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CONTRACTED MODULAR DIOPHANTINE INEQUALITIES AND NUMERICAL SEMIGROUPS

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Abstract. We study the set $\mathrm{T}(a,b,c)$ of all integer solutions to the Diophantine inequality $ax \bmod b \leqslant x-c$, with a,b,c nonnegative integers and $b \neq 0$. We obtain an exact formula for the cardinality of $\mathbb{N} \setminus \mathrm{T}(a,b,c)$ and give an algorithm to decide whether or not a numerical semigroup can be represented as $\mathrm{T}(a,b,c) \cup \{0\}$.

1. Introduction

Given two nonnegative integers m and n, with $n \neq 0$, we denote by $m \bmod n$ the remainder of the division of m by n.

Following the notation in [6], [4] and [3], a modular Diophantine inequality is an expression of the form $ax \bmod b \leqslant x$, with a,b nonnegative integers and $b \neq 0$. The set of integer solutions to this inequality is denoted by $\mathrm{S}(a,b)$ and it turns out to be a numerical semigroup, that is, a subset of the set of nonnegative integers $\mathbb{N}$ that is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ has finitely many elements. A numerical semigroup S is said to be modular if there exist nonnegative integers a,b with $b \neq 0$ such that $S = \mathrm{S}(a,b)$.

A contracted modular Diophantine inequality is an expression of the form $ax \bmod b \leqslant x-c$, with a,b,c nonnegative integers such that $b \neq 0$. We denote by $\mathrm{T}(a,b,c)$ the set of integer solutions to this inequality. As we show in Section 2., $\mathrm{T}(a,b,c) \cup \{0\}$ is a numerical semigroup which we call contracted modular. Accordingly, a numerical semigroup S is said to be contracted modular if there exist nonnegative integers a,b,c, with $b \neq 0$, such that $S = \mathrm{T}(a,b,c) \cup \{0\}$. We call this expression a contracted modular representation of S.

Given a set X, we denote by $\#X$ the cardinality of X.

The main result is this paper is Theorem 14 which exhibit a formula for $\#(\mathbb{N} \setminus \mathrm{T}(a,b,c))$ in terms of a,b,c. We also give a deterministic algorithm to decide whether a numerical semigroup is contracted modular or not.

By using this algorithm we have explored many examples and this has led us to conjecture that the class of contracted modular numerical semigroups is included in the

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class of system proportionally modular numerical semigroups. System proportionally modular numerical semigroups are introduced in [7] and recently it has been proved that they are exactly those numerical semigroups which admit a Toms' decomposition (see [8, 5]).

2. Preliminaries and basic results

Along this paper and unless otherwise stated we will assume a, b, c to be nonnegative integers such that $b \neq 0$.

It is clear that $\mathrm{T}(a,b,c) \subseteq \mathrm{S}(a,b)$. The following result, which proof is immediate from definitions, establishes when equality holds.

LEMMA 1. *The following conditions are equivalent:*
(1) $\mathrm{T}(a,b,c) = \mathrm{S}(a,b)$.
(2) $0 \in \mathrm{T}(a,b,c)$.
(3) $c = 0$.

PROPOSITION 2. $\mathrm{T}(a,b,c) \cup \{0\}$ *is a numerical semigroup.*

Proof. Call $S = \mathrm{T}(a,b,c)$ and suppose that $x, y \in S$. Then $ax \bmod b \leqslant x - c$ and $ay \bmod b \leqslant y - c$. This implies that $a(x+y) \bmod b \leqslant (ax \bmod b) + (ay \bmod b) \leqslant x - c + y - c \leqslant x + y - c$, whence $x + y \in S$, and so S is closed under addition.

Since $ax \bmod b \leqslant b - 1$ for any integer x, we see that if $x \geqslant b + c - 1$, then $ax \bmod b \leqslant x - c$, and so $x \in S$. This implies that $\mathbb{N} \setminus S$ is finite. $\square$

If S is a numerical semigroup, the least positive integer in S is called the multiplicity of S and is represented by $\mathrm{m}(S)$. The largest integer not in S is called the Frobenius number of S and it is denoted by $\mathrm{g}(S)$. This integer has been widely studied in literature (see for instance [2]). The numbers $\mathrm{m}(S)$, $\mathrm{g}(S)$ and $\#(\mathbb{N} \setminus S)$ are three important invariants for a numerical semigroup S (see also [1]).

