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GENERALIZED (ρ,θ)–η–B–VEXITY AND GENERALIZED (ρ,θ)–η–B–PREINVEXITY

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Abstract. In this paper we study some relations between (ρ,θ) - η -B-vexity and (ρ,θ) - η -B-vex sets in a real Banach space X. In particular we also establish some of the relations between (ρ,θ) - η -B-invexity and (ρ,θ) - η -preinvexity in a real Banach space X.

1. Introduction

Bector et al.[1] considered a class of functions called B-vex functions by generalizing the convexity definition of these functions. These functions are quite similar to the strong pseudo convex functions introduced by Bector [2] and the (α,λ)-convex functins given by Castagnoli et al.[4]. Hanson [5] introduced the class of invex functions. Hanson's paper gave a new direction of research giving rise to a great deal of additional results. This notion of η-invexity was originally introduced by Hanson [5], who showed that for a nonlinear programming problem whose objective and constrained functions are η-invex (all with respect to the same η), the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient. Later, Kaul et al.[7] named them η-convex and defined η-pseudo-convex and η-quasi-convex functions. They established the relations between convex, pseudo-convex and quasi-convex functions. Ben-Israel et al.[3] and Hanson et al.[6] introduced a class of functions which are called preinvex by Weir et al.[9] as a generalization of convexity.

In this paper we study some of the properties of (ρ,θ) - η - B -vex functions and extend the class of (ρ,θ) - η - B -vex functions to (ρ,θ) - η -pseudo- B -vex and (ρ,θ) - η -quasi- B -vex functions. We also establish some relations between (ρ,θ) - η - B -vexity, (ρ,θ) - η - B -invexity and (ρ,θ) - η -preinvexity.

2. Notations and preliminaries

Let X be a real Banach space and D be a non empty open convex subset of X. We define the following concepts which we need in sequel.

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DEFINITION 2.1. A numerical function $f : D \subseteq X \to R$ is said to be (ρ,θ)-B-vex at $y \in D$ if there exists a function $b(x,y,\lambda) : D \times D \times [0,1] \to R_+$, $\theta : D \times D \to X$ and $\rho \in R$ such that $f(\lambda x + (1-\lambda)y) \leqslant \lambda b(x,y,\lambda)f(x) + (1-\lambda b(x,y,\lambda))f(y) + \rho\|\theta(x,y)\|^2$ for $0 \leqslant \lambda \leqslant 1$ and $\forall\ x \in X$. f is said to be (ρ,θ)-B-vex on D if it is (ρ,θ)-B-vex at each $y \in D$.

DEFINITION 2.2. Given $S \subseteq D \times R$, S is said to be (ρ,θ)-B-vex set if (x,α) and $(y,\beta) \in S$ imply that $(\lambda x + (1-\lambda)y, \lambda b\alpha + (1-\lambda b)\beta + \rho\|\theta(x,y)\|^2) \in S$, $0 \leqslant \lambda \leqslant 1$. The epigraph $E(f)$ is defined by

$$E(f) = \{(x,\alpha) \mid x \in D, \alpha \in R, f(x) \leqslant \alpha\}.$$

DEFINITION 2.3. Let $y \in D$. The set D is said to be (ρ,θ)-η-invex at y with respect to η if for each $x \in D$,

$$y + \lambda\eta(x,y) + \rho\|\theta(x,y)\|^2 \in D, \quad 0 \leqslant \lambda \leqslant 1.$$

D is said to be (ρ,θ)-η-invex set with respect to η if D is invex at each $y \in D$ with respect to same η.

DEFINITION 2.4. A numerical function $f : D \subseteq X \to R$ defined on a non empty subset D of X which is invex at $y \in D$ is said to be (ρ,θ)-η-B-preinvex with respect to η at $y \in D$ if there exists $b : D \times D \times [0,1] \to R_+$ such that
$f(y + \lambda\eta(x,y)) \leqslant \lambda b(x,y,\lambda)f(x) + (1-\lambda b(x,y,\lambda))f(y) + \rho\|\theta(x,y)\|^2 \ \ \forall\ x \in D$, $0 \leqslant \lambda \leqslant 1$. f is said to be (ρ,θ)-η-B-preinvex with respect to η on D if it is (ρ,θ)-η-B-preinvex at each $y \in D$ with respect to same η.

