Mathematical
Inequalities
& Applications
Volume 10, Number 3 (2007), 499–516

ON THE LOG–CONVEXITY OF TWO–PARAMETER HOMOGENEOUS FUNCTIONS

ZHEN-HANG YANG

(*communicated by Z. Páles*)

Abstract. Suppose $f(x,y)$ is a positive homogeneous function defined on $\mathbb{U}(\subseteq \mathbb{R}_+ \times \mathbb{R}_+)$, then call $\left(\frac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{\frac{1}{p-q}}$ two-parameter homogeneous function and denote by $\mathcal{H}_f(a,b;p,q)$. If $f(x,y)$ is third differentiable, then the log-convexity with respect to parameters p and q of $\mathcal{H}_f(p,q)$ depend on the sign of $J=(x-y)(xI)_x$, where $I=(\ln f)_{xy}$. As applications a group of chains of inequalities for homogeneous means are established, which generalize, strengthen and unify Tong-po Ling 's and Stolarsky's inequalities, and a reversed chain of inequalities for exponential mean (identic mean) is derived, which contains a reversed Stolarsky's inequality. Several estimations of lower and upper bounds of extended mean are presented.

1. Introduction and main results

The so-called extended mean values between two unequal positive numbers a and b were defined first by K. B. Stolarsky in [14] as

$$E(a,b;p,q)=\begin{cases}\left(\dfrac{q}{p}\dfrac{a^p-b^p}{a^q-b^q}\right)^{\frac{1}{p-q}} & p\neq q, pq\neq 0\\ \left(\dfrac{1}{p}\dfrac{a^p-b^p}{\ln a-\ln b}\right)^{\frac{1}{p}} & p\neq 0, q=0\\ \left(\dfrac{1}{q}\dfrac{a^q-b^q}{\ln a-\ln b}\right)^{\frac{1}{q}} & p=0, q\neq 0\\ \exp\left(\dfrac{a^p\ln a-b^p\ln b}{a^p-b^p}-\dfrac{1}{p}\right) & p=q\neq 0\\ \sqrt{ab} & p=q=0\end{cases}. \tag{1.1}$$

As the generalized power-mean, C. Gini obtained a similar two-parameter type mean in [3]. That is:

$$G(a,b;p,q)=\begin{cases}\left(\dfrac{a^p+b^p}{a^q+b^q}\right)^{\frac{1}{p-q}} & p\neq q\\ \exp(\dfrac{a^p\ln a+b^p\ln b}{a^p+b^p}) & p=q.\end{cases} \tag{1.2}$$

Mathematics subject classification (2000): 26B25, 26D07, 26E60,26A48.

Key words and phrases: two-parameter homogeneous function, homogeneous mean, logarithmically convexity, chain of inequalities, estimation of lower and upper bounds.

Observe the above two-parameter type means, we find that their forms are both $\left(\frac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{\frac{1}{p-q}}$, where $f(x,y)=\frac{x-y}{\ln x-\ln y}(x,y>0,x\neq y)$, $A(x,y)=\frac{x+y}{2}(x,y>0)$. Obviously, they are both homogeneous functions of x and y. Consequently, we have the following definition.

DEFINITION 1. Assume $f:\ \mathbb{U}(\subseteqq\mathbb{R}_+\times\mathbb{R}_+)\to\mathbb{R}_+$ is an n-order homogeneous function for variables x and y, and is continuous and first order partial derivatives exist, $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$ with $a\neq b$, $(p,q)\in\mathbb{R}\times\mathbb{R}$.

If $(1,1)\notin\mathbb{U}$, then define that

$$\mathcal{H}_f(a,b;p,q) = \left(\frac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{\frac{1}{p-q}}\ (p\neq q, pq\neq 0), \tag{1.3}$$

$$\mathcal{H}_f(a,b;p,p) = \lim_{q\to p}\mathcal{H}_f(a,b;p,q) \tag{1.4}$$

$$= \exp\left(\frac{a^pf_x(a^p,b^p)\ln a+b^pf_y(a^p,b^p)\ln b}{f(a^p,b^p)}\right)\ (p=q\neq 0), \tag{1.5}$$

$f_x(x,y)$ and $f_y(x,y)$ denote partial derivative with respect to 1st and 2nd variable of $f(x,y)$ respectively.

