Mathematical
Inequalities
& Applications
Volume 10, Number 2 (2007), 365–370

# ON A SUMMABILITY FACTOR THEOREM

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(*communicated by J. Marshall Ash*)

*Abstract.* Let $1 < k \leqslant s < \infty$. In the present paper we obtain sufficient conditions for a series $\sum a_n$, which is absolutely summable of order $k$ by a weighted mean method to be such that $\sum a_n\lambda_n$ is absolutely summable of order $s$ by a triangular matrix. As corollary of this result we obtain an inclusion theorem.

## 1. Introduction

In a recent paper (see, [6]) the author obtained necessary conditions for the series $\sum a_n\lambda_n$ to be absolutely summable of order $s$ by a triangular matrix whenever the series $\sum a_n$, is absolutely summable of order $k$ by a weighted mean matrix

In this paper we obtain sufficient conditions for a series $\sum a_n$, which is absolutely summable of order $k$ by a weighted mean method, to be such that $\sum a_n\lambda_n$ is absolutely summable of order $s$ by a triangular matrix.

Let $T$ be a lower lower triangular matrix, $\{s_n\}$ a sequence.Then

$$T_n := \sum_{k=0}^{n} t_{nk}s_k. \tag{1.1}$$

A series $\sum a_n$ is said to be summable $|T|_k, k \geqslant 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|\Delta T_{n-1}|^k < \infty. \tag{1.2}$$

Given any lower triangular matrix $T$ one can associate the matrices $\bar{T}$ and $\hat{T}$, with entries defined by

$$\bar{t}_{nk} = \sum_{i=k}^{n} t_{ni}, \quad n, i = 0, 1, 2, ..., \qquad \hat{t}_{nk} = \bar{t}_{nk} - \bar{t}_{n-1,k}, \quad n = 1, 2, ....$$

respectively.

*Mathematics subject classification* (2000): 40F05, 40D25, 40G99.
*Key words and phrases*: absolute summability, weighted mean matrix, summability factor.


Paper MIA-10-34

With $s_n := \sum_{i=0}^{n} a_n\lambda_n$,

$$\begin{aligned} t_n &= \sum_{k=0}^{n} t_{nk}s_k = \sum_{k=0}^{n} t_{nk}\sum_{i=0}^{k} a_i\lambda_i \\ &= \sum_{i=0}^{n} a_i\lambda_i \sum_{k=i}^{n} t_{nk} = \sum_{i=0}^{n} \bar{t}_{nk}a_i\lambda_i. \\ Y_n := t_n - t_{n-1} &= \sum_{i=0}^{n} \bar{t}_{ni}a_i\lambda_i - \sum_{i=0}^{n-1} \bar{t}_{n-1,i}a_i\lambda_i \\ &= \sum_{i=0}^{n} \hat{t}_{ni}a_i\lambda_i, \quad \text{since} \quad \bar{t}_{n-1,n} = 0. \end{aligned} \tag{1.3}$$

We shall call $T$ a triangle if $T$ is lower triangular and $t_{nn} \neq 0$ for each n.

We shall always assume that $\{p_n\}$ is a positive sequence with $P_n \to \infty$. Also, $\Delta_\nu \hat{t}_{n\nu} := \hat{t}_{n\nu} - \hat{t}_{n,\nu+1}$.

THEOREM 1.1. *Let $1 < k \leqslant s < \infty$. Let $\{\lambda_n\}$ be a sequence of constants, $T$ be a triangle with bounded entries such that $T$ and $\{p_n\}$ satisfy*

$(i)$ $t_{\nu\nu}\lambda_\nu = O\Big(\Big(\dfrac{p_\nu}{P_\nu}\Big)\nu^{1/s-1/k}\Big)$,

$(ii)$ $(n|X_n|)^{s-k} = O(1)$,

$(iii)$ $\displaystyle\sum_{\nu=1}^{n-1} |\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)| = O(|t_{nn}\lambda_n|)$,

$(iv)$ $\displaystyle\sum_{n=\nu+1}^{\infty} (n|t_{nn}\lambda_n|)^{s-1}|\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)| = O(\nu^{s-1}|t_{\nu\nu}\lambda_\nu|^s)$,

$(v)$ $\displaystyle\sum_{\nu=1}^{n-1} |t_{\nu\nu}\lambda_\nu||\hat{t}_{n,\nu}\lambda_\nu| = O(|t_{nn}\lambda_n|)$, *and*