DEFINITION 2.5. The function $f : D \subseteq X \to R$ is said to be (ρ,θ)-η-B-invex with respect to η at $y \in D$ if there exists $b : D \times D \to R_+$ such that

$$b(x,y)[f(x) - f(y)] \geqslant (\nabla f(y), \eta(x,y)) + \rho\|\theta(x,y)\|^2 \forall\, x \in X.$$

f is said to be (ρ,θ)-η-B-invex with respect to η on D if it is (ρ,θ)-η-B-invex at each $y \in D$ with respect to same η.

DEFINITION 2.6. The function $f : D \subseteq X \to R$ is said to be (ρ,θ)-η-prequasi-invex at $y \in D$ with respect to η if D is invex with respect to η, $\theta : D \times D \to X$, $\rho \in R$ and for each $x \in D$,

$$f(x) \leqslant f(y) \Rightarrow f(y + \lambda\eta(x,y)) \leqslant (1-\lambda)f(y) + \lambda f(x) + \rho\|\theta(x,y)\|^2, 0 \leqslant \lambda \leqslant 1.$$

We say that f is (ρ,θ)-η-prequasi-invex on D with respect to η if D is invex with respect to η and prequasi-invex at each $y \in D$ with respect to same η.

DEFINITION 2.7. Suppose that $S \subseteq X \times$ R. We say that S is (ρ,θ)-η-B-invex set with respect to η, b_1, b_2 if (x,α), $(y,\beta) \in S$ imply $(y + \lambda\eta(x,y), b_1\alpha + b_2\beta + \rho\|\theta(x,y)\|^2) \in S$ for $0 \leqslant \lambda \leqslant 1$, $b_1 + b_2 = 1$.

3. Generalized (ρ,θ) - B -vexity

In this section we study the relationship between (ρ,θ) - B -vexity function and (ρ,θ) - B -vexity set. Also we study the relation between (ρ,θ) - η - B -preinvexity and (ρ,θ) - η - B -invex set.

THEOREM 3.1. *A numerical function f defined on a convex set D is (ρ,θ) - B -vex if and only if $E(f)$ is a (ρ,θ) - B -vex set in $X\times R$.*

Proof. Assume that f is (ρ,θ) - B -vex on D. Let (x,α) and $(y,\beta)\in E(f)$. It follows that,

$$f(x)\leqslant\alpha,\ f(y)\leqslant\beta.$$

Since f is (ρ,θ) - B -vex on D, for $\lambda\in[0,1]$, we have

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &\leqslant \lambda bf(x)+(1-\lambda b)f(y)+\rho\|\theta(x,y)\|^2\\ &\leqslant \lambda b\alpha+(1-\lambda b)\beta+\rho\|\theta(x,y)\|^2. \end{aligned}$$

Thus

$$(\lambda x+(1-\lambda)y,\lambda b\alpha+(1-\lambda b)\beta+\rho\|\theta(x,y)\|^2)\in E(f)\text{ for }\lambda\in[0,1].$$

Conversely, suppose $E(f)$ is a (ρ,θ) - B -vex set. Let $x,y\in D$. Then

$$(x,f(x))\in E(f)\text{ and }(y,f(y))\in E(f).$$

Therefore, for $\lambda\in[0,1]$,

$$(\lambda x+(1-\lambda)y,\lambda bf(x)+(1-\lambda b)f(y)+\rho\|\theta(x,y)\|^2)\in E(f).$$

Hence

$$f(\lambda x+(1-\lambda)y)\leqslant\lambda bf(x)+(1-\lambda b)f(y)+\rho\|\theta(x,y)\|^2\text{ for }\lambda\in[0,1].$$

So f is (ρ,θ) - B -vex function on D. □

THEOREM 3.2. *Suppose $(S_i)_{i\in I}$ is a family of (ρ,θ) - B -vex sets in $D\times R$. Then their intersection $\cap_{i\in I}S_i$ is also a (ρ,θ) - B -vex set.*