If $(1,1)\in\mathbb{U}$, then define further

$$\mathcal{H}_f(a,b;p,0) = \left(\frac{f(a^p,b^p)}{f(1,1)}\right)^{\frac{1}{p}}\ (p\neq 0, q=0), \tag{1.6}$$

$$\mathcal{H}_f(a,b;0,q) = \left(\frac{f(a^q,b^q)}{f(1,1)}\right)^{\frac{1}{q}}\ (p=0, q\neq 0), \tag{1.7}$$

$$\mathcal{H}_f(a,b;0,0) = \lim_{p\to 0}\mathcal{H}_f(a,b;p,0) = a^{\frac{f_x(1,1)}{f(1,1)}}b^{\frac{f_y(1,1)}{f(1,1)}}\ (p=q=0). \tag{1.8}$$

From Lemma 1, $\mathcal{H}_f(a,b;p,q)$ is still a homogeneous function of positive numbers a and b. We call it a homogeneous function of positive numbers a and b with two parameters p and q, in short, we call it two-parameter homogeneous functions.

The following properties of $\mathcal{H}_f(p,q)$ are obvious by some simple calculations:

Property 1. $\mathcal{H}_f(a,b;p,q)$ are symmetric with respect to p,q i.e.

$$\mathcal{H}_f(a,b;p,q)=\mathcal{H}_f(a,b;q,p). \tag{1.9}$$

Property 2. Define that

$$G_f(x,y):=\exp\left(\frac{xf_x(x,y)\ln x+yf_y(x,y)\ln y}{f(x,y)}\right), \tag{1.10}$$

then

$$\mathcal{H}_f(a,b;p,p)=G_f^{\frac{1}{p}}(a^p,b^p). \tag{1.11}$$

Property 3. Set $T(t)=\ln f(a^t,b^t)$, then

$$T'(t)=\frac{a^tf_x(a^t,b^t)\ln a+b^tf_y(a^t,b^t)\ln b}{f(a^t,b^t)}=\ln G_f^{\frac{1}{t}}(a^t,b^t), \tag{1.12}$$

where $t \neq 0$ if $(1,1) \notin \mathbb{U}$.

Property 4. If $T'(t)$ is continuous on $[p,q]$, then

$$\ln \mathcal{H}_f(p,q) = \frac{1}{p-q}\int_q^p T'(t)dt = \frac{1}{p-q}\int_q^p \ln G_{f,t} dt. \tag{1.13}$$

Property 5. If $f(x,y) = f(y,x)$ for all $(x,y) \in \mathbb{U}$, then

$$\mathcal{H}_f(t,-t) = G^n, \tag{1.14}$$

$$T(t) - T(-t) = 2nt \ln G, \tag{1.15}$$

where $G = \sqrt{ab}$ if $(1,1) \notin \mathbb{U}$.

In the case of not being confused, we set

$$\mathcal{H}_f = \mathcal{H}_f(p,q) = \mathcal{H}_f(a,b;p,q) = \left(\frac{f(p)}{f(q)}\right)^{\frac{1}{p-q}},$$

$$G_{f,p} = G_{f,p}(a,b) = G_f^{\frac{1}{p}}(a^p,b^p) = \mathcal{H}_f(p,p).$$

By Definition 1, we have $\mathcal{H}_L(a,b;p,q) = E(a,b;p,q)$ and $\mathcal{H}_A(a,b;p,q) = G(a,b;p,q)$, which show that the conception of two-parameter homogeneous function greatly develope the extension of the conception of extended mean and Gini mean. Nevertheless, the two-parameter homogeneous function is not a mean in general, e.g. for the case $f(x,y) = |x-y|(x,y > 0, x \neq y)$ (see case 4 in section 4).

