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THE DUNKL–WILLIAMS INEQUALITY IN AN INNER PRODUCT SPACE

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(*communicated by L. Maligranda*)

Abstract. We prove a refinement of the well known Dunkl-Williams Inequality in an inner product space.

1. In a normed linear space

The Dunkl-Williams Inequality [1] for nonzero x, y in a normed linear space is:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{4\,\|x-y\|}{\|x\| + \|y\|}. \tag{1}$$

A number of refinements of (1) have been obtained by various authors. The Maligranda Inequality [2],

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{|\|x\| - \|y\|| + \|x-y\|}{\max(\|x\|\,, \|y\|)}\,,$$

appears to be the sharpest currently known. (The Massera-Schäffer inequality,

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{2\,\|x-y\|}{\max(\|x\|\,, \|y\|)}\,,$$

though not as sharp as Maligranda's, does sharpen (1) and actually preceded it [3].)

We begin by proving a reverse Maligranda Inequality, which we will use below. This can be deduced from [2], though it is not explicitly mentioned there. We take a different approach, which makes the current investigation self contained.

THEOREM 1. *For nonzero x, y in a normed linear space, we have*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geqslant \frac{\|x-y\| - |\,\|y\| - \|x\|\,|}{\min(\|x\|\,, \|y\|)}.$$

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Proof. We have

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 = \frac{1}{\|x\|^2 \|y\|^2} \| \, \|y\| \, x - \|x\| \, x - (\|x\| \, y - \|x\| \, x) \, \|^2$$
$$\geqslant \frac{1}{\|x\|^2 \|y\|^2} (\| \, \|y\| \, x - \|x\| \, x \, \| - \| \, \|x\| \, y - \|x\| \, x \, \| \,)^2$$
$$= \frac{\|x\|^2}{\|x\|^2 \|y\|^2} \left(\, |\|y\| - \|x\|| - \|x - y\| \, \right)^2$$
$$= \frac{1}{\|y\|^2} \left(\, \|x - y\| - | \, \|y\| - \|x\| \, | \, \right)^2.$$

Now doing the same, except "$+ \|y\| \, y - \|y\| \, y$" instead of "$- \|x\| \, x + \|x\| \, x$" gives

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 \geqslant \frac{1}{\|x\|^2} (\|x - y\| - |\|y\| - \|x\| \, |)^2. \quad \square$$

2. In an inner product space

The Dunkl-Williams Inequality [1] for nonzero x, y in an inner product space is:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{2 \, \|x - y\|}{\|x\| + \|y\|}. \tag{2}$$

In fact, (2) characterizes an inner product space: $\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{2\|x-y\|}{\|x\|+\|y\|}$ in a normed linear space X implies that X is an inner product space [4]. Taking $x = (1, 0)$, $y = (0, 2)$ in $\mathbb{R}^2$, for example, shows that refinements of (2) analogous to those of (1) mentioned in Section 1 above are not available: $\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \sqrt{2}$, but $\frac{\|x-y\|}{\max(\|x\|,\|y\|)} = \sqrt{5}/2 \cong 1.12$. We prove a different sort of refinement.

THEOREM 2. *For nonzero x, y in an inner product space, we have*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{(\|x\| - \|y\|)^2}{(\|x\| + \|y\|)^2} + \sqrt{\frac{4 \, \|x - y\|^2}{(\|x\| + \|y\|)^2} + \frac{(\|x\| - \|y\|)^4}{(\|x\| + \|y\|)^4} - 4 \frac{(\|x\| - \|y\|)^2}{(\|x\| + \|y\|)^2}}.$$

Proof. We write $\alpha = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$. Theorem 1 reads

$$\alpha \geqslant \frac{\|x - y\| - | \, \|y\| - \|x\| \, |}{\min(\|x\| \, , \|y\|)}.$$