COROLLARY 3. *Let* $S = \mathrm{T}(a,b,c) \cup \{0\}$. *Then:*
(1) $\mathrm{g}(S) \leqslant b + c - 2$.
(2) $c \leqslant \mathrm{m}(S)$.

Proof.
(1) It is a consequence of the fact that any integer $x \geqslant b + c - 1$ belongs to S.
(2) If $x \in \mathrm{T}(a,b,c)$, since $ax \bmod b \geqslant 0$, we have $0 \leqslant x - c$. $\square$

If z is an integer, we denote by $\{z, \rightarrow\}$ the set of all integers greater than or equal to z. Clearly $\{0\} \cup \{n, \rightarrow\}$ is a numerical semigroup for any nonnegative integer n. We say that a numerical semigroup S is a halfline, if $S = \{0\} \cup \{n, \rightarrow\}$ for some nonnegative integer n. Given a real number x, we set $\lceil x \rceil = \min\{z \in \mathbb{Z} \mid z \geqslant x\}$ and $\lfloor x \rfloor = \max\{z \in \mathbb{Z} \mid z \leqslant x\}$ as usual.

As we show in the next result, any numerical semigroup which is a halfline is also contracted modular.

LEMMA 4. *Let* b, c *be nonnegative integers, with* $b \neq 0$. *Then:*
(1) $\mathrm{T}(0,b,c) = \{c, \rightarrow\}$.
(2) $\mathrm{T}(1,b,c) = \{\lceil \frac{c}{b} \rceil b, \rightarrow\}$.

Proof. (1) It is clear from definitions.

(2) If $x \in \mathrm{T}(1,b,c)$, then $x \bmod b \leqslant x-c$. This implies that there exists $k \in \mathbb{N}$ such that $0 \leqslant x-kb \leqslant x-c$. From this we deduce that $k \geqslant \lceil \frac{c}{b} \rceil$ and $x \geqslant kb$, and so $x \geqslant \lceil \frac{c}{b} \rceil b$. Conversely, if $x \geqslant \lceil \frac{c}{b} \rceil b$, then $x \bmod b \leqslant x - \lceil \frac{c}{b} \rceil b \leqslant x - \frac{c}{b} b = x-c$. This means that $x \in \mathrm{T}(1,b,c)$. $\square$

Note that $\mathrm{T}(a,b,c) = \mathrm{T}(a \bmod b, b, c)$, so we can suppose without any loss of generality that $a < b$. Moreover, by Lemma 4 if $\mathrm{T}(a,b,c) \cup \{0\}$ is not a halfline, we can also assume that $2 \leqslant a \leqslant b-1$.

If A and B are two subsets of $\mathbb{Z}$, we call $A+B = \{a+b \mid a \in A,\ b \in B\}$.

PROPOSITION 5. *If $c = bq+r$, with $q,r \in \mathbb{N}$, then* $\mathrm{T}(a,b,c) = \{c-r\} + \mathrm{T}(a,b,r)$.

Proof. Note that if $x \in \mathrm{T}(a,b,c)$, then $x \geqslant c$ and so $x = y+c$ for some $y \in \mathbb{N}$. Hence the proof reduces to show that $y+c \in \mathrm{T}(a,b,c)$ if and only if $y+r \in \mathrm{T}(a,b,r)$. But this is immediate because $a(y+c) \bmod b = a(y+qb+r) \bmod b = a(y+r) \bmod b$, and so $a(y+c) \bmod b \leqslant y$ if and only if $a(y+r) \bmod b \leqslant y$. $\square$

In view of Proposition 5 we can restrict to the case $c \leqslant b-1$. This assumption will be crucial to prove Theorem 14 in the next section.

A semigroup is a nonempty set with an associative and commutative binary operation on it. A nonempty subset I of a semigroup S is said to be an ideal if $S+I \subseteq I$. An ideal I is principal if there exists $s \in S$ such that $I = \{s\}+S$.

