Mathematical
Inequalities
& Applications
Volume 10, Number 2 (2007), 389–401

A NEW CLASS OF GENERALIZED NONLINEAR MULTI–VALUED QUASI–VARIATIONAL–LIKE INCLUSIONS WITH H–MONOTONE MAPPINGS

YEOL JE CHO AND HENG-YOU LAN

(*communicated by R. U. Verma*)

Abstract. In this paper, we introduce and study a new class of generalized nonlinear multi-valued quasi-variational-like inclusions with H-monotone operators in Hilbert spaces. By using the resolvent operator method associated with H-monotone operator due to Fang and Huang, we construct a new iterative algorithm for solving this kind of nonlinear multi-valued variational inclusions. We also prove the existence of solutions for the nonlinear multi-valued variational inclusions and the convergence of iterative sequences generated by the algorithm. Our results improve and generalize many known corresponding results.

1. Introduction

It is well known that variational inclusion is an important generalization of variational inequality, which has wide applications in the pure and applied sciences and has been studied extensively by many authors (see, for example, [1, 2, 4-17, 21-24, 27] and the references therein).

In 2003, Fang and Huang [7] first introduced the notion of the H-monotonicity in the context of solving some nonlinear inclusion systems in Hilbert space settings. This notion does impact greatly the theory of maximal monotone mappings in terms of applications to problems from several fields. Furthermore, Jin [13] and Verma [24] used the generalized resolvent operator technique to studying a general class of nonlinear variational inclusion problems involving H-monotone mappings in different space settings. Very recently, Verma [24] considered a class of nonlinear variational inclusion problems which generalize to the case of the relaxed monotone mappings in Hilbert spaces, and studied the solvability of the nonlinear variational inclusions based on the resolvent operator technique.

On the other hand, Hassouni and Moudafi [9] introduced and studied a class of variational inclusions and developed a perturbed algorithm for finding approximate solutions of the variational inclusions. Adly [1], Huang [10-12], Ding [4], and Ding and

Mathematics subject classification (2000): 49J40, 47J20, 65B05.

Key words and phrases: generalized nonlinear multi-valued variational inclusion with H-monotone operator, resolvent operator technique, relaxed cocoercive mapping, iterative algorithm with errors, existence and convergence.

This work was supported by the Korea Research Foundation Grant (KRF-2004-041-C00033) and the Educational Science Foundation of Sichuan, Sichuan of China (2004C018).

Paper MIA-10-37

Luo [6] have obtained some important extensions of the results in [9] in various different assumptions. We observe that all authors, in [1, 4, 9, 10], assume that the functionals included in variational inclusions or generalized quasi-variational inclusions are proper convex and lower semicontinuous. Recently, Ding [5], Salahuddin and Rais [22], suggested and analyzed a kind of iterative schemes for solving generalized nonlinear quasi-variational like inclusions with nonconvex functionals on Hilbert spaces. In 2004, Lan, Kim and Huang [15] introduced a new kind of generalized nonlinear quasi variational inclusions involving non-monotone set-valued mappings with noncompact values and constructed some new iterative algorithms for solving this class of generalized nonlinear quasi variational inclusions in Hilbert spaces.

Inspired and motivated by the recent works [3, 5, 15, 24-26], we introduce and study a new class of generalized nonlinear multi-valued quasi-variational-like inclusions with H-monotone operators in Hilbert spaces. By using the resolvent operator method associated with H-monotone operator due to Fang and Huang, we construct a new iterative algorithm for solving this kind of nonlinear multi-valued variational inclusions. We also prove the existence of solutions for the nonlinear multi-valued variational inclusions and the convergence of iterative sequences generated by the algorithm. Our results improve and generalize many known corresponding results.

2. Preliminaries

Throughout this paper, we suppose that E is a real Hilbert space with dual space endowed with an norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let 2^E denote the family of all the nonempty subsets of E and $C(E)$ be the family of all nonempty closed subsets of E.

For a given element $f\in E$, single- valued mappings $p,g : E\to E$ and $N : E\times E\to E$, set-valued mappings $T,Q : E\to 2^E$ and H-monotone mapping $M(\cdot,t) : E\to 2^E$ with $g(E)\bigcap \mathrm{Dom}M(\cdot,t)\neq\emptyset$ for each fixed $t\in E$, we consider the following problem:

Find $x,u,w\in E$ such that $u\in T(x), w\in Q(x)$ and

$$f\in N(p(x),u)+M(g(x),w). \tag{2.1}$$

This problem is called a generalized nonlinear multi-valued quasi-variational-like inclusion involving H-monotone mapping.

Some examples of the problem (2.1):

(1) If $M(\cdot,t)=\triangle\varphi(\cdot,t)$ for all $t\in E$, where $\varphi : E\times E\to R\cup\{+\infty\}$ is a proper functional such that for each fixed $t\in E$, $\varphi(\cdot,t) : E\to R\cup\{+\infty\}$ is a lower semicontinuous and subdifferentiable on E and $\triangle\varphi(\cdot,t)$ denotes the subdifferential of $\varphi(\cdot,t)$, then the problem (2.1) reduces to the problem of finding $x,u,w\in E$ such that $u\in T(x), w\in Q(x)$ and

$$\langle f-N(p(x),u),y-g(x))\rangle\geqslant\varphi(g(x),w)-\varphi(y,w),\quad\forall\, y\in E, \tag{2.2}$$

which is called a generalized nonlinear multi-valued quasi-variational-like inclusion.