Proof. Given that $(S_i)_{i\in I}$ is a family of (ρ,θ) - B -vex set. To show that $\cap_{i\in I}S_i$ is a (ρ,θ) - B -vex set. Let (x,α) and (y,β) $\in\cap_{i\in I}S_i$. Since S_i is a (ρ,θ) - B -vex set, for each $i\in I$,

$$\begin{aligned} &(\lambda x+(1-\lambda)y,\lambda bf(x)+(1-\lambda b)f(y)+\rho\|\theta(x,y)\|^2)\in S_i\ \forall\, i\in I\\ &\qquad\Rightarrow(\lambda x+(1-\lambda)y,\lambda bf(x)+(1-\lambda b)f(y)+\rho\|\theta(x,y)\|^2)\in\cap_{i\in I}S_i. \end{aligned}$$

Hence $\cap_{i\in I}S_i$ is a (ρ,θ) - B -vex set. □

THEOREM 3.3. *Assume that $(f_i)_{i\in I}$ is a family of numerical functions which are (ρ,θ) - B -vex and bounded above on a convex set D. Then the numerical function $f(x)=sup_{i\in I}f(x_i)$ is a (ρ,θ) - B -vex function on D.*

Proof. Suppose each $(f_i)_{i\in I}$ is a (ρ,θ)-B- vex function on D then its epigraph

$$E(f_i)=\{(x,\alpha):x\in X,\alpha\in R,f_i(x)\leqslant\alpha\}$$

is a (ρ,θ)-B-vex set in $D\times R$. So their intersection

$$\begin{aligned}\cap_{i\in I}E(f_i)&=\{\ (x,\alpha):x\in X,\alpha\in R,f_i(x)\leqslant\alpha\ \forall\ i\in I\}\\&=\{\ (x,\alpha):x\in X,\alpha\in R,f(x)\leqslant\alpha\}\end{aligned}$$

is also a (ρ,θ)-B-vex set by the Theorem 3.2. It follows that this intersection is the epigraph of f. Hence f is a (ρ,θ)-B-vex function on D. □

It is noted that

(a) Every differentiable B-vex function f is (ρ,θ)-pseudo-$\bar{B}$-vex function, where

$$\bar{b}(x,y)=\lim_{\lambda\to0+}b(x,y,\lambda).$$

But the converse is not necessarily true. We show this by an example.

EXAMPLE 3.1. Define $f:(-1,1)\to R$ by

$$f(x)=x+x^3.$$

Define $\theta:D\times D\to R$ by

$$\theta(x,y)=\begin{cases}\sqrt{x-y} & \text{if } x>y,\\ 0 & \text{if } x\leqslant y\end{cases}$$

and $\rho=$ -1. Define $b:D\times D\times[0,1]\to R_+$ by

$$b(x,y,\lambda)=\begin{cases}1-\lambda & \text{if } x\geqslant y,\\ 0 & x<y.\end{cases}$$

Then

$$\bar{b}(x,y)=\begin{cases}1 & \text{if } x\geqslant y,\\ 0 & x<y.\end{cases}$$

First we have to show that f is (ρ,θ)-pseudo-$\bar{B}$-vex function, i.e. to show that

$$(\nabla f(y),x-y)+\rho\|\theta(x,y)\|^2\geqslant0\Rightarrow\bar{b}(x,y)f(x)\geqslant\bar{b}(x,y)f(y).$$

Then

$$(\nabla f(y),x-y)+\rho\|\theta(x,y)\|^2=(1+3y^2,x-y)-(x-y)=(x-y)3y^2\geqslant0\ \forall\ x\ ,y.$$

It follows that $\bar{b}(x,y)(f(x)-f(y))=\bar{b}(x,y)[(x-y)+(x^3-y^3)]\geqslant0\ \forall\ x,\ y$. Taking $x=\frac{-1}{4},\ y=\frac{-1}{2},\ \lambda=\frac{1}{2}$. Then

$$f(\lambda x+(1-\lambda y))>\lambda b(x,y,\lambda)f(x)+(1-\lambda b(x,y,\lambda))f(y).$$

Hence f is not B-vex function.