COROLLARY 6. *If $c = bq+r$, with $q,r \in \mathbb{N}$ and $qb \geqslant r$, then $\mathrm{T}(a,b,c)$ is a principal ideal of* $\mathrm{T}(a,b,r)$.

Proof. By Proposition 5, it is enough to show that $c-r \in \mathrm{T}(a,b,c)$. But this is clear because $a(qb) \bmod b = 0 \leqslant qb-r$ and so $c-r = qb \in \mathrm{T}(a,b,r)$. $\square$

3. The number of gaps of a contracted modular Diophantine inequality

Any element of the set $\mathbb{N} \setminus \mathrm{T}(a,b,c)$ is said to be a gap of the contracted modular Diophantine inequality $ax \bmod b \leqslant x-c$.

This section is devoted to prove Theorem 14 which supplies a formula for the cardinality of the set $\mathbb{N} \setminus \mathrm{T}(a,b,c)$ in terms of a,b and c.

If n,m are integers with not both of them equal to 0, we represent by $\gcd(m,n)$ the greatest common divisor of m and n, as usual. In this section a and b will denote positive integers such that $2 \leqslant a \leqslant b-1$. We call $d = \gcd(a-1,b)$ and $d' = \gcd(a,b)$. Since $\gcd(\frac{a-1}{d}, \frac{b}{d}) = 1$, there exists a unique integer $u \in \{1,\ldots,\frac{b}{d}-1\}$ verifying that $\frac{a-1}{d} u \bmod \frac{b}{d} = 1$ (u is the inverse of $\frac{a-1}{d}$ modulo $\frac{b}{d}$).

LEMMA 7. ([6], Theorem 12) *Under the previous conditions,* $\#(\mathbb{N} \setminus \mathrm{S}(a,b)) = \frac{b+1-d-d'}{2}$.

The following result, which proof is immediate from definitions, presents the set $\mathrm{T}(a,b,c)$ as being obtained from the modular numerical semigroup $\mathrm{S}(a,b)$ by removing some of its elements. This can be thought of as if the original numerical semigroup

$\mathrm{S}(a,b)$ were shorten or contracted to the numerical semigroup $\mathrm{T}(a,b,c)\cup\{0\}$. This justifies the terminology we use in this paper.

LEMMA 8. *If* $c \geqslant 1$, *then* $\mathrm{T}(a,b,c) = \mathrm{S}(a,b) \setminus \{x \in \mathbb{N} \mid ax \bmod b = x - i$ *for some* i *such that* $0 \leqslant i \leqslant c-1\}$.

We note that if $ax \bmod b = x - i$ for some integer x, then i must be a multiple of d. From this observation we can refine the statement of Lemma 8 as follows.

LEMMA 9. *If* $c \geqslant 1$, *then* $\mathrm{T}(a,b,c) = \mathrm{S}(a,b) \setminus \{x \in \mathbb{N} \mid ax \bmod b = x - kd$ *for some* k *such that* $0 \leqslant k \leqslant \lfloor \frac{c-1}{d} \rfloor\}$.

For any nonnegative integer k we call $H_k = \{x \in \mathbb{N} \mid ax \bmod b = x - kd\}$. Hence by Lemma 9 we see that the set $\mathrm{T}(a,b,c)$ can be obtained from $\mathrm{S}(a,b)$ by iteratively removing all the elements in the set H_k for every k from 0 to $\lfloor \frac{c-1}{d} \rfloor$. In the next lemma we explicitly determine the elements of H_k.

LEMMA 10. *Let* k *be an integer. Then* $ax \bmod b = x - kd$ *if and only if* $kd \leqslant x \leqslant kd + b - 1$ *and* $x = (\frac{b}{d} - u)k \bmod \frac{b}{d} + t\frac{b}{d}$ *for some integer* t.

Proof. Clearly, $ax \bmod b = x - kd$ if and only if $0 \leqslant x - kd \leqslant b - 1$ and $ax \equiv x - kd \bmod b$, that is, $0 \leqslant x - kd \leqslant b-1$ and $\frac{a-1}{d}x \equiv -k \bmod \frac{b}{d}$. Hence $ax \bmod b = x - kd$ if and only if $kd \leqslant x \leqslant kd + b - 1$ and $x = (\frac{b}{d} - u)k \bmod \frac{b}{d} + t\frac{b}{d}$ for some integer t. $\square$

LEMMA 11. *For any integer* z, $\#\{t \in \mathbb{Z} \mid z \leqslant t\frac{b}{d} \leqslant z + b - 1\} = d$.