(2) If $T,Q : E\to E$ are single-valued mappings, then the problem (2.1) becomes to the following generalized nonlinear variational inclusion:

Find $x \in E$ such that

$$f \in N(p(x), T(x)) + M(g(x), Q(x)). \tag{2.3}$$

The problem (2.3) was studied by Verma [24] when $p = g = I$, the identity mapping and $N(x,y) = S(x)$, $M(x,z) = M(x)$ for all $x,y,z \in E$, where $S : E \to E$ is a single-valued and $M : E \to 2^E$ is a H-monotone mapping.

(3) If $M(x,y) \equiv M(y)$ for all $x,y \in E$, where $M : E \to 2^E$ is a maximal monotone mapping, then the problem (2.1) is equivalent to finding $x \in E, u \in T(x)$ such that $g(x) \in \mathrm{Dom}(M)$ and

$$f \in N(p(x), u) + M(g(x)). \tag{2.4}$$

This problem is called the generalized set-valued mixed variational inequality, which was studied by Liu and Li [17] when $f = 0$. Furthermore, if $f = 0$ and $p : E \to 2^E$ is a set-valued mapping, then the problem (2.4) reduces to the variational inclusion problem by Huang [10].

(4) If $g \equiv I$ and $N(u,v) = F(u) + Q(v)$ for all $u,v \in E$, where $F, Q : E \to E$ are two mappings, then the problem (2.4) reduces to the following nonlinear variational inclusion problem:

$$f \in F(p(x)) + Q(u) + M(x), \tag{2.5}$$

which is called the generalized set-valued variational inclusion problem.

(5) If $p \equiv I$ and $Q \equiv 0$, then the problem (2.5) is equivalent to finding $x \in X$ such that

$$f \in F(x) + M(x).$$

This problem was introduced and studied by Jung and Morales [14] in Banach spaces.

(6) If $M \equiv \partial\phi$ is the subdifferential of a proper convex lower semicontinuous functional $\phi : E \to R \cup \{+\infty\}$, $N(u,v) = u - v$ for all $u,v \in E$ and $T E \to E$ is a single-valued mapping, then the problem (2.4) reduces to finding $x \in E$ such that

$$\langle p(x) - T(x) - f, y - g(x)\rangle + \phi(y) - \phi(g(x)) \geqslant 0, \quad \forall y \in E, \tag{2.6}$$

which is called the variational inclusion problem in Hilbert spaces considered by Hassouni and Moudafi [9].

(7) If $T \equiv 0$, $g \equiv I$, $f = 0$ and K is a nonempty closed convex subset of E, then the problem (2.6) becomes to finding $x \in K$ such that

$$\langle p(x), y - x\rangle + \phi(y) - \phi(x) \geqslant 0, \quad \forall x \in K,$$

which was studied by Verma [26].

REMARK 2.1. For a suitable choice of $f, N, \varphi, p, g, T, Q$ and M, a number of classes of variational inequalities, complementarity problems and variational inclusions can be obtained as special cases of the generalized nonlinear variational inclusions (2.1)-(2.3) (see, for example, [5, 6, 15, 22, 24] and the references therein).

In the sequel, we give some concepts and lemmas.

DEFINITION 2.1. Let $H: E \to E$ be an any mapping. A mapping $g: E \to E$ is said to be

(i) r-strongly monotone, if there exists a constant $r > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geqslant r\|x - y\|^2, \quad \forall x, y \in E;$$

(ii) c-strongly monotone with respect to H, if there exists a constant $c > 0$ such that

$$\langle g(x) - g(y), H(x) - H(y) \rangle \geqslant c\|x - y\|^2, \quad \forall x, y \in E;$$

(iii) α-relaxed monotone, if there exists a constant $\alpha > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geqslant -\alpha\|x - y\|^2, \quad \forall x, y \in E;$$

(iv) σ-cocoercive with respect to H, if there exists a constant $\sigma > 0$ such that

$$\langle g(x) - g(y), H(x) - H(y) \rangle \geqslant \sigma\|g(x) - g(y)\|^2, \quad \forall x, y \in E;$$

(v) ϱ-relaxed cocoercive with respect to H, if there exists a constant $\varrho > 0$ such that

$$\langle g(x) - g(y), H(x) - H(y) \rangle \geqslant -\varrho\|g(x) - g(y)\|^2, \quad \forall x, y \in E;$$

(vi) (δ, γ)-relaxed cocoercive with respect to H, if there exist constants $\delta > 0$ and $\gamma > 0$ such that

$$\langle g(x) - g(y), H(x) - H(y) \rangle \geqslant -\delta\|g(x) - g(y)\|^2 + \gamma\|x - y\|^2, \quad \forall x, y \in E;$$

(vii) β-Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|g(x) - g(y)\| \leqslant \beta\|x - y\|, \quad \forall x, y \in E.$$

REMARK 2.2. If $H = I$, then (iv) and (v) of Definition 2.1 reduce to the definitions of cocoerciveness and relaxed cocoerciveness, respectively. Further, (vi) of Definition 2.1 collapses to (ii) of Definition 2.1 when $\delta = 0$ and $\gamma = c$.

EXAMPLE 2.1. ([24]) Let $g: E \to E$ be a nonexpansive mapping. Then $I - g$ is $\frac{1}{2}$-cocoercive and γ-relaxed cocoercive for $\frac{1}{2} > -\gamma$, where $\gamma > 0$.

DEFINITION 2.2. A multi-valued operator $A: E \to 2^E$ is said to be

(i) monotone if

$$\langle u - v, x - y \rangle \geqslant 0, \quad \forall x, y \in E, u \in A(x), v \in A(y);$$

(ii) maximal monotone if A is monotone and $(I + \rho A)(E) = E$ for all $\rho > 0$.

DEFINITION 2.3. ([7, 8]) Let $H: E \to E$ and $M: E \to 2^E$ be any mappings on a Hilbert space E. The map M is said to be H-monotone if M is monotone and $(H + \rho M)(E) = E$ for all $\rho > 0$.