$(vi)$ $\displaystyle\sum_{n=\nu+1}^{\infty} (n|t_{nn}\lambda_n|)^{s-1}|\hat{t}_{n,\nu}\lambda_\nu| = O((\nu|t_{\nu,\nu}\lambda_\nu|)^{s-1})$,

*where $X_n$ is defined in formula $(1.4)$.*

*Then $\sum a_n\lambda_n$ is summable $|T|_s$ whenever $\sum a_n$ is summable $|\bar{N},p_n|_k$.*

*Proof.* Let $\{u_n\}$ denote the sequence of $(\bar{N},p_n)$ means of the series $\sum a_n$.

$$\begin{aligned} u_n &= \frac{1}{P_n}\sum_{i=0}^{n} p_i s_i = \frac{1}{P_n}\sum_{i=0}^{n} p_i \sum_{\nu=0}^{i} a_\nu \\ &= \frac{1}{P_n}\sum_{\nu=0}^{n} a_\nu (P_n - P_{\nu-1}) \end{aligned}$$

Thus
$$X_n = u_n - u_{n-1} = \sum_{v=0}^{n}\Big(1-\frac{P_{v-1}}{P_n}\Big)a_v - \sum_{v=0}^{n-1}\Big(1-\frac{P_{v-1}}{P_{n-1}}\Big)a_v$$
$$= \frac{p_n}{P_n}a_n + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_{v-1}a_v \tag{1.4}$$
$$= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}a_v.$$

Therefore,
$$\frac{P_nP_{n-1}X_n}{p_n} = \sum_{v=1}^{n}P_{v-1}a_v$$
$$\frac{P_{n-1}P_{n-2}X_{n-1}}{p_{n-1}} = \sum_{v=1}^{n-1}P_{v-1}a_v$$
$$\frac{P_nP_{n-1}X_n}{p_n} - \frac{P_{n-1}P_{n-2}X_{n-1}}{p_{n-1}} = P_{n-1}a_n.$$

Hence,
$$a_n = \frac{P_nX_n}{p_n} - \frac{P_{n-2}X_{n-1}}{p_{n-1}}, \qquad n>0. \tag{1.5}$$

Substituting (1.5) into (1.3) gives
$$Y_n = \sum_{v=1}^{n}\hat{t}_{nv}\lambda_v a_v$$
$$= \sum_{v=1}^{n}\hat{t}_{nv}\lambda_v\Big(\frac{X_vP_v}{p_v} - \frac{X_{v-1}P_{v-2}}{p_{v-1}}\Big)$$
$$= \sum_{v=1}^{n}\hat{t}_{nv}\lambda_v\frac{X_vP_v}{p_v} - \sum_{v=1}^{n}\hat{t}_{nv}\lambda_v\frac{X_{v-1}P_{v-2}}{p_{v-1}}$$
$$= \sum_{v=1}^{n}\hat{t}_{nv}\lambda_v\frac{X_vP_v}{p_v} - \sum_{v=0}^{n-1}\hat{t}_{n,v+1}\lambda_{v+1}\frac{X_vP_{v-1}}{p_v}$$
$$= \frac{\hat{t}_{nn}\lambda_nX_nP_n}{p_n} + \sum_{v=1}^{n-1}(\hat{t}_{nv}\lambda_vP_v - \hat{t}_{n,v+1}\lambda_{v+1}P_{v-1})\frac{X_v}{p_v}.$$

We may write
$$\hat{t}_{nv}\lambda_vP_v - \hat{t}_{n,v+1}\lambda_{v+1}P_{v-1} = P_{v-1}(\lambda_v\hat{t}_{nv} - \lambda_{v+1}\hat{t}_{n,v+1}) + \hat{t}_{nv}\lambda_vp_v.$$

Therefore
$$Y_n = \frac{P_nt_{nn}\lambda_nX_n}{p_n} + \sum_{v=1}^{n-1}\Big(\frac{P_{v-1}}{p_v}\Delta_v(\hat{t}_{n,v}\lambda_v) + \lambda_v\hat{t}_{n,v}\Big)X_v$$
$$= T_{n1} + T_{n2} + T_{n3}, \qquad \text{say.}$$

From Minkowski's inequality it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{s-1}|T_{ni}|^s < \infty, \quad i = 1,2,3.$$