Proof. First we note that $z \leqslant t\frac{b}{d}$ if and only if $t \geqslant \lceil \frac{dz}{b} \rceil$. We prove for any $l \in \mathbb{N}$, that $(\lceil \frac{dz}{b} \rceil + l)\frac{b}{d} \leqslant z + b - 1$ if and only if $l + 1 \leqslant d$. To achieve this, we distinguish two cases:

1. Suppose that $\lceil \frac{dz}{b} \rceil = \lfloor \frac{dz}{b} \rfloor + 1$. Then $(\lceil \frac{dz}{b} \rceil + l)\frac{b}{d} \leqslant z + b - 1$ if and only if $(\lfloor \frac{dz}{b} \rfloor + l + 1)\frac{b}{d} \leqslant z + b - 1$, and this is equivalent to $dz - dz \bmod b + b + bl \leqslant db - d + dz$. So $(\lceil \frac{dz}{b} \rceil + l)\frac{b}{d} \leqslant z + b - 1$ if and only if $l + 1 \leqslant d + \frac{dz \bmod b - d}{b}$. We note that the hypothesis for this case also implies that $dz \bmod b \neq 0$, and so $dz \bmod b = d(z \bmod \frac{b}{d})$ is a multiple of d which is strictly contained between 0 and b. This implies that $0 \leqslant \frac{dz \bmod b - d}{b} < 1$ and so the inequality $l + 1 \leqslant d + \frac{dz \bmod b - d}{b}$ is equivalent to $l + 1 \leqslant d$.
2. If $\lceil \frac{dz}{b} \rceil \neq \lfloor \frac{dz}{b} \rfloor + 1$, then $\lceil \frac{dz}{b} \rceil = \frac{dz}{b}$. This implies that $(\lceil \frac{dz}{b} \rceil + l)\frac{b}{d} \leqslant z + b - 1$ if and only if $(\frac{dz}{b} + l)\frac{b}{d} \leqslant z + b - 1$, and this is equivalent to $l + 1 \leqslant d$. $\square$

LEMMA 12. *For any integer* k, $\#\{x \in \mathbb{N} \mid ax \bmod b = x - kd\} = d$.

Proof. By Lemma 10, $\#\{x \in \mathbb{N} \mid ax \bmod b = x - kd\} = \#\{t \in \mathbb{Z} \mid z \leqslant t\frac{b}{d} \leqslant z + b - 1\}$, with $z = kd - (\frac{b}{d} - u)k \bmod \frac{b}{d}$. To conclude the proof, simply apply Lemma 11. $\square$

LEMMA 13. *Let* $c, k, \overline{k}$ *be integers such that* $1 \leqslant c \leqslant b - 1$ *and* $0 \leqslant k \leqslant \overline{k} \leqslant \lfloor \frac{c-1}{d} \rfloor$. *If* $(\frac{b}{d} - u)k \bmod \frac{b}{d} = (\frac{b}{d} - u)\overline{k} \bmod \frac{b}{d}$, *then* $k = \overline{k}$.

Proof. From the hypothesis $(\frac{b}{d}-u)k \bmod \frac{b}{d} = (\frac{b}{d}-u)\overline{k} \bmod \frac{b}{d}$ we obtain that $(\frac{b}{d}-u)(k-\overline{k}) \equiv 0 \bmod \frac{b}{d}$. As $\gcd(\frac{b}{d}-u, \frac{b}{d}) = 1$, we get that $k \equiv \overline{k} \bmod \frac{b}{d}$. Since we are assuming that $c \leqslant b-1$, we also have $\lfloor \frac{c-1}{d} \rfloor < \frac{b}{d}$. From this we deduce that $0 \leqslant \overline{k} - k < \frac{b}{d}$ and so $\overline{k} - k = 0$. □

Finally, we are in position to prove the main result in this section.