EXAMPLE 2.2. ([19]) Let $H: E \to E$ be m-strongly monotone and $f: E \to R$ be locally Lipschitz such that ∂f is α-relaxed monotone. Then ∂f is H-monotone, that is $H + \partial f$ is maximal monotone for $m - \alpha > 0$, where $m, \alpha > 0$.

Note that, if H is strictly monotone and M is H-monotone, then M is maximal monotone. Let the resolvent operator $J^{\rho}_{H,M}$ be defined by

$$J^{\rho}_{H,M}(x) = (H+\rho M)^{-1}(x), \quad \forall x \in E. \tag{2.7}$$

REMARK 2.3. If $H = I$, then the definition of I-monotone operators is that of maximal monotone operators. In fact, the class of H-monotone operators has close relation with that of maximal monotone operators.

EXAMPLE 2.3. ([7]) Let $H : E \to E$ be a strictly monotone single-valued operator and $M : E \to 2^E$ an H-monotone operator. Then M is maximal monotone.

EXAMPLE 2.4. ([7]) Let $M : E \to 2^E$ be a maximal monotone operator and $H : E \to E$ be a bounded, cocoercive, hemi-continuous and monotone operator. Then M is H-monotone.

The following example shows that a maximal monotone operator need not be H-monotone for some H.

EXAMPLE 2.5. ([7]) Let $E = R$, $M = I$ and $H(x) = x^2$ for all $x \in E$. Then it is easy to see that I is maximal monotone and the range of $H+I$ is $[-\frac{1}{4}, +\infty)$. Therefore, I is not H-monotone.

REMARK 2.4. When $H = I$, (2.7) reduces to the definition of the resolvent operator of a maximal monotone operator (see [20]).

LEMMA 2.1. ([7]) *Let $H : E \to E$ be a r-strongly monotone operator and $M : E \to 2^E$ be an H-monotone operator. Then the resolvent operator $J^{\rho}_{H,M}$ is $\frac{1}{r}$-Lipschitz continuous, i.e.,*

$$\|J^{\rho}_{H,M}(x) - J^{\rho}_{H,M}(y)\| \leqslant \frac{1}{r}\|x-y\|, \quad \forall x, y \in E.$$

DEFINITION 2.4. Let $T : E \to 2^E$ be a multi-valued mapping. For all $x, y \in E$, the mapping $N(\cdot,\cdot) : E \times E \to E$ is called to be

(i) τ-Lipschitz continuous with respect to the first argument, if there exists a constant $\tau > 0$ such that

$$\|N(x,\cdot) - N(y,\cdot)\| \leqslant \tau\|x-y\| \quad \forall x, y \in E;$$

(ii) T is said to be ζ-$\hat{\mathbf{H}}$-Lipschitz continuous, if there exists a constant $\zeta > 0$ such that

$$\hat{\mathbf{H}}(T(x), T(y)) \leqslant \zeta\|x-y\|, \quad \forall x, y \in E,$$

where $\hat{\mathbf{H}} : 2^E \times 2^E \to (-\infty, +\infty) \cup \{+\infty\}$ is the Hausdorff pseudo-metric, i.e.,

$$\hat{\mathbf{H}}(A,B) = \max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{x\in B}\inf_{y\in A}\|x-y\|\}, \quad \forall A, B \in 2^E.$$

Note that if the domain of $\hat{\mathbf{H}}$ is restricted to closed bounded subsets, then $\hat{\mathbf{H}}$ is the Hausdorff metric.

In a similar way, we can define Lipschitz continuity of the mapping $N(\cdot,\cdot)$ with respect to the second argument.

3. Main results

We first transfer the generalized nonlinear quasi-variational-like inclusion problem (2.1) into a fixed point problem. From the definition of $J^{\rho}_{H,M(\cdot,w)}$, it is easy to prove the following Lemma.

LEMMA 3.1. *The (x,u,w) is a solution of problem (2.1), if and only if it satisfies the following relation*

$$g(x) = J^{\rho}_{H,M(\cdot,w)}[H(g(x)) - \rho(N(p(x),u) - f)], \tag{3.1}$$

where $J^{\rho}_{H,M(\cdot,w)} = (H + \rho M(\cdot,w))^{-1}$ and $\rho > 0$ is a constant.

REMARK 3.1. The equality (3.1) can be written as

$$x = (1-\lambda)x + \lambda\{x - g(x) + J^{\rho}_{H,M(\cdot,w)}[H(g(x)) - \rho(N(p(x),u) - f)]\},$$

where $0 < \lambda \leqslant 1$ is a parameter and $\rho > 0$ is a constant. This fixed point formulation enables us to suggest the following iterative algorithm.

Algorithm 3.1. Let $p, g : E \to E, N : E \times E \to E$ and $T, Q : E \to 2^E$ be nonlinear mappings. Suppose that $M : E \times E \to 2^E$ is a multi-valued mappings such that for each fixed $t \in E$, $M(\cdot,t)$ is H-monotone with $p(E) \bigcap \mathrm{Dom} M(\cdot,t) \neq \emptyset$. For any given $x_0 \in E$, we choose $u_0 \in T(x_0), w_0 \in Q(x_0)$ and let

$$x_1 = (1-\lambda)x_0 + \lambda\{x_0 - g(x_0) + J^{\rho}_{H,M(\cdot,w_0)}[H(g(x_0)) - \rho(N(p(x_0),u_0) - f)]\} + \lambda e_0.$$