As special cases of the two-parameter homogeneous functions, the extended mean and Gini mean have been researched by various authors in [19, 18, 16, 14, 13, 12, 11, 10, 7, 6, 4, 3, 9]. It is worth mentioning that Qi Feng studied the log-convexity for the parameters of the extended mean in [10], and pointed out the two-parameters mean is a log-concave function with respect to either parameter p or q on interval $(0,+\infty)$ and is a log-convex function on interval $(-\infty,0)$. This is a very interesting and useful result.

The aim of this paper is to investigate the log-convexity with respect to the parameters of the two-parameter homogeneous functions, and get the following results:

THEOREM 1. *Let $f(x,y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$ and be third order differentiable. If*

$$J = (x-y)(xI)_x < (>)0, \text{ where } I = (\ln f)_{xy}, \tag{1.16}$$

then $\mathcal{H}_f(p,q)$ is strictly log-convex (log-concave) with respect to either p or q on $(0,+\infty)$, and log-concave (log-convex) on $(-\infty,0)$.

REMARK 1. This is a generalization of Qi Feng's result on the log-convexity of extended mean values (see [10]).

COROLLARY 1. *The conditions are the same as Theorem 1's. If (1.16) holds, then $\mathcal{H}_f(p,1-p)$ is strictly decreasing (increasing) in p on $(0,\frac{1}{2})$, increasing (decreasing) on $(\frac{1}{2},1)$.*

If $f(x,y)$ is symmetric with respect to x and y further, then the above monotone interval can be extended from $(0,\frac{1}{2})$ to $(-\infty,0)$ and $(0,\frac{1}{2})$, and $(\frac{1}{2},1)$ to $(\frac{1}{2},1)$ and $(1,+\infty)$, respectively.

COROLLARY 2. *The conditions are the same as Theorem* 1*'s. If* (1.16) *holds, then for* $p,q \in (0,+\infty)$ *with* $p \neq q$*, there is*

$$G_{f,\frac{p+q}{2}} < (>)\mathcal{H}_f(p,q) < (>)\sqrt{G_{f,p}G_{f,q}}. \tag{1.17}$$

For $p,q \in (-\infty,0)$ *with* $p \neq q$*, the inequality* (1.17) *is reversed.*

If $f(x,y)$ *is defined on* $\mathbb{R}_+ \times \mathbb{R}_+$ *and is symmetric with respect to* x *and* y *further, then substituting* $p+q>0$ *for* $p,q \in (0,+\infty)$ *and* $p+q<0$ *for* $p,q \in (-\infty,0)$*,* (1.17) *is also true, respectively.*

2. Lemmas

To prove Theorem 1, Corollary 1 and 2, we need the following lemmas, in which Lemma 1 and 2 are from section 3 in [20].

LEMMA 1. *Let* $f(x,y)$*,* $g(x,y)$ *be* n*,* m*-order homogenous functions over* Ω *respectively. Then* $f\cdot g$*,* f/g*,* $(g \neq 0)$ *are* $n+m, n-m$*-order homogenous functions over* Ω *respectively.*

If for a certain p *with* $(x^p,y^p) \in \Omega$*, and* $f^p(x,y)$ *exists, then* $f(x^p,y^p)$*,* $f^p(x,y)$ *are both* np*-order homogeneous functions over* Ω*.*

LEMMA 2. *Let* $f(x,y)$ *be an* n*-order homogeneous function over* Ω *and* f_x*,* f_y *both exist. Then* f_x*,* f_y *are both* $n-1$*-order homogeneous functions over* Ω*. Furthermore we have*

$$xf_x + yf_y = nf. \tag{2.1}$$

Particularly, when $n=1$ *and* $f(x,y)$ *is first order differentiable over* Ω*, then*

$$\begin{aligned} xf_x + yf_y &= f, & (2.2)\\ xf_{xx} + yf_{xy} &= 0, & (2.3)\\ xf_{xy} + yf_{yy} &= 0. & (2.4) \end{aligned}$$