THEOREM 14. *Let a, b and c be nonnegative integers such that $a < b$. Then*

$$\#(\mathbb{N} \setminus \mathrm{T}(a,b,c)) = \frac{b+1-d-d'}{2} + \Big(\Big\lfloor \frac{c-1}{d} \Big\rfloor + 1\Big)d.$$

Proof. If $a = 0$ or $a = 1$, the formula in the lemma is an easy consequence of Lemma 4. So we will assume that $2 \leqslant a \leqslant b-1$. We consider three cases:

1. If $c = 0$, then $\mathrm{T}(a,b,c) = \mathrm{S}(a,b)$. In this case the result follows straightly from Lemma 7.
2. Suppose that $1 \leqslant c \leqslant b-1$. By Lemma 9 we know that $\mathrm{S}(a,b)$ is the disjoint union of $\mathrm{T}(a,b,c)$ and $\cup_{k=0}^{\lfloor \frac{c-1}{d} \rfloor} H_k$. This implies that

$$\#(\mathbb{N} \setminus \mathrm{T}(a,b,c)) = \#(\mathbb{N} \setminus \mathrm{S}(a,b)) + \#\Big(\bigcup_{k=0}^{\lfloor \frac{c-1}{d} \rfloor} H_k \Big).$$

   By applying Lemmas 10 and 13, we get $H_k \cap H_{\overline{k}} = \emptyset$ for any $k, \overline{k}$ such that $0 \leqslant k < \overline{k} \leqslant \lfloor \frac{c-1}{d} \rfloor$, and this leads to

$$\#\Big(\bigcup_{k=0}^{\lfloor \frac{c-1}{d} \rfloor} H_k \Big) = \sum_{k=0}^{\lfloor \frac{c-1}{d} \rfloor} \# H_k.$$

   For this case now the result follows from Lemma 12.
3. If $c \geqslant b$, then $c = qb + r$ with $r = c \bmod b < b$. By the cases (1) and (2) we have

$$\#(\mathbb{N} \setminus \mathrm{T}(a,b,r)) = \frac{b+1-d-d'}{2} + (\Big\lfloor \frac{r-1}{d} \Big\rfloor + 1)d.$$

   Now by applying Proposition 5, we obtain that

$$\#(\mathbb{N} \setminus \mathrm{T}(a,b,c)) = (c-r) + \#(\mathbb{N} \setminus \mathrm{T}(a,b,r)) =$$

$$(c-r) + \frac{b+1-d-d'}{2} + (\Big\lfloor \frac{r-1}{d} \Big\rfloor + 1)d.$$

   By using that b is a multiple of d and that $c - r = qb$, we deduce that $c - r$ is also a multiple of d and so

$$\#(\mathbb{N} \setminus \mathrm{T}(a,b,c)) = \frac{b+1-d-d'}{2} + \Big(\frac{c-r}{d} + \Big\lfloor \frac{r-1}{d} \Big\rfloor + 1\Big)d =$$

$$\frac{b+1-d-d'}{2} + \Big(\Big\lfloor \frac{c-1}{d} \Big\rfloor + 1\Big)d. \qquad \square$$

EXAMPLE 15. Let us apply the formula in Theorem 14 to count the number of gaps of the Diophantine inequality $7x \bmod 35 \leqslant x-4$.

We have $a=7, b=35, c=4$, $d=\gcd(a-1,b)=1$ and $d'=\gcd(a,b)=7$. We get $\#(\mathbb{N}\setminus \mathrm{T}(7,35,4))=\frac{35+1-1-7}{2}+(\lfloor\frac{4-1}{1}\rfloor+1)\cdot 1=18$. In fact one can check that $\mathbb{N}\setminus \mathrm{T}(7,35,4)=\{0,1,2,3,4,6,7,8,9,12,13,14,17,18,19,23,24,29\}$.

4. An algorithm to decide whether a numerical semigroup is contracted modular or not

Let S be a numerical semigroup. If S is a halfline, say $S=\{0\}\cup\{m,\rightarrow\}$, then from Lemma 4 we know that $S=\mathrm{T}(0,b,m)\cup\{0\}$ for any positive integer b, so in this case S is contracted modular. If S is not a halfline, then S is contracted modular if and only if there exist nonnegative integers a,b,c such that $2\leqslant a\leqslant b-1$ and $S=\mathrm{T}(a,b,c)\cup\{0\}$.