Using condition $(i)$,

$$J_1 := \sum_{n=1}^{\infty} n^{s-1}|T_{n1}|^s = \sum_{n=1}^{\infty} n^{s-1}\left|\frac{t_{nn}\lambda_n P_n}{p_n} X_n\right|^s$$
$$= O(1)\sum_{n=1}^{\infty} n^{s-1}(n^{1/s-1/k})^s|X_n|^s$$
$$= O(1)\sum_{n=1}^{\infty} n^{k-1}|X_n|^k(n^{s-s/k-k+1}|X_n|^{s-k}).$$

But, from $(ii)$

$$n^{s-s/k-k+1}|X_n|^{s-k} = (n^{1-1/k}|X_n|)^{s-k}$$
$$= O((n|X_n|)^{s-k}) = O(1).$$

Hence $J_1 = O(1)$.

Using $(i)$, Hölder's inequality, $(ii)$, $(iii)$ and $(iv)$

$$J_2 := \sum_{n=1}^{\infty} n^{s-1}|T_{n2}|^s = \sum_{n=1}^{\infty} n^{s-1}\left|\sum_{\nu=1}^{n-1}\left(\frac{P_{\nu-1}}{p_\nu}\right)\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)X_\nu\right|^s$$
$$= O(1)\sum_{n=1}^{\infty} n^{s-1}\left(\sum_{\nu=1}^{n-1}\nu^{1/s-1/k}|t_{\nu\nu}\lambda_\nu|^{-1}|\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)||X_\nu|\right)^s$$
$$= O(1)\sum_{n=1}^{\infty} n^{s-1}\left(\sum_{\nu=1}^{n-1}\nu^{1-s/k}|t_{\nu\nu}\lambda_\nu|^{-s}|\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)||X_\nu|^s\right)\times\left(\sum_{\nu=1}^{n-1}|\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)|\right)^{s-1}$$
$$= O(1)\sum_{n=1}^{\infty}(n|t_{nn}\lambda_n|)^{s-1}\sum_{\nu=1}^{n-1}\nu^{1-s/k}|t_{\nu\nu}\lambda_\nu|^{-s}|\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)||X_\nu|^s$$
$$= O(1)\sum_{\nu=1}^{\infty}\nu^{1-s/k}|t_{\nu\nu}\lambda_\nu|^{-s}|X_\nu|^s\sum_{n=\nu+1}^{\infty}(n|t_{nn}\lambda_n|)^{s-1}|\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)|$$
$$= O(1)\sum_{\nu=1}^{\infty}\nu^{1-s/k}|t_{\nu\nu}\lambda_\nu|^{-s}|X_\nu|^s\nu^{s-1}|t_{\nu\nu}\lambda_\nu|^s$$
$$= O(1)\sum_{\nu=1}^{\infty}\nu^{s-s/k}|X_\nu|^s$$
$$= O(1)\sum_{\nu=1}^{\infty}\nu^{k-1}|X_\nu|^k(\nu^{s-s/k-k+1}|X_\nu|^{s-k})$$
$$= O(1)\sum_{\nu=1}^{\infty}\nu^{k-1}|X_\nu|^k = O(1),$$

since $\sum a_n$ is summable $|\bar{N},p_n|_k$.