LEMMA 3. *Let* $f(x,y)$ *be a positive* n*-order homogenous function defined on* $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$ *and be second order differentiable. Then*

$$T''(t) = -xyI(\ln b - \ln a)^2, \text{ where } I = (\ln f)_{xy}, \tag{2.5}$$

where $x=a^t, y=b^t$*.*

Proof. Since $f(x,y)$ is a positive n-order homogeneous function, from equation (2.1), we obtain

$$x(\ln f)_x + y(\ln f)_y = n \quad \text{or} \quad x(\ln f)_x = n - y(\ln f)_y. \tag{2.6}$$

By Property 2, there is

$$\begin{aligned} T'(t) &= \frac{a^t f_x(a^t,b^t)\ln a + b^t f_y(a^t,b^t)\ln b}{f(a^t,b^t)}\\ &= \frac{xf_x(x,y)\ln a + yf_y(x,y)\ln b}{f(x,y)}\\ &= x(\ln f)_x \ln a + y(\ln f)_y \ln b\\ &= n\ln a + y(\ln f)_y(\ln b - \ln a). \end{aligned} \tag{2.7}$$

Notice that $y(\ln f)_y$ is a 0-order homogeneous function, so

$$x\left(y(\ln f)_y\right)_x + y\left(y(\ln f)_y\right)_y = 0, \quad \text{or} \quad y\left(y(\ln f)_y\right)_y = -x\left(y(\ln f)_y\right)_x .$$

Hence

$$\begin{aligned}
T''(t) &= (\ln b - \ln a)\left(\frac{\partial y(\ln f)_y}{\partial x}\frac{dx}{dt} + \frac{\partial y(\ln f)_y}{\partial y}\frac{dy}{dt}\right) \\
&= (\ln b - \ln a)\left(\left(y(\ln f)_y\right)_x a^t \ln a + y\left(y(\ln f)_y\right)_y b^t \ln b\right) \\
&= (\ln b - \ln a)\left(x\left(y(\ln f)_y\right)_x \ln a - x\left(y(\ln f)_y\right)_x \ln b\right) \\
&= -(\ln b - \ln a)^2 x\left(y(\ln f)_y\right)_x \\
&= -xy(\ln f)_{xy}(\ln b - \ln a)^2 \\
&= -xyI(\ln b - \ln a)^2.
\end{aligned}$$

The proof is completed. □

LEMMA 4. *Let $f(x,y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$ and be third order differentiable. Then*

$$T'''(t) = -Ct^{-3}J, \tag{2.8}$$

where $J = (x-y)(x\,I\,)_x, t \neq 0, C = xy(x-y)^{-1}(\ln x - \ln y)^3 > 0.$

Proof. From Lemma 1 and 2, we understand that $I = (\ln f)_{xy} = (ff_{xy} - f_x f_y)/f^2$ is a -2-order homogeneous function of x and y, thus xyI is a 0-order homogeneous function. By (2.1), we get