LEMMA 16. *Suppose that a,b are integers such that $2\leqslant a\leqslant b-1$. Then $\gcd(a-1,b)+\gcd(a,b)\leqslant \frac{b}{2}+2$.*

Proof. Call $d=\gcd(a-1,b)$ and $d'=\gcd(a,b)$. The hypothesis $a\leqslant b-1$, implies that $d\leqslant\frac{b}{2}$ and $d'\leqslant\frac{b}{2}$. Clearly the lemma is true when $d=1$ or $d'=1$. Hence we assume $d\geqslant 2$ and $d'\geqslant 2$. Since $\gcd(d,d')=1$, we have that b is a multiple of dd' and in particular $dd'\leqslant b$. This implies that $d+d'\leqslant d+\frac{b}{d}$. To conclude the proof, it is enough to see that $d+\frac{b}{d}\leqslant 2+\frac{b}{2}$, which is true when $d\leqslant\frac{b}{2}$. $\square$

LEMMA 17. *Let a,b,c be nonnegative integers such that $2\leqslant a\leqslant b-1$. Then $b\leqslant 4\#(\mathbb{N}\setminus\mathrm{T}(a,b,c))+2$.*

Proof. Call $d=\gcd(a-1,b)$ and $d'=\gcd(a,b)$. From Theorem 14 we know that $\#(\mathbb{N}\setminus\mathrm{T}(a,b,c))\geqslant\frac{b+1-d-d'}{2}$ and so $2\#(\mathbb{N}\setminus\mathrm{T}(a,b,c))\geqslant b+1-d-d'$. By Lemma 16 we have $2\#(\mathbb{N}\setminus\mathrm{T}(a,b,c))\geqslant b+1-\frac{b}{d}-2$ and thus $b\leqslant 4\#(\mathbb{N}\setminus\mathrm{T}(a,b,c))+2$. $\square$

As a consequence of Corollary 3 and Lemma 17 we obtain the following result.

PROPOSITION 18. *Let S be a numerical semigroup. If $S=\mathrm{T}(a,b,c)\cup\{0\}$ for some nonnegative integers a,b,c such that $2\leqslant a\leqslant b-1$, then $c\leqslant\mathrm{m}(S)$ and $\mathrm{g}(S)-\mathrm{m}(S)+2\leqslant b\leqslant 4\#(\mathbb{N}\setminus S)+6$.*

COROLLARY 19. *Let S be a contracted modular numerical semigroup which is not a halfline. Then there exists only a finite number of triples $(a,b,c)\in\mathbb{N}^3$ such that $2\leqslant a\leqslant b-1$ and $S=\mathrm{T}(a,b,c)\cup\{0\}$.*

As we saw in Lemma 4, any numerical semigroup which is a halfline is also a contracted modular numerical semigroup and admits an infinite number of contracted modular representations.

Given $n_1,\dots,n_p\in\mathbb{N}$, we set $\langle n_1,\dots,n_p\rangle=\{\lambda_1 n_1+\cdots+\lambda_p n_p\mid \lambda_1,\dots,\lambda_p\in\mathbb{N}\}$. Then $\langle n_1,\dots,n_p\rangle$ is a numerical semigroup if and only if the greatest common divisor of $n_1,\dots,n_p$ is equal to 1. In this case, $\langle n_1,\dots,n_p\rangle$ is said to be the numerical semigroup generated by $n_1,\dots,n_p$.

Now we are in conditions to give the announced procedure.

ALGORITHM 20.

Input: a numerical semigroup S, generated by $n_1 < \cdots < n_p$.

Output: "yes" if S is contracted modular and "no" if S is not contracted modular.

(1) If $\{n_1, n_1+1, \ldots, 2n_1-1\} \subseteq \{n_1, \ldots, n_p\}$, then return "yes".

(2) Compute $\mathrm{g}(S), \mathrm{m}(S)$ and $\#(\mathbb{N} \setminus S)$ from the given generators $n_1, \ldots, n_p$ of S.

(3) Let $A = \{(a,b,c) \in \mathbb{N}^3 \mid \mathrm{g}(S) - \mathrm{m}(S) + 2 \leqslant b \leqslant 4\#(\mathbb{N} \setminus S) + 6,\ c \leqslant \mathrm{m}(S)$ and $2 \leqslant a \leqslant b-1\}$.