(b) Every pseudo-convex function is (ρ,θ)-pseudo-B-vex function but the converse is not necessarily true when $b(x,y)=0$ for some $x,y\in D$.

EXAMPLE 3.2. Define $f:(-1,1)\to R$ by

$$f(x)=x^3.$$

Define $\theta : D\times D\to R$ by

$$\theta(x,y)=\begin{cases}\sqrt{y-x} & \text{if } y>x,\\ 0 & \text{if } x\geqslant y.\end{cases}$$

Taking $\rho=-1$. Define $b:D\times D\to R_+$ by

$$b(x,y)=\begin{cases}1 & \text{if } xy\neq 0,\\ 0 & xy=0.\end{cases}$$

We have to show that f is (ρ,θ)-pseudo-B-vex function, i.e. to show that

$$(\nabla f(y),x-y)+\rho\|\theta(x,y)\|^2\geqslant 0\Rightarrow b(x,y)f(x)\geqslant b(x,y)f(y).$$

Then $(\nabla f(y),x-y)+\rho\|\theta(x,y)\|^2 = 3y^2(x-y)\geqslant 0\ \forall\, x\,,y.$

It follows that $b(x,y)(f(x)-f(y))=(x-y)+(x^3-y^3)\geqslant 0\ \forall\, x,\ y.$

We show that f is not pseudo-convex function. $(\nabla f(y),x-y)\geqslant 0$ but $f(x)<f(y)$ at $x=\frac{-1}{3},\ y=0.$

Hence f is not pseudo-convex function.

(c) Every invex function f with respect to η is (ρ,θ)-η-B-invex with respect to the same η, where $b(x,y)=1$. If $b(x,y)\neq 1$, then there exist functions which are (ρ,θ)-η-B-invex with respect to η but not invex with respect to same η.

EXAMPLE 3.3. Define $f:(0,\frac{\pi}{2})\to R$ by

$$f(x)=\sin x.$$

Define $\theta : D\times D\to R$ by

$$\theta(x,y)=\begin{cases}\sqrt{x-y} & \text{if } x\geqslant y,\\ 0 & \text{if } x<y.\end{cases}$$

Taking $\rho=-1$. Define $b:D\times D\to R_+$ by

$$b(x,y)=\begin{cases}2 & \text{if } x\geqslant y,\\ 0 & \text{if } x<y.\end{cases}$$

Define $\eta : D\times D\to R$ by

$$\eta(x,y)=x-y.$$

First we show that f is (ρ,θ)-η-B-invex, i.e. to show that

$$b(x,y)(f(x)-f(y))\geqslant(\nabla f(y),\eta(x,y))+\rho\|\theta(x,y)\|^2.$$

Then

$$\begin{aligned}&b(x,y)(\sin x-\sin y)-(\cos y,x-y)+(x-y)\\&\qquad=2b(x,y)\sin(\frac{x-y}{2})\cos(\frac{x+y}{2})-(x-y)(\cos y-1)\geqslant 0\ \forall\, x,y.\end{aligned}$$

Hence f is (ρ,θ)-η-B-invex but f is not invex with respect to η because

$$(\nabla f(y),\eta(x,y)) > f(x)-f(y)$$

at $x=\frac{\pi}{4}$ and $y=\frac{\pi}{6}$.

(d) Every pseudo-invex function f with respect to η is (ρ,θ)-η-pseudo-B-invex function with respect to same η but the converse is not necessarily true when $b(x,y)=0$ for some $x,y\in D$.