$$x(xyI)_x + y(xyI)_y = 0, \text{ or } \quad y(xyI)_y = -x(xyI)_x. \tag{2.9}$$

By Lemma 3 and notice $x = a^t, y = b^t$, and then

$$\begin{aligned}
T'''(t) &= \frac{dT''(t)}{dt} = \frac{d\left(-xyI(\ln b - \ln a)^2\right)}{dt} \\
&= -(\ln b - \ln a)^2\left(\frac{\partial(xyI)}{\partial x}\frac{dx}{dt} + \frac{\partial(xyI)}{\partial y}\frac{dy}{dt}\right) \\
&= -(\ln b - \ln a)^2\left(a^t \ln a \cdot (xyI)_x + b^t \ln b \cdot (xyI)_y\right) \\
&= -(\ln b - \ln a)^2\left((x(xyI)_x \ln a + y \ln b (xyI)_y \ln b)\right) \\
&= -(\ln b - \ln a)^2\left(x(xyI)_x\right)(\ln a - \ln b) \\
&= (\ln b - \ln a)^3 xy(xI)_x \\
&= xy\frac{(\ln b - \ln a)^3}{x-y}\left((x-y)(xI)_x\right) \\
&= -xy\frac{(\ln x - \ln y)^3}{t^3(x-y)}\left((x-y)(xI)_x\right) \\
&= -Ct^{-3}J. \qquad \square
\end{aligned}$$

LEMMA 5. *The conditions of this Lemma are the same as Lemma* 3*, and* $f(x,y)$ *is symmetric with respect to* x *and* y*, then the following equations hold:*

$$T'(t)+T'(-t) = 2n\ln G, \tag{2.10}$$

$$T''(-t) = T''(t). \tag{2.11}$$

Proof. By direct calculating first and second derivative to variable t in two sides of equation (1.15) respectively, the equations (2.10) and (2.11) are derived immediately.
The proof is completed. □

REMARK 2. If $(1,1)\in\mathbb{U}$, i.e. $T'(0)$ exists, then $T'(0)=n\ln G$, thus the (2.10) can be rewritten as

$$T'(t)+T'(-t)=2T'(0). \tag{2.12}$$

The following lemma is from Péter Czinder and Zsolt Páles [2], which are applied in Stolarsky and Gini means. In fact it will be also applied in the two-parameter homogeneous functions in section 4.

LEMMA 6. *Let* $f : \mathcal{J}\to R$ *be symmetric with respect to an element* $m\in\overline{\mathcal{J}}$*. Furthermore, suppose that* f *is convex over the interval* $\mathcal{J}\cap(-\infty,m]$ *and concave over* $\mathcal{J}\cap[m,+\infty)$*. Then, for any interval* $[p,q]\subset\mathcal{J}$

$$f\left(\frac{p+q}{2}\right)\leqslant(\geqslant)\frac{1}{p-q}\int_q^p f(t)dt\leqslant(\geqslant)\frac{f(p)+f(q)}{2} \tag{2.13}$$

holds if $\frac{p+q}{2}\leqslant(\geqslant)m$*.*

In (2.13) *the reversed inequalities are valid if* f *is concave over the interval* $\mathcal{J}\cap[-\infty,m)$ *and convex over* $\mathcal{J}\cap[m,+\infty)$*.*

3. Proofs of main results

Next we will prove Theorem 1 and Corollaries 1-2.

Proof of Theorem 1. It needs only to prove the convexity of $\ln\mathcal{H}_f$ for p .

1) when $p\neq q$, $\ln\mathcal{H}_f=\dfrac{T(p)-T(q)}{p-q}$,

$$\frac{\partial\ln\mathcal{H}_f}{\partial p} = \frac{(p-q)T'(p)-T(p)+T(q)}{(p-q)^2}=\frac{g(p,q)}{(p-q)^2}, \tag{3.1}$$

$$\frac{\partial g(p,q)}{\partial p} = (p-q)T''(p) \tag{3.2}$$

$$\frac{\partial^2\ln\mathcal{H}_f}{\partial p^2} = \frac{(p-q)g_p(p,q)-2g(p,q)}{(p-q)^3}=\frac{k(p,q)}{(p-q)^3}, \tag{3.3}$$

$$\frac{\partial k(p,q)}{\partial p} = (p-q)^2T'''(p). \tag{3.4}$$

Notice $k(q,q)=0$, Obviously, if $T'''(p)>0$, then

$$\frac{\partial^2 \ln \mathcal{H}_f}{\partial p^2}=\frac{k(p,q)}{(p-q)^3}>0,$$

i.e. $\ln \mathcal{H}_f$ is convex in p. If $T'''(p)<0$, then it is reversed.