(4) If $S = \mathrm{T}(a,b,c) \cup \{0\}$ for some triple $(a,b,c) \in A$, then return "yes". Otherwise, return "no".

EXAMPLE 21. Let us apply Algorithm 20 to the numerical semigroup $S = \langle 4,6,7 \rangle$.

First we note that S is not a halfline. One can easily check that $m(S) = 4, \mathrm{g}(S) = 9$ and $\#(\mathbb{N} \setminus S) = 5$. Next we consider the set $A = \{(a,b,c) \in \mathbb{N}^3 \mid 7 \leqslant b \leqslant 26,\ 2 \leqslant a \leqslant b-1$ and $0 \leqslant c \leqslant 4\}$. It can be tested that no triple $(a,b,c) \in A$ fullfils the condition $S = \mathrm{T}(a,b,c) \cup \{0\}$. Thus S is not contracted modular.

EXAMPLE 22. Let $S = \langle 7,10,12,15,16,18 \rangle$. Now $m(S) = 7, \mathrm{g}(S) = 13$ and $\#(\mathbb{N} \setminus S) = 10$.

By applying Algorithm 20 to S we get that S is contracted modular. In fact all the contracted modular representations of S are $S = \mathrm{T}(7,16,3) \cup \{0\}$ and $S = \mathrm{T}(7,16,4) \cup \{0\}$.

EXAMPLE 23. For $S = \langle 6,7,11,15 \rangle$ we have $m(S) = 6, \mathrm{g}(S) = 16$ and $\#(\mathbb{N} \setminus S) = 9$.

After applying Algorithm 20 to S we get that S is not contracted modular.

5. Some remarks

Following the notation in [7], a numerical semigroup is proportionally modular, if it is the set of solutions to a Diophantine inequality of the type $ax \bmod b \leqslant cx$, with a,b,c nonnegative integers and $b \neq 0$. We put $\mathrm{S}(a,b,c) = \{x \in \mathbb{N} \mid ax \bmod b \leqslant cx\}$. A numerical semigroup is said to be system proportionally modular, if it can be written as an intersection of proportionally modular numerical semigroups, that is, if it is the set of solutions to a Diophantine inequation system

$$\begin{cases} a_1 x \bmod b_1 \leqslant c_1 x, \\ \qquad \vdots \\ a_r x \bmod b_r \leqslant c_r x. \end{cases}$$

It has been recently proved in [5] that the class of system proportionally modular numerical semigroups coincides with the class of numerical semigroups having a Toms' decomposition. These semigroups appear as positive cones of the K_0-group of certain C^*-algebras (see [8]).

REMARK 24.

1. As we got in Example 22, the numerical semigroup $S = \langle 7, 10, 12, 15, 16, 18\rangle$ is contracted modular. By using Algorithm 24 in [7] it can be checked that S is not proportionally modular. This shows that the class of contracted modular numerical semigroups is not included in the class of proportionally modular numerical semigroups. We also note that S is system proportionally modular. This can be seen by applying Algorithm 27 in [7]. In fact

$$S = \langle 5,6,7\rangle \cap \langle 6,7,8,9,10\rangle \cap \langle 7,8,9,10,11,12\rangle$$
$$= \mathrm{S}(3,14,1) \cap \mathrm{S}(5,30,2) \cap \mathrm{S}(12,84,5).$$

2. We saw in Example 23 that the numerical semigroup $S = \langle 6,7,11,15\rangle$ is not contracted modular. Now by applying Algorithm 24 in [7] to S we get that S is proportionally modular. In fact, it can be checked that $S = \mathrm{S}(7,38,2)$. This implies that the class of proportionally modular numerical semigroups is not included in the class of contracted modular numerical semigroups. Now it is obvious that S is system proportionally modular.

3. According to Example 21 above, the numerical semigroup $S = \langle 4,6,7\rangle$ is not contracted modular. It can be tested by using Algorithm 27 in [7] that S is not system proportionally modular.

We propose the following open problem: prove that any contracted modular numerical semigroup is system proportionally modular.

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