By Hölder's inequality, $(ii)$, $(v)$ and $(vi)$, we have

$$\begin{aligned}
J_3 := \sum_{n=1}^{\infty} n^{s-1}|T_{n3}|^s &= \sum_{n=1}^{\infty} n^{s-1}\Big|\sum_{\nu=1}^{n-1}\hat{t}_{n,\nu}\lambda_\nu X_\nu\Big|^s \\
&\leqslant \sum_{n=1}^{\infty} n^{s-1}\Big(\sum_{\nu=1}^{n-1}|\hat{t}_{n,\nu}\lambda_\nu||X_\nu|\Big)^s \\
&\leqslant \sum_{n=1}^{\infty} n^{s-1}\Big(\sum_{\nu=1}^{n-1}|t_{\nu\nu}\lambda_\nu|^{1-s}|\hat{t}_{n,\nu}\lambda_\nu||X_\nu|^s\Big)\times\Big(\sum_{\nu=1}^{n-1}|t_{\nu\nu}\lambda_\nu||\hat{t}_{n,\nu}\lambda_\nu|\Big)^{s-1} \\
&= O(1)\sum_{n=1}^{\infty}(n|t_{nn}\lambda_n|)^{s-1}\sum_{\nu=1}^{n-1}|t_{\nu\nu}\lambda_\nu|^{1-s}|\hat{t}_{n,\nu}\lambda_\nu||X_\nu|^s \\
&= O(1)\sum_{\nu=1}^{\infty}|t_{\nu\nu}\lambda_\nu|^{1-s}|X_\nu|^s\sum_{n=\nu+1}^{\infty}(n|t_{nn}\lambda_n|)^{s-1}|\hat{t}_{n,\nu}\lambda_\nu| \\
&= O(1)\sum_{\nu=1}^{\infty}|t_{\nu\nu}\lambda_\nu|^{1-s}|X_\nu|^s(\nu|t_{\nu\nu}\lambda_\nu|)^{s-1} \\
&= O(1)\sum_{\nu=1}^{\infty}\nu^{s-1}|X_\nu|^s \\
&= O(1)\sum_{\nu=1}^{\infty}\nu^{k-1}|X_\nu|^k(\nu|X_\nu|)^{s-k} \\
&= O(1)\sum_{\nu=1}^{\infty}\nu^{k-1}|X_\nu|^k = O(1).
\end{aligned}$$

We now state sufficient conditions when $k = s$.

THEOREM 1.2. *Let $\{\lambda_n\}$ be a sequence of constants, $T$ a triangle with bounded entries such that $T$ and $\{p_n\}$ satisfy*

$(i)$ $t_{nn}\lambda_n = O\Big(\dfrac{p_n}{P_n}\Big)$,

$(ii)$ $\displaystyle\sum_{\nu=1}^{n-1}|\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)| = O(|t_{nn}\lambda_n|)$,

$(iii)$ $\displaystyle\sum_{n=\nu+1}^{\infty}(n|t_{nn}\lambda_n|)^{k-1}|\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)| = O(\nu^{k-1}|t_{\nu\nu}\lambda_\nu|^k)$,

$(iv)$ $\displaystyle\sum_{\nu=1}^{n-1}|t_{\nu\nu}\lambda_\nu||\hat{t}_{n\nu}\lambda_\nu| = O(|t_{nn}\lambda_n|)$,

$(v)$ $\displaystyle\sum_{n=\nu+1}^{\infty}(n|t_{nn}\lambda_n|)^{k-1}|\hat{t}_{n,\nu}\lambda_\nu| = O((\nu|t_{\nu\nu}\lambda_\nu|)^{k-1})$.

*Then $\sum a_n$ summable $|\bar{N},p_n|_k$ implies $\sum\lambda_n a_n$ is summable $|T|_k, k \geqslant 1$.*

Every summability factor theorem of this type implies an inclusion theorem by setting each $\lambda_n = 1$. We have

COROLLARY 1.1. ([3]) *Let* $1 < k \leqslant s < \infty$. *Let* $T$ *be a triangle with bounded entries such that* $T$ *and* $\{p_n\}$ *satisfy*

$(i)$ $t_{\nu\nu} = O\left(\left(\frac{p_\nu}{P_\nu}\right)\nu^{1/s-1/k}\right)$,

$(ii)$ $(n|X_n|)^{s-k} = O(1)$,

$(iii)$ $\sum_{\nu=1}^{n-1} |\Delta_\nu(\hat{t}_{n\nu})| = O(|t_{nn}|)$,

$(iv)$ $\sum_{n=\nu+1}^{\infty} (n|t_{nn}|)^{s-1}|\Delta_\nu(\hat{t}_{n\nu})| = O(\nu^{s-1}|t_{\nu\nu}|^s)$,

$(v)$ $\sum_{\nu=1}^{n-1} |t_{\nu\nu}||\hat{t}_{n,\nu}| = O(|t_{nn}|)$, *and*

$(vi)$ $\sum_{n=\nu+1}^{\infty} (n|t_{nn}|)^{s-1}|\hat{t}_{n,\nu}| = O((\nu|t_{\nu\nu}|)^{s-1})$.

*Acknowlegment.* The author offers his sincerest gratitude to Prof. B. E. Rhoades for his kind interest and valuable advice in the preparation of this paper and is grateful to the referee for his valuable suggestions.

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(Received January 18, 2006)

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