Since $u_0 \in T(x_0)$ and $w_0 \in Q(x_0)$, for any $x_1 \in E$, by Nadler [18], there exist $u_1 \in T(x_1), w_1 \in Q(x_1)$ such that

$$\|u_0 - u_1\| \leqslant (1+1)\hat{\mathbf{H}}(T(x_0), T(x_1)),$$
$$\|w_0 - w_1\| \leqslant (1+1)\hat{\mathbf{H}}(Q(x_0), Q(x_1)).$$

Let

$$x_2 = (1-\lambda)x_1 + \lambda[x_1 - g(x_1) + J^{\rho}_{H,M(\cdot,w_1)}[H(g(x_1)) - \rho(N(p(x_1),u_1) - f)] + \lambda e_1.$$

Continuing this way, we can obtain sequences $\{x_n\}, \{u_n\}, \{w_n\}$ satisfying

$$\begin{cases} x_{n+1} = (1-\lambda)x_n + \lambda\{x_n - g(x_n) + J^{\rho}_{H,M(\cdot,w_n)} \\ \qquad [H(g(x_n)) - \rho(N(p(x_n),u_n) - f)]\} + \lambda e_n, \\ u_n \in T(x_n), \|u_n - u_{n+1}\| \leqslant (1+(n+1)^{-1})\hat{\mathbf{H}}(T(x_n), T(x_{n+1})), \\ w_n \in Q(x_n), \|w_n - w_{n+1}\| \leqslant (1+(n+1)^{-1})\hat{\mathbf{H}}(M(x_n), M(x_{n+1})), \\ n = 0, 1, 2, \cdots, \end{cases} \tag{3.2}$$

where $0 < \lambda \leqslant 1$ and $\rho > 0$ are both constants, $e_n \in E (n \geqslant 0)$ is an error to take into account a possible inexact computation of the resolvent operator point.

If T, Q are the same as of problem (2.3), then Algorithm 3.1 reduces to the following algorithm:

Algorithm 3.2. For any given $x_0 \in E, u_0 \in T(x_0), w_0 \in Q(x_0)$, we can obtain iterative sequences $\{x_n\}, \{u_n\}$ and $\{w_n\}$ as follows:

$$x_{n+1} = (1-\lambda)x_n+\lambda\{x_n-g(x_n)+J^{\rho}_{H,M(\cdot,Q(x_n))}[H(g(x_n))-\rho(N(p(x_n),T(x_n))-f)]\}+\lambda e_n.$$

REMARK 3.2. If we choose suitable λ, e_n, N, f, T, Q, p, g and M, then Algorithms 3.1 and 3.2 can be degenerated to a number of algorithms involving many known algorithms which due to classes of variational inequalities, complementarity problems, and variational inclusions (see, for example, [5, 6, 9, 15, 22]).

Now we prove the existence of a solution of problem (2.1) and the convergence of Algorithm 3.1.

THEOREM 3.1. *Let $T, Q : E \to C(E)$ be ξ-$\hat{\mathbf{H}}$-Lipschitz continuous and ζ-$\hat{\mathbf{H}}$-Lipschitz continuous, respectively. Suppose that $H : E \to E$ is r-strongly monotone and σ-Lipschitz continuous, for each fixed $w \in E$, $M(\cdot, w) : E \to 2^E$ be a H-monotone mapping. Let p be τ-Lipschitz continuous, g be α-strongly monotone and β-Lipschitz continuous, and $N : E \times E \to E$ be (γ, c)-relaxed cocoercive with respect to H in the first argument and Lipschitz continuous with respect to both arguments with constants $\delta > 0$ and $\epsilon > 0$, respectively. If there exist constants $\rho > 0$ and $\mu > 0$ such that for each $x, y, t \in E$,*

$$\|J^{\rho}_{H,M(\cdot,x)}(t) - J^{\rho}_{H,M(\cdot,y)}(t)\| \leqslant \mu\|x - y\| \tag{3.3}$$

and

$$\begin{cases} k = \sqrt{1 - 2\alpha + \beta^2} + \mu\zeta < 1, \\ \rho < \dfrac{r(1-k)}{\epsilon\xi}, \quad \delta\tau - \epsilon\xi > 0, \\ |\rho - \dfrac{c\beta^2 - \gamma\tau^2 - r\epsilon\xi(1-k)}{\delta^2\tau^2 - \epsilon^2\xi^2}| \\ \qquad < \dfrac{\sqrt{[c\beta^2-\gamma\tau^2-r\epsilon\xi(1-k)]^2-(\delta^2\tau^2-\epsilon^2\xi^2)[\sigma^2\beta^2-r^2(1-k)^2]}}{\delta^2\tau^2-\epsilon^2\xi^2}, \\ c\beta^2 > \gamma\tau^2 + r\epsilon\xi(1-k) + \sqrt{(\delta^2\tau^2 - \epsilon^2\xi^2)[\sigma^2\beta^2 - r^2(1-k)^2]}, \\ \sum_{i=1}^{\infty} \|e_i - e_{i-1}\| \, \kappa^{-i} < \infty, \ \forall \kappa \in (0,1), \quad \lim_{n\to\infty} e_n = 0, \end{cases} \tag{3.4}$$

then the iterative sequences $\{x_n\}, \{u_n\}$ and $\{w_n\}$ generated by Algorithm 3.1 converge strongly to x^*, u^* and w^*, respectively, and (x^*, u^*, w^*) is a solution of problem (2.1).