From Lemma 4, when $J=(x-y)(xI)_x<0$, if $p\in(0,+\infty)$, then $T'''(p)=-Cp^{-3}J>0$. While $p\in(-\infty,0)$, then $T'''(p)=-Cp^{-3}J<0$.

In the same way, when $J=(x-y)(xI)_x>0$, if $p\in(0,+\infty)$, then $T'''(p)=-Cp^{-3}J<0$. While $p\in(-\infty,0)$, then $T'''(p)=-Cp^{-3}J>0$.

2) when $p=q$, from (1.12) we have

$$\ln \mathcal{H}_f=T'(p), \tag{3.5}$$

and then

$$\frac{\partial^2 \ln \mathcal{H}_f}{\partial p^2}=T'''(p)=-Cp^{-3}J. \tag{3.6}$$

The result in part 2) can be proved in the same way as part 1), of which details are omitted.

Combining 1) with 2), we complete the proof of this Theorem immediately. □

Proof of Corollary 1.1. It proves only the case of $J=(x-y)(xI)_x<0$.

1) For $p\in(\frac{1}{2},1)$. Assume $p_1,p_2\in(\frac{1}{2},1)$ with $p_1<p_2$, set

$$\alpha=\frac{p_2-p_1}{2p_2-1},\beta=\frac{p_2+p_1-1}{2p_2-1},$$

then $\alpha\ ,\beta>0$, $\alpha+\beta=1$ and $\alpha p_2+\beta(1-p_2)=1-p_1$. By Theorem 1, $\mathcal{H}_f(p,q)$ is log-convex in p on $(0,+\infty)$, and then

$$\begin{aligned}
\mathcal{H}_f(p_2,1-p_2) &= \left(\tfrac{f(p_2)}{f(1-p_2)}\right)^{\frac{1}{2p_2-1}}\\
&= \left(\tfrac{f(p_2)}{f(p_1)}\right)^{\frac{1}{2p_2-1}}\left(\tfrac{f(1-p_2)}{f(p_1)}\right)^{\frac{-1}{2p_2-1}}\\
&= (\mathcal{H}_f(p_2,p_1))^{\frac{p_2-p_1}{2p_2-1}}(\mathcal{H}_f(1-p_2,p_1))^{\frac{p_2+p_1-1}{2p_2-1}}\\
&= \mathcal{H}_f^{\alpha}(p_2,p_1)\mathcal{H}_f^{\beta}(1-p_2,p_1)\\
&> \mathcal{H}_f(\alpha p_2+\beta(1-p_2),p_1)\\
&= \mathcal{H}_f(1-p_1,p_1),
\end{aligned}$$

i.e. $\mathcal{H}_f(p,1-p)$ is strictly increasing in p on $(\frac{1}{2},1)$.

If $p\in(0,\frac{1}{2})$, Assume $p_1,p_2\in(\frac{1}{2},1)$ with $p_1<p_2$, then $1-p_2,1-p_1\in(\frac{1}{2},1)$ and $1-p_2<1-p_1$, so there is

$$\mathcal{H}_f(1-p_1,1-(1-p_1))>\mathcal{H}_f(1-p_2,1-(1-p_2)), \tag{3.7}$$

i.e. $\mathcal{H}_f(p_2,1-p_2)<\mathcal{H}_f(p_1,1-p_1)$. It shows that $\mathcal{H}_f(p,1-p)$ is strictly decreasing in p on $(0,\frac{1}{2})$.

