e^{i\theta}}\right) \\ &\begin{cases} \geqslant q-l+\dfrac{1}{2\delta} & \text{when } \delta > 0 \\ \leqslant q-l+\dfrac{1}{2\delta} & \text{when } \delta < 0. \end{cases}\end{aligned} \tag{2.18}$$

But the inequalities in (2.18) contradict the inequalities in (2.12). Hence $|w(z)| < 1$ for all $z \in \mathbf{U}$, and therefore (2.16) yields

$$\left| \left[\frac{(q-l)!}{q!} \cdot z^{l-q} \cdot \mathcal{D}^l \left(\frac{f(z)}{g(z)} \right) \right]^{\delta} - 1 \right| = |w(z)| < 1,$$

which evidently implies (2.13). The desired proof is thus completed.

The following Theorem 3 (below) which was earlier stated by Irmak and Raina (see [3]) can easily be obtained, when we take $l := 0$ in Theorem 2 under certain conditions.

THEOREM 3. *Let* $z \in \mathbf{U}$; $\delta \in \mathbf{R}^*$; $0 \leqslant \alpha < p$; $n, m, p \in \mathbf{N}$; $q = n - m \in \mathbf{N}$; $f(z) \in \mathcal{T}(n)$ *and* $g(z) \in \mathcal{T}(m)$. *If* $f(z)$ *satisfies the inequality:*

$$\Re e \left(\frac{zf'(z)}{f(z)} \right) \begin{cases} < q + \alpha + \dfrac{1}{2\delta} & \textit{when } \delta > 0,\ g(z) \in \mathcal{S}_m(\alpha) \\ > q + \alpha + \dfrac{1}{2\delta} & \textit{when } \delta < 0,\ g(z) \notin \mathcal{S}_m(\alpha), \end{cases}$$

then

$$\Re e \left\{ \left(z^{-q} \frac{f(z)}{g(z)} \right)^{\delta} \right\} > 0.$$

THEOREM 4. *Let* $z \in \mathbf{U}$; $\delta \in \mathbf{R}^*$; $n, m, p \in \mathbf{N}$; $l \in \mathbf{N}_0$; $q - l = n - m \in \mathbf{N}$; $f(z) \in \mathcal{T}(n)$ *and* $g(z) \in \mathcal{T}(m)$. *If*

$$\Re e \left\{ \frac{z\mathcal{D}^{1+l} \left(\frac{f'(z)}{g'(z)} \right)}{\mathcal{D}^l \left(\frac{f'(z)}{g'(z)} \right)} \right\} \begin{cases} < q - l + \dfrac{1}{2\delta} & \textit{when } \delta > 0 \\ > q - l + \dfrac{1}{2\delta} & \textit{when } \delta < 0, \end{cases} \tag{2.19}$$

then

$$\Re e \left\{ \left[\frac{(q-l)!}{q!} \cdot z^{l-q} \cdot \mathcal{D}^l \left(\frac{m}{n} \frac{f'(z)}{g'(z)} \right) \right]^{\delta} \right\} > 0, \tag{2.20}$$

where the operator $\mathcal{D}^l\{\cdot\}$ *is defined by* (2.14).

Proof. Let $f(z) \in \mathcal{T}(n)$ and $g(z) \in \mathcal{T}(m)$ with $n - m \in \mathbf{N}$. Since

$$\frac{m}{n} \frac{f'(z)}{g'(z)} = z^q + k_1 z^{1+q} + k_2 z^{2+q} + \cdots \in \mathcal{T}(q) \quad (q = n - m \in \mathbf{N}),$$

and also using the definition of $\mathcal{D}^l\{\cdot\}$, given by (2.14), we again define a function $w(z)$ by

$$\left[\frac{(q-l)!}{q!} \cdot z^{l-q} \cdot \mathcal{D}^l \left(\frac{m}{n} \cdot \frac{f'(z)}{g'(z)} \right) \right]^{\delta} = 1 + w(z) \quad (z \in \mathbf{U}; \delta \in \mathbf{R}^*),$$

then by appealing to the same technique as in the proof of Theorem 2, we arrive at the assertion (2.20) of Theorem 3 under the conditions stated with (2.19).