Proof. From (3.2), we have

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&= \|(1-\lambda)x_n+\lambda[x_n-g(x_n)+J^{\rho}_{H,M(\cdot,w_n)}(H(g(x_n))-\rho(N(p(x_n),u_n)-f))] \\
&\quad +\lambda e_n-\{(1-\lambda)x_{n-1}+\lambda[x_{n-1}-g(x_{n-1}) \\
&\qquad +J^{\rho}_{H,M(\cdot,w_{n-1})}(H(g(x_{n-1}))-\rho(N(p(x_{n-1}),u_{n-1})-f))]+\lambda e_{n-1}\}\| \\
&\leqslant (1-\lambda)\|x_n-x_{n-1}\|+\lambda\|x_n-x_{n-1}-(g(x_n)-g(x_{n-1}))\| \\
&\quad +\lambda\|J^{\rho}_{H,M(\cdot,w_n)}(H(g(x_n))-\rho(N(p(x_n),u_n)-f)) \\
&\quad -J^{\rho}_{H,M(\cdot,w_{n-1})}(H(g(x_{n-1}))-\rho(N(p(x_{n-1}),u_{n-1})-f))\|+\lambda\|e_n-e_{n-1}\|.
\end{aligned} \tag{3.5}$$

By Lemma 2.1 and condition (3.3), we can get

$$\begin{aligned}
&\|J^{\rho}_{H,M(\cdot,w_n)}(H(g(x_n))-\rho(N(p(x_n),u_n)-f)) \\
&\quad -J^{\rho}_{H,M(\cdot,w_{n-1})}(H(g(x_{n-1}))-\rho(N(p(x_{n-1}),u_{n-1})-f))\| \\
&\leqslant \|J^{\rho}_{H,M(\cdot,w_n)}(H(g(x_n))-\rho(N(p(x_n),u_n)-f)) \\
&\quad -J^{\rho}_{H,M(\cdot,w_n)}(H(g(x_{n-1}))-\rho(N(p(x_{n-1}),u_{n-1})-f))\| \\
&\quad +\|J^{\rho}_{H,M(\cdot,w_n)}(H(g(x_{n-1}))-\rho(N(p(x_{n-1}),u_{n-1})-f)) \\
&\quad -J^{\rho}_{H,M(\cdot,w_{n-1})}(H(g(x_{n-1}))-\rho(N(p(x_{n-1}),u_{n-1})-f))\| \\
&\leqslant \frac{1}{r}\|H(g(x_n))-H(g(x_{n-1}))-\rho[N(p(x_n),u_n)-N(p(x_{n-1}),u_{n-1})]\| \\
&\quad +\mu\|w_n-w_{n-1}\| \\
&\leqslant \frac{1}{r}\|H(g(x_n))-H(g(x_{n-1}))-\rho[N(p(x_n),u_n)-N(p(x_{n-1}),u_n)]\| \\
&\quad +\frac{\rho}{r}\|N(p(x_{n-1}),u_n)-N(p(x_{n-1}),u_{n-1})\|+\mu\|w_n-w_{n-1}\|.
\end{aligned} \tag{3.6}$$

Since g is α-strongly monotone and β-Lipschitz continuous, we have

$$\begin{aligned}
&\|x_n-x_{n-1}-(g(x_n)-g(x_{n-1}))\|^2 \\
&\quad = \|x_n-x_{n-1}\|^2-2\langle x_n-x_{n-1},g(x_n)-g(x_{n-1})\rangle+\|g(x_n)-g(x_{n-1})\|^2 \\
&\quad \leqslant (1-2\alpha+\beta^2)\|x_n-x_{n-1}\|^2.
\end{aligned} \tag{3.7}$$

Since T is ξ-$\hat{\mathbf{H}}$-Lipschitz continuous, Q is ζ-$\hat{\mathbf{H}}$-Lipschitz continuous, H is σ-Lipschitz continuous, p is τ-Lipschitz continuous and $N(\cdot,\cdot)$ is (γ,c)-relaxed cocoercive with respect to H in the first argument and Lipschitz continuous with respect to both arguments with constants $\delta>0$ and $\epsilon>0$, respectively, now we obtain

$$\begin{aligned}
\|N(p(x_{n-1}),u_n)-N(p(x_{n-1}),u_{n-1})\| &\leqslant \epsilon\|u_n-u_{n-1}\| \\
&\leqslant \epsilon(1+n^{-1})\hat{\mathbf{H}}(T(x_n),T(x_{n-1})) \\
&\leqslant \epsilon\xi(1+n^{-1})\|x_n-x_{n-1}\|,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}\|w_n - w_{n-1}\| &\leqslant (1+n^{-1})\hat{\mathbf{H}}(Q(x_n), Q(x_{n-1})) \\ &\leqslant \zeta(1+n^{-1})\|x_n - x_{n-1}\|\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\|H(g(x_n)) - H(g(x_{n-1})) - \rho[N(p(x_n), u_n) - N(p(x_{n-1}), u_n)]\|^2 \\ &\quad = \|H(g(x_n)) - H(g(x_{n-1}))\|^2 + \rho^2\|N(p(x_n), u_n) - N(p(x_{n-1}), u_n)\|^2 \\ &\qquad - 2\rho\langle N(p(x_n), u_n) - N(p(x_{n-1}), u_n), H(g(x_n)) - H(g(x_{n-1}))\rangle \\ &\quad \leqslant \sigma^2\|g(x_n) - g(x_{n-1})\|^2 + \rho^2\delta^2\|p(x_n) - p(x_{n-1})\|^2 \\ &\qquad - 2\rho[-\gamma\|p(x_n) - p(x_{n-1})\|^2 + c\|g(x_n) - g(x_{n-1})\|^2] \\ &\quad = (\sigma^2 - 2\rho c)\|g(x_n) - g(x_{n-1})\|^2 + (\rho^2\delta^2 + 2\rho\gamma)\|p(x_n) - p(x_{n-1})\|^2 \\ &\quad \leqslant [(\sigma^2 - 2\rho c)\beta^2 + (\rho^2\delta^2 + 2\rho\gamma)\tau^2]\|x_n - x_{n-1}\|^2.\end{aligned} \tag{3.10}$$