EXAMPLE 3.4. Define $f:(0,\frac{\pi}{2})\to R$ by

$$f(x)=\cos x.$$

Define $\theta: D\times D\to R$ by

$$\theta(x,y)=\begin{cases}\sqrt{y-x} & \text{if } y>x,\\ 0 & \text{if } y\leqslant x.\end{cases}$$

Taking $\rho=-1$. Define $b: D\times D\to R_+$ by

$$b(x,y)=\begin{cases}0 & \text{if } x\geqslant y,\\ xy & \text{if } x<y.\end{cases}$$

Define $\eta: D\times D\to R$ by

$$\eta(x,y)=y-x.$$

First we show that f is (ρ,θ)-η-pseudo-B-invex, i.e. to show that

$$(\nabla f(y),\eta(x,y))+\rho\|\theta(x,y)\|^2\geqslant 0\Rightarrow b(x,y)f(x)\geqslant b(x,y)f(y).$$

Now

$$\begin{aligned}(\nabla f(y),\eta(x,y))+\rho\|\theta(x,y)\|^2 &= (-\sin y)(y-x)-(y-x)\\ &= -(y-x)(\sin y+1)\geqslant 0\ \forall\, x,y.\end{aligned}$$

Then $b(x,y)(f(x)-f(y))\geqslant 0\ \forall\, x,y$. Hence f is (ρ,θ)-η-pseudo-B-invex with respect to η but f is not pseudo-invex with respect to η because $(\nabla f(y),\eta(x,y))\geqslant 0$ but $f(x)<f(y)$ at $x=\frac{\pi}{3}$, $y=\frac{\pi}{6}$.

(e) Every quasi-invex function f with respect to η is (ρ,θ)-η-quasi-B-invex function with respect to same η but the converse is not necessarily true.

EXAMPLE 3.5. Define $f:(0,\frac{\pi}{2})\to R$ by

$$f(x)=\sin x.$$

Define $\theta: D\times D\to R$ by

$$\theta(x,y)=\begin{cases}\sqrt{y-x} & \text{if } y>x,\\ 0 & \text{if } y\leqslant x.\end{cases}$$

Taking $\rho = -1$. Define $b : D \times D \to R_+$ by

$$b(x,y) = \begin{cases} 0 & \text{if } x \geqslant y, \\ xy & \text{if } x < y. \end{cases}$$

Define $\eta : D \times D \to R$ by

$$\eta(x,y) = y - x.$$

First we show that f is (ρ,θ) - η -quasi- B -invex, i.e. to show that

$$b(x,y)(f(x) - f(y)) \leqslant 0 \Rightarrow (\nabla f(y), \eta(x,y)) + \rho\|\theta(x,y)\|^2 \leqslant 0.$$

Now

$$\begin{aligned} b(x,y)(f(x) - f(y)) &= b(x,y)(\sin x - \sin y) \\ &= 2b(x,y)\sin(\frac{x-y}{2})\cos(\frac{x+y}{2}) \leqslant 0 \;\forall\, x,y. \end{aligned}$$

Then

$$\begin{aligned} (\nabla f(y), \eta(x,y)) + \rho\|\theta(x,y)\|^2 &= (\cos y, y - x) - (y - x) \\ &= (y - x)(\cos y - 1) \leqslant 0 \;\forall\, x,y. \end{aligned}$$

Hence f is (ρ,θ) - η -quasi- B -invex with respect to η but f is not quasi-invex with respect to η because

$$f(x) \leqslant f(y) \; but \; (\nabla f(y), \eta(x,y)) > 0$$

at $x = \frac{\pi}{6}$, $y = \frac{\pi}{3}$.

(f) Every differentiable B -vex function is (ρ,θ) - η - $\bar{B}$ -invex with respect to η where

$$\bar{b}(x,y) = \lim_{\lambda \to 0+} b(x,y,\lambda).$$

But the converse is not necessarily true.

EXAMPLE 3.6. Define $f : (0, \frac{\pi}{2}) \to R$ by

$$f(x) = \sin x.$$

Define $\theta : D \times D \to R$ by

$$\theta(x,y) = \begin{cases} \sqrt{x-y} & \text{if } x > y, \\ 0 & \text{if } x \leqslant y. \end{cases}$$

Taking $\rho = -1$. Define $b : D \times D \times [0,1] \to R_+$ by

$$b(x,y,\lambda) = \begin{cases} 1 & \text{if } x \geqslant y, \\ \lambda & x < y. \end{cases}$$

Then

$$\bar{b}(x,y) = \begin{cases} 1 & \text{if } x \geqslant y, \\ 0 & x < y. \end{cases}$$