Now $| \, \|y\| - \|x\| \, | - \|x - y\| + 2 \min(\|x\| \, , \|y\|) = \|y\| + \|x\| - \|x - y\|$, so that

$$\|y\| + \|x\| - \|x - y\| \geqslant (2 - \alpha) \min(\|x\| \, , \|y\|).$$

A simple computation reveals that $\frac{\langle x,y\rangle}{\|x\|\|y\|} = 1 - \frac{1}{2}\alpha^2$, and this together with the Law of Cosines gives

$$\alpha^2 = \frac{\|x-y\|^2 - (\,\|y\| - \|x\|\,)^2}{\|x\|\,\|y\|}.$$

Therefore we have

$$\begin{aligned}
&\|x-y\|^2 - \left(\frac{\|y\|+\|x\|}{2}\right)^2 \alpha^2 \\
&= \frac{(\|y\|-\|x\|)^2}{4\,\|x\|\,\|y\|}\left[(\|x\|+\|y\|)^2 - \|x-y\|^2\right] \\
&= \frac{(\|y\|-\|x\|)^2}{4\,\|x\|\,\|y\|}(\|x\|+\|y\|-\|x-y\|)(\|x\|+\|y\|+\|x-y\|) \\
&\geqslant \frac{(\|y\|-\|x\|)^2}{4\,\|x\|\,\|y\|}(\|x\|+\|y\|-\|x-y\|)(\|x\|+\|y\|+|\|x\|-\|y\||) \\
&= \frac{(\|y\|-\|x\|)^2}{4\,\|x\|\,\|y\|}[\|x\|+\|y\|-\|x-y\|]\,2\max(\|x\|\,,\|y\|) \\
&\geqslant \frac{(\|y\|-\|x\|)^2}{4\,\|x\|\,\|y\|}(2-\alpha)\min(\|x\|\,,\|y\|)2\max(\|x\|\,,\|y\|) \\
&= \frac{(\|y\|-\|x\|)^2}{2}(2-\alpha).
\end{aligned}$$

Thus

$$\|x-y\|^2 - \left(\frac{\|y\|+\|x\|}{2}\right)^2 \alpha^2 \geqslant \frac{(\|y\|-\|x\|)^2}{2}(2-\alpha),$$

or

$$\alpha^2 - 2\left(\frac{\|y\|-\|x\|}{\|y\|+\|x\|}\right)^2 \alpha + \frac{4\left[(\|y\|-\|x\|)^2 - \|x-y\|^2\right]}{(\|y\|+\|x\|)^2} \leqslant 0.$$

Therefore, α is no larger than the positive root of the quadratic

$$\lambda^2 - 2\left(\frac{\|y\|-\|x\|}{\|y\|+\|x\|}\right)^2 \lambda + \frac{4\left[(\|y\|-\|x\|)^2 - \|x-y\|^2\right]}{(\|y\|+\|x\|)^2}.$$

That is,

$$\alpha \leqslant \frac{(\|x\|-\|y\|)^2}{(\|x\|+\|y\|)^2} + \sqrt{\frac{4\,\|x-y\|^2}{(\|x\|+\|y\|)^2} + \frac{(\|x\|-\|y\|)^4}{(\|x\|+\|y\|)^4} - 4\frac{(\|x\|-\|y\|)^2}{(\|x\|+\|y\|)^2}}. \quad \square$$

NOTES. As with (2), equality holds in Theorem 2 if and only if either $\|x\| = \|y\|$ or $\|x-y\| = \|x\| + \|y\|$. Theorem 2 is indeed a refinement of (2); one can see this by direct verification, or by seeing that the quadratic above is nonnegative, upon substituting $\lambda = \frac{2\|x-y\|}{\|x\|+\|y\|}$. Following the latter course, one can verify the following.

COROLLARY 3. *For nonzero x, y in an inner product space, we have*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{2\,\|x-y\|}{\|x\|+\|y\|} - t,$$

where

$$0 \leqslant t = \frac{2\left(\frac{\|x\|-\|y\|}{\|x\|+\|y\|}\right)^2\left(2-\frac{2\|x-y\|}{\|x\|+\|y\|}\right)}{\frac{2\|x-y\|}{\|x\|+\|y\|}-\left(\frac{\|x\|-\|y\|}{\|x\|+\|y\|}\right)^2+\sqrt{\frac{4\|x-y\|^2}{(\|x\|+\|y\|)^2}+\frac{(\|x\|-\|y\|)^4}{(\|x\|+\|y\|)^4}-4\frac{(\|x\|-\|y\|)^2}{(\|x\|+\|y\|)^2}}}.$$

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