From (3.5)-(3.10), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\leqslant [1 - \lambda + \lambda(\sqrt{1 - 2\alpha + \beta^2} + r^{-1}\sqrt{(\sigma^2 - 2\rho c)\beta^2 + (\rho^2\delta^2 + 2\rho\gamma)\tau^2} \\ &\qquad + \rho\epsilon\xi r^{-1}(1+n^{-1}) + \mu\zeta(1+n^{-1}))]\|x_n - x_{n-1}\| + \lambda\|e_n - e_{n-1}\| \\ &= s_n\|x_n - x_{n-1}\| + \lambda\|e_n - e_{n-1}\|,\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}s_n &= 1 - \lambda[1 - (k_n + h_n(\rho))], \\ k_n &= \sqrt{1 - 2\alpha + \beta^2} + \mu\zeta(1 + n^{-1}), \\ h_n(\rho) &= \rho\epsilon\xi r^{-1}(1 + n^{-1}) + r^{-1}\sqrt{(\sigma^2 - 2\rho c)\beta^2 + (\rho^2\delta^2 + 2\rho\gamma)\tau^2}.\end{aligned}$$

Letting $s = 1 - \lambda[1 - (k + h(\rho))]$, where

$$\begin{aligned}k &= \sqrt{1 - 2\alpha + \beta^2} + \mu\zeta, \\ h(\rho) &= \rho\epsilon\xi r^{-1} + r^{-1}\sqrt{(\sigma^2 - 2\rho c)\beta^2 + (\rho^2\delta^2 + 2\rho\gamma)\tau^2},\end{aligned}$$

then we have

$$k_n \to k, \quad h_n(\rho) \to h(\rho), \quad s_n \to s, \quad \text{as } n \to \infty.$$

From condition (3.4), we know that $0 < s < 1$ and hence there exist $n_0 > 0$ and $s_0 \in (s, 1)$ such that $s_n \leqslant s_0$ for all $n \geqslant n_0$. Therefore, by (3.11), we have

$$\|x_{n+1} - x_n\| \leqslant s_0\|x_n - x_{n-1}\| + \lambda\|e_n - e_{n-1}\|, \quad \forall n \geqslant n_0. \tag{3.12}$$

(3.12) implies that

$$\|x_{n+1} - x_n\| \leqslant s_0^{n-n_0}\|x_{n_0+1} - x_{n_0}\| + \lambda\sum_{j=1}^{n-n_0} s_0^{j-1} t_{n-(j-1)}, \quad \forall n \geqslant n_0,$$

where $t_n = \|e_n - e_{n-1}\|$ for all $n > n_0$. Hence, for any $m \geqslant n > n_0$, we have

$$
\begin{aligned}
\|x_m - x_n\| &\leqslant \sum_{i=n}^{m-1} \|x_{i+1} - x_i\| \\
&\leqslant \sum_{i=n}^{m-1} s_0^{i-n_0} \|x_{n_0+1} - x_{n_0}\| + \lambda \sum_{i=n}^{m-1} [\sum_{j=1}^{i-n_0} s_0^{j-1} t_{i-(j-1)}] \\
&\leqslant \sum_{i=n}^{m-1} s_0^{i-n_0} \|x_{n_0+1} - x_{n_0}\| + \lambda \sum_{i=n}^{m-1} s_0^{i} [\sum_{j=1}^{i-n_0} \frac{t_{i-(j-1)}}{s_0^{i-(j-1)}}].
\end{aligned} \tag{3.13}
$$

Since $\sum_{i=1}^{\infty} t_i\ \kappa^{-i} < \infty,\ \forall \kappa \in (0,1)$ and $s_0 < 1$, it follows from (3.13) that $\|x_m - x_n\| \to 0$, as $n \to \infty$, and so $\{x_n\}$ is a Cauchy sequence in E. Thus, there exists $x^* \in E$ such that $x_n \to x^*$ as $n \to \infty$.

Now we prove that $u_n \to u^* \in T(x^*)$ and $w_n \to w^* \in Q(x^*)$. In fact, it follows from (3.8) and (3.9) that $\{u_n\}$ and $\{w_n\}$ are also Cauchy sequences in E. Let $u_n \to u^*$ and $w_n \to w^*$, respectively. In the sequel, we will show that $u^* \in T(x^*)$ and $w^* \in Q(x^*)$. Noting $u_n \in T(x_n)$, we have

$$
\begin{aligned}
d(u^*, T(x^*)) &= \inf\{\|u_n - y\| : y \in T(x^*)\} \\
&\leqslant \|u^* - u_n\| + d(u_n, T(x^*)) \\
&\leqslant \|u^* - u_n\| + \hat{\mathbf{H}}(T(x_n), T(x^*)) \\
&\leqslant \|u^* - u_n\| + \xi\|x_n - x^*\| \to 0.
\end{aligned}
$$

Hence $d(u^*, T(x^*)) = 0$ and therefore $u^* \in T(x^*)$. Similarly, we can prove that $w^* \in Q(x^*)$. Since

$$
x_{n+1} = (1-\lambda)x_n + \lambda[x_n - g(x_n) + J^{\rho}_{H,M(\cdot,w_n)}(H(g(x_n)) - \rho(N(p(x_n),u_n) - f))] + \lambda e_n,
$$

we can obtain

$$
g(x^*) = J^{\rho}_{H,M(\cdot,w^*)}(H(g(x^*)) - \rho(N(p(x^*),u^*) - f)).
$$

By Lemma 3.1, (x^*, u^*, w^*) is a solution of problem (2.1). This completes the proof. □

From Theorem 3.1, we have the following results.