Define $\eta : D \times D \to R$ by

$$\eta(x,y) = x - y.$$

First we have to show that f is (ρ,θ)-$\bar{B}$-invex function, i.e. to show that

$$\bar{b}(x,y)(f(x)-f(y)) \geqslant (\nabla f(y), x-y) + \rho\|\theta(x,y)\|^2.$$

Then

$$\begin{aligned}&\bar{b}(x,y)(\sin x - \sin y) - (\cos y, y-x) + (x-y)\\ &\qquad = 2\sin(\frac{x-y}{2})\cos(\frac{x+y}{2}) - (x-y)(\cos y - 1) \geqslant 0\ \forall\, x,y.\end{aligned}$$

Thus f is (ρ,θ)-$\bar{B}$-invex function but f is not B-vex function since

$$f(\lambda x + (1-\lambda y)) > \lambda b(x,y,\lambda) f(x) + (1-\lambda b(x,y,\lambda) f(y))$$

at $x=\frac{\pi}{3}$, $y=\frac{\pi}{6}$ and $\lambda=\frac{1}{2}$.

(g) Every B-invex function f with respect to η is (ρ,θ)-η-quasi-B-invex function with respect to same η but the converse is not necessarily true.

EXAMPLE 3.7. Define $f:(0,\frac{\pi}{2}) \to R$ by

$$f(x) = \sin x.$$

Define $\theta : D\times D \to R$ by

$$\theta(x,y) = \begin{cases} \sqrt{y-x} & \text{if } y > x,\\ 0 & \text{if } y \leqslant x.\end{cases}$$

Taking $\rho = -1$. Define $b : D\times D \to R_+$ by

$$b(x,y) = \begin{cases} 0 & \text{if } x \geqslant y,\\ xy & \text{if } x < y.\end{cases}$$

Define $\eta : D\times D \to R$ by

$$\eta(x,y) = y - x.$$

First we show that f is (ρ,θ)-η-quasi-B-invex, i.e. to show that

$$b(x,y)(f(x)-f(y)) \leqslant 0 \Rightarrow (\nabla f(y), \eta(x,y)) + \rho\|\theta(x,y)\|^2 \leqslant 0.$$

Now

$$\begin{aligned} b(x,y)(f(x)-f(y)) &= b(x,y)(\sin x - \sin y)\\ &= 2b(x,y)\sin(\frac{x-y}{2})\cos(\frac{x+y}{2}) \leqslant 0\ \forall\, x,y.\end{aligned}$$

Then

$$\begin{aligned} (\nabla f(y), \eta(x,y)) + \rho\|\theta(x,y)\|^2 &= (\cos y, y-x) - (y-x)\\ &= (y-x)(\cos y - 1) \leqslant 0\ \forall\, x,y.\end{aligned}$$

Hence f is (ρ,θ) - η -quasi- B -invex with respect to η but f is not B -invex with respect to η because

$$b(x,y)(f(x)-f(y)) < (\nabla f(y),\eta(x,y))$$

at $x=\frac{\pi}{6},\ y=\frac{\pi}{3}$

Some Properties of (ρ,θ) - η - B -preinvex functions:

PROPOSITION 3.1. *Suppose that f is (ρ,θ) - η - B -preinvex (strictly) on D with respect to η, b_1, b_2, then f is (ρ,θ) - η -prequasi-invex (strictly) on D with respect to same η.*

Proof. Given that

$$f(y+\lambda\eta(x,y)) \leqslant b_1 f(x)+b_2 f(y)+\rho\|\theta(x,y)\|^2(<),\ b_1+b_2=1, \forall\, x,y\in D. \quad (1)$$

To show that,

$$f(x)\leqslant f(y) \Rightarrow f(y+\lambda\eta(x,y)) \leqslant f(y)+\rho\|\theta(x,y)\|^2(<).$$

Since $f(x)\leqslant f(y)$, from (1), we have

$$f(y+\lambda\eta(x,y)) \leqslant f(y)+\rho\|\theta(x,y)\|^2(\text{since } b_1+b_2=1).$$

Hence f is (ρ,θ) - η -prequasi-invex (strictly) on D with respect to η. □

Now we will study the characterization of (ρ,θ) - η - B -preinvex functions in terms of (ρ,θ) - η - B -invex set.