The following Theorem 5 (below) which was earlier given by Irmak and Raina in [3] can easily be obtained when we put $l := 0$ under certain conditions.

THEOREM 5. *Let* $z \in \mathbf{U}$; $\delta \in \mathbf{R}^*$; $0 \leqslant \alpha < p$; $n, m, p \in \mathbf{N}$; $q = n - m \in \mathbf{N}$; $f(z) \in \mathcal{T}(n)$ *and* $g(z) \in \mathcal{T}(m)$. *If* $f(z)$ *satisfies the inequality:*

$$\Re e\left(\frac{zf''(z)}{f'(z)}\right)\begin{cases} < q+\alpha-1+\dfrac{1}{2\delta} & \text{when } \delta > 0,\ g(z) \in \mathcal{K}_m(\alpha) \\ > q+\alpha-1+\dfrac{1}{2\delta} & \text{when } \delta < 0,\ g(z) \notin \mathcal{K}_m(\alpha), \end{cases} \tag{2.21}$$

then

$$\Re e\left\{\left(z^{-q}\frac{m}{n}\frac{f'(z)}{g'(z)}\right)^{\delta}\right\} > 0, \tag{2.22}$$

where the value of

$$\left(z^{-q}\frac{m}{n}\frac{f'(z)}{g'(z)}\right)^{\delta}$$

is taken its principal value.

Finally, if we select suitable values of the parameters in the general class $\Omega_w(p;\alpha,\lambda)$, then we obtain several useful results consisting of the subclasses

$$\mathcal{V}_w(p;\alpha) \ \ (w \in \mathbf{C}^*; 0 \leqslant \alpha < p; p \in \mathbf{N}),$$

$$\mathcal{W}_w(p;\alpha) \ \ (w \in \mathbf{C}^*; 0 \leqslant \alpha < p; p \in \mathbf{N}),$$

$$\mathcal{U}_w(\alpha,\lambda) \ \ (w \in \mathbf{C}^*; 0 \leqslant \alpha < 1),$$

$$\mathcal{TSK}_{\lambda}^{\delta}(p;\alpha) \ \ (\delta \in \mathbf{R}^*; 0 \leqslant \alpha < p; p \in \mathbf{N}) \ (see[3]),$$

$$\mathcal{TS}^{\delta}(p;\alpha) \quad (\delta \in \mathbf{R}^*; 0 \leqslant \alpha < p; p \in \mathbf{N}),$$

$$\mathcal{TK}^{\delta}(p;\alpha) \quad (\delta \in \mathbf{R}^*; 0 \leqslant \alpha < p; p \in \mathbf{N}),$$

$$\mathcal{T}_{\lambda}(p;\alpha) \quad (0 \leqslant \lambda \leqslant 1; 0 \leqslant \alpha < p; p \in \mathbf{N}) \ \ (see[6]),$$

$$\mathcal{S}(p;\alpha), \mathcal{K}(p;\alpha), \ \ (0 \leqslant \alpha < p; p \in \mathbf{N}),$$

$$\mathcal{S}(\alpha), \mathcal{K}(\alpha) \ \ (0 \leqslant \alpha < 1),$$

$$\mathcal{S}(p),\ \mathcal{K}(p),\ \mathcal{S} \text{ and } \mathcal{K}.$$

Acknowledgements. The first author of this paper was supported by TÜBITAK (The Scientific and Technical Research Council of Turkey) and Başkent University (Ankara, Turkey). He would like to acknowledge the helping attitude of Professor Mehmet Haberal, Rector of Başkent University, who generously supports scientific researches in all aspects. The second author was also supported by All India Council of Technical Education (Govt. of India), New Delhi.

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(Received December 14, 2004)

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