THEOREM 3.2. *Let $T, Q : E \to E$ be ξ-Lipschitz continuous and ζ-Lipschitz continuous, respectively. Suppose that f, g, p, N and M are the same as in Theorem* 3.1 *and conditions* (3.3) *and* (3.4) *hold, then the iterative sequences $\{x_n\}$ generated by Algorithm* 3.2 *converge strongly to the unique solution x^* of problem* (2.3).

Proof. By Theorem 3.1, the problem (2.3) has a solution $x^* \in E$ and $x_n \to x^* (n \to \infty)$, where $\{x_n\}$ is the iterative sequences generated by Algorithm 3.2. Now we prove that x^* is a unique solution of the problem (2.3). In fact, if x is also a solution of the problem (2.3), then

$$g(x) = J^{\rho}_{H,M(\cdot,Q(x))}(H(g(x)) - \rho(N(p(x),T(x)) - f)).$$

It follows that

$$\begin{aligned}
\|x^* - x\| &= \|x^* - x - (g(x^*) - g(x)) + J^{\rho}_{H,M(\cdot,Q(x^*))}(H(g(x^*)) - \rho(N(p(x^*),T(x^*)) - f)) \\
&\quad - J^{\rho}_{H,M(\cdot,Q(x))}(H(g(x)) - \rho(N(p(x),T(x)) - f))\| \\
&\leqslant \|x^* - x - (g(x^*) - g(x))\| \\
&\quad + \|J^{\rho}_{H,M(\cdot,Q(x^*))}(H(g(x^*)) - \rho(N(p(x^*),T(x^*)) - f)) \\
&\quad - J^{\rho}_{H,M(\cdot,Q(x^*))}(H(g(x)) - \rho(N(p(x),T(x)) - f))\| \\
&\quad + \|J^{\rho}_{H,M(\cdot,Q(x^*))}(H(g(x)) - \rho(N(p(x),T(x)) - f)) \\
&\quad - J^{\rho}_{H,M(\cdot,Q(x))}(H(g(x)) - \rho(N(p(x),T(x)) - f))\| \\
&\leqslant \sqrt{1 - 2\alpha + \beta^2}\|x^* - x\| + \mu\|Q(x^*) - Q(x)\| \\
&\quad + \frac{1}{r}\|H(g(x^*)) - H(g(x)) - \rho[N(p(x^*),T(x^*)) - N(p(x),T(x^*))]\| \\
&\quad + \frac{\rho}{r}\|N(p(x),T(x^*)) - N(p(x),T(x))\| \\
&\leqslant \theta\|x^* - x\|,
\end{aligned}$$

where

$$\theta = \sqrt{1 - 2\alpha + \beta^2} + \mu\zeta + \frac{1}{r}\sqrt{(\sigma^2 - 2\rho c)\beta^2 + (\rho^2\delta^2 + 2\rho\gamma)\tau^2} + \frac{\rho\epsilon\xi}{r}.$$

It follows from (3.4) that $0 < \theta < 1$ and so $x^* = x$. This complete the proof. □

THEOREM 3.3. *Let $T, Q : E \to C(E)$ be ξ-$\hat{\mathbf{H}}$-Lipschitz continuous and ζ-$\hat{\mathbf{H}}$-Lipschitz continuous, respectively. Suppose that $H : E \to E$ is r-strongly monotone and σ-Lipschitz continuous, for each fixed $w \in E$, $M(\cdot,w) : E \to 2^E$ be a H-monotone mapping. Let p be τ-Lipschitz continuous, g be α-strongly monotone and β-Lipschitz continuous, and $N : E \times E \to E$ be c-strongly monotone with respect to H in the first argument and Lipschitz continuous with respect to both arguments with constants $\delta > 0$ and $\epsilon > 0$, respectively. If there exist constants $\rho > 0$ and $\mu > 0$ such that for each $x, y, t \in E$,*

$$\|J^{\rho}_{H,M(\cdot,x)}(t) - J^{\rho}_{H,M(\cdot,y)}(t)\| \leqslant \mu\|x - y\| \tag{3.14}$$

and

$$\begin{cases}
k = \sqrt{1 - 2\alpha + \beta^2} + \mu\zeta < 1, \\
\rho\epsilon\xi + \sqrt{\sigma^2\beta^2 - 2\rho c\beta^2 + \rho^2\delta^2\tau^2} < r(1 - k), \\
\sum_{i=1}^{\infty}\|e_i - e_{i-1}\|\,\kappa^{-i} < \infty,\ \forall\kappa \in (0,1), \quad \lim_{n\to\infty} e_n = 0,
\end{cases} \tag{3.15}$$

then the iterative sequences $\{x_n\}, \{u_n\}$ *and* $\{w_n\}$ *generated by Algorithm* 3.1 *converge strongly to* x^*, u^* *and* w^**, respectively, and* (x^*, u^*, w^*) *is a solution of problem* (2.1)*.*

THEOREM 3.4. *Let* $T, Q : E \to E$ *be* ξ*-Lipschitz continuous and* ζ*-Lipschitz continuous, respectively. Suppose that* f, g, p, N *and* M *are the same as in Theorem* 3.3 *and conditions* (3.14) *and* (3.15) *hold, then the iterative sequences* $\{x_n\}$ *generated by Algorithm* 3.2 *converge strongly to the unique solution* x^* *of problem* (2.3)*.*

REMARK 3.3. If we choose suitable f, N, T, Q, p, g and M, then Theorems 3.1-3.4 can be degenerated to many known results of (generalized) variational inequalities as special cases (see, for example, [5, 15, 24] and the references therein).

REMARK 3.4. We can construct a new perturbed iterative algorithm for solving the generalized nonlinear variational inclusion (2.3) and prove the convergence and stability of the iterative sequence generated by the perturbed iterative algorithm. For more details, we refer to [10, 15] and the references therein.