THEOREM 3.4. *A numerical function f defined on an invex set $D\subseteq X$ is (ρ,θ) - η - B -preinvex with respect to η, b_1, b_2 if and only if $E(f)$ is (ρ,θ) - η - B -invex set with respect to η, b_1, b_2.*

Proof. Assume that f is (ρ,θ) - η - B -preinvex on D with respect to η, b_1, b_2. Let (x,α) and $(y,\beta)\in E(f)$. Then

$$f(x)\leqslant\alpha \text{ and } f(y)\leqslant\beta.$$

By hypothesis of f on D, we have , for $0\leqslant\lambda\leqslant 1$

$$\begin{aligned} f(y+\lambda\eta(x,y)) &\leqslant b_1 f(x)+b_2 f(y)+\rho\|\theta(x,y)\|^2\\ &\leqslant b_1\alpha+b_2\beta+\rho\|\theta(x,y)\|^2\\ \Rightarrow (y+\lambda\eta(x,y), b_1\alpha+b_2\beta+\rho\|\theta(x,y)\|^2) &\in E(f), 0\leqslant\lambda\leqslant 1.\end{aligned}$$

Thus $E(f)$ is (ρ,θ) - η - B -invex set with respect to η, b_1, b_2.
Conversely, suppose $E(f)$ is (ρ,θ) - η - B -invex set with respect to η, b_1, b_2, and let $x,y\in D$, then

$$\begin{gathered}(x,f(x))\in E(f), (y,f(y))\in E(f)\\ \Rightarrow\ (y+\lambda\eta(x,y), b_1 f(x)+b_2 f(y)+\rho\|\theta(x,y)\|^2)\in E(f)\\ 0\leqslant\lambda\leqslant 1, b_1+b_2=1\\ \Rightarrow\ f(y+\lambda\eta(x,y))\leqslant b_1 f(x)+b_2 f(y)+\rho\|\theta(x,y)\|^2.\end{gathered}$$

Hence f is (ρ,θ) - η - B -preinvex with respect to η, b_1, b_2. □

REFERENCES

[1] C. R. BECTOR, C. SINGH, *B-vex Functions*, Journal of Optimization Theory and Applications, **71**, (1991), 237–257.
[2] C. R. BECTOR, *Mathematical Analysis of Some Nonlinear Programming Problems*, Ph.D. Thesis, Indian Institute of Technology, Kanpur, India, 1968.
[3] A. BEN-ISRAEL, B. MOND, *What is Invexity*?, Journal of the Australian Mathematical Society, Series B, **28**, (1986), 1–9.
[4] E. CASTAGNOLI, P. MAZZOLENI, *About Derivatives of Some Generalized Concave Functions*, Journal of Information and Optimization Sciences, **10**, (1989), 53–65.
[5] M. A. HANSON, *On Sufficiency of The Kuhn-Tucker Conditions*, Journal of Mathematical Analysis and Applications, **80**, (1981), 545–550.
[6] M. A. HANSON, B. MOND, *Convex Transformable Programming Problems and Invexity*, Journal of Information and Optimization Sciences, **8**, (1987), 201–207.
[7] R. N. KAUL, S. KAUR, *Optimality Criteria in Nonlinear Programming Involving Nonconvex Functions*, Journal of Mathematical Analysis and Applications, **105**, (1985), 104–112.
[8] R. PINI, C. SINGH, *Generalized Convexity and Generalized Monotonocity*, Journal of Information and Optimization Sciences, **20**, (2) (1999), 215–233.
[9] T. WEIR, W. JEYAKUMAR, *Strong and Weak Invexity in Mathematical Programming*, Methods of Operations Research, **55**, (1985), 109–125.
[10] T. WEIR, B. MOND, *Preinvex Functions in Multiple Objective Optimization*, Journal of Mathematical Analysis and Applications, **136**, (1988), 29–38.

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