REFERENCES

[1] S. ADLY, *Perturbed algorithm and sensitivity analysis for a general class of variational inclusions*, J. Math. Anal. Appl., **201**, (1996), 609–630.

[2] R. AHMAD, Q. H. ANSARI, *An iterative algorithm for generalized nonlinear variational inclusions*, Appl. Math. Lett., **13**, (5) (2000), 23–26.

[3] Y. J. CHO, J. H. KIM, N. J. HUANG AND S. M. KANG, *Ishikawa and Mann iterative processes with errors for generalized strongly nonlinear implicit quasi-variational inequalities*, Publ. Math. Debrecen, **58**, (2001), 635–649.

[4] X. P. DING, *Perturbed proximal point algorithm for generalized quasivariational inclusions*, J. Math. Anal. Appl., **210**, (1) (1997), 88–101.

[5] X. P. DING, *Generalized quasi-variational-like inclusions with nonconvex functionals*, Appl. Math. & Comput., **122**, (3) (2001), 267–282.

[6] X. P. DING, C. L. LUO, *Perturbed proximal point algorithms for general quasi-variational-like inclusions*, J. Comput. Appl. Math., **113**, (2000), 153–165.

[7] Y. P. FANG, N. J. HUANG, *H-monotone operator and resolvent operator technique for variational inclusions*, Applied Mathematics and Computation, **145**, (2003), 795–803.

[8] Y. P. FANG, N. J. HUANG, *H-monotone operators and system of variational inclusions*, Communications on Applied Nonlinear Analysis, **11**, (2004), 93–101.

[9] A. HASSOUNI, A. MOUDAFI, *perturbed algorithm for variational inclusions*, J. Math. Anal. Appl., **185**, (3) (1994), 706–712.

[10] N. J. HUANG, *Mann and Ishikawa type perturbed iterative algorithms for generalized nonlinear implicit quasi-variational inclusions*, Computers Math. Appl., **35**, (10) (1998), 1–7.

[11] N. J. HUANG, *Generalized nonlinear variational inclusions with noncompact valued mappings*, Appl. Math. Lett., **9**, (3) (1996), 25–29.

[12] N. J. HUANG, *Generalized nonlinear implicit quasivariational inclusion and an application to implicit variational inequalities*, Z. Angew. Math. Mech., **79**, (8) (1999), 569–575.

[13] M. M. JIN, *Perturbed algorithm and stability for strongly nonlinear quasi-variational inclusion involving H-monotone operators*, Math. Inequal. Appl., **9**, (4) (2006), 771–779.

[14] J. S. JUNG, C. H. MORALES, *The Mann process for perturbed m-accretive opertators in Banach spaces*, Nonlinear Anal. **46**, (2) (2001), 231–243.

[15] H. Y. LAN, J. K. KIM AND N. J. HUANG, *On the generalized nonlinear quasi-variational inclusions involving non-monotone set-valued mappings*, Nonlinear Funct. Anal. & Appl., **9**, (3) (2004), 451–465.

[16] C. H. LEE, Q. H. ANSARI AND J. C. YAO, *A perturbed algorithm for strongly nonlinear variational-like inclusions*, Bull. Austral. Math. Soc., **62**, (2000), 417–426.

[17] L. W. LIU, Y. Q. LI, *On generalized set-valued variational inclusions*, J. Math. Anal. Appl., **261**, (1) (2001), 231–240.

[18] S. B. NALDER, *Multi-valued contraction mappings*, Pacific J. Math., **30**, (1969), 475–488.
[19] Z. NANIEWICZ, P. D. PANAGIOTOPOULOS, *Mathematical Theory of Hemi-variational Inequalities and Applications*, Marcel Dekker, New York, 1995.
[20] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications II: Monotone Operators*, Springer-Verlag, Berlin, 1985.
[21] M. A. NOOR, K. I. NOOR AND T. M. RASSIAS, *Invitation to variational inequalities*, in: Analysis, Geometry and Groups: A Riemann Legacy Volume, Hadronic, FL, (1993), 373–448.
[22] SALAHUDDIN, A. RAIS, *Generalized multivalued nonlinear quasi-variational like inclusions*, Nonlinear Anal. Forum, **6**, (2) (2001), 409–416.
[23] S. H. SHIM, S. M. KANG, N. J. HUANG AND Y. J. CHO, *Perturbed iterative algorithms with errors for completely generalized strongly nonlinear implicit quasivariational inclusions*, J. Inequal. Appl., **5**, (4) (2000), 381–395.
[24] R. U. VERMA, *Nonlinear H-monotone variational inclusions and resolvent operator technique*, Int. J. Pure & Appl. Math. Sci., **2**, (1) (2005), 53–57.
[25] R. U. VERMA, *Generalized system for relaxed cocoercive variational inequalities and projection methods*, Journal of Optimization Theory and Applications, **121**, (2004), 203–210.
[26] R. U. VERMA, *Partially relaxed cocoercive variational inequalities and auxiliary problem principle*, Journal of Applied Mathematics and Stochatic Analysis, **17**, (2) (2004), 143–148.
[27] GEORGE X. Z. YUAN, *KKM Theory and Applications in Nonlinear Analysis*, Marcel Dekker, New York, 1999.

(Received September 1, 2005)

Yeol Je Cho
Department of Mathematics Education and the RINS
College of Education
Gyeongsang National University
Chinju 660-701
Korea
e-mail: yjcho@gsnu.ac.kr

Corresponding author: Heng-you Lan
Department of Mathematics
Sichuan University of Science & Engineering
Zigong, Sichuan 643000
P. R. China
e-mail: hengyoulan@163.com