2) If $p \in (1, +\infty)$ and $f(x,y)$ is symmetric with respect to x and y. Set

$$\alpha = \frac{p_2 - p_1}{p_2 - p_1 + 1}, \beta = \frac{1}{p_2 - p_1 + 1} \text{ with } 1 < p_1 < p_2, \tag{3.8}$$

then α , $\beta > 0$, $\alpha + \beta = 1$ and

$$\begin{aligned} \alpha p_2 + \beta(p_1 - 1) &= p_2 - 1, &(3.9)\\ \alpha(p_1 - 1) + \beta p_2 &= p_1. &(3.10)\end{aligned}$$

By the log-convexity of $\mathcal{H}_f(p,q)$ in p on $(0, +\infty)$, we have

$$\begin{cases} \mathcal{H}_f^{\alpha}(p_2, 1-p_2)\mathcal{H}_f^{\beta}(p_1 - 1, 1 - p_2) > \mathcal{H}_f(p_2 - 1, 1 - p_2); \\ \mathcal{H}_f^{\alpha}(p_1 - 1, -p_1)\mathcal{H}_f^{\beta}(p_2, -p_1) > \mathcal{H}_f(p_1, -p_1). \end{cases} \tag{3.11}$$

Notice that

$$\begin{aligned} \mathcal{H}_f(p_1 - 1, 1 - p_2) &= \left(\frac{f(p_1-1)}{f(1-p_2)}\right)^{\frac{1}{p_2+p_1-2}} = G^{\frac{2n(p_1-1)}{p_2+p_1-2}} \left(\frac{f(1-p_1)}{f(1-p_2)}\right)^{\frac{1}{p_2+p_1-2}}, \\ \mathcal{H}_f(p_2 - 1, 1 - p_2) &= \mathcal{H}_f(p_1, -p_1) = G^n, \\ \mathcal{H}_f(p_1 - 1, -p_1) &= G^{2n}\mathcal{H}_f^{-1}(p_1, 1 - p_1), \\ \mathcal{H}_f(p_2, -p_1) &= \left(\frac{f(p_2)}{f(-p_1)}\right)^{\frac{1}{p_2+p_1}} = G^{\frac{2np_1}{p_2+p_1}} \left(\frac{f(p_2)}{f(p_1)}\right)^{\frac{1}{p_2+p_1}}, \end{aligned}$$

and then (3.11) is equivalent to

$$\begin{cases} \mathcal{H}_f^{\alpha}(p_2, 1 - p_2)G^{\frac{2\beta n(p_1-1)}{p_2+p_1-2}} \left(\frac{f(1-p_1)}{f(1-p_2)}\right)^{\frac{\beta}{p_2+p_1-2}} > G^n, \\ G^{2\alpha n}\mathcal{H}_f^{-\alpha}(p_1, 1 - p_1)G^{\frac{2n\beta p_1}{p_2+p_1}} \left(\frac{f(p_2)}{f(p_1)}\right)^{\frac{\beta}{p_2+p_1}} > G^n. \end{cases} \tag{3.12}$$

Taking the $\frac{p_2+p_1-2}{\beta}$ -th , $\frac{p_2+p_1}{\beta}$ -th power of the two sides in the above two inequalities, respectively, then

$$\begin{cases} \mathcal{H}_f^{\alpha(p_2+p_1-2)}(p_2, 1 - p_2)G^{2\beta n(p_1-1)} \left(\frac{f(1-p_1)}{f(1-p_2)}\right)^{\beta} > G^{n(p_2+p_1-2)}, \\ G^{2\alpha n(p_2+p_1)}\mathcal{H}_f^{-\alpha(p_2+p_1)}(p_1, 1 - p_1)G^{2n\beta p_1} \left(\frac{f(p_2)}{f(p_1)}\right)^{\beta} > G^{n(p_2+p_1)}. \end{cases} \tag{3.13}$$

Let the left sides of two inequalities in 3.13 multiply each other, the right sides do also, we have

$$\begin{aligned} &\mathcal{H}_f^{\alpha(p_2+p_1-2)}(p_2, 1 - p_2)\mathcal{H}_f^{-\alpha(p_2+p_1)}(p_1, 1 - p_1) \left(\frac{f(1-p_1)}{f(1-p_2)}\frac{f(p_2)}{f(p_1)}\right)^{\beta} \\ &\quad > G^{2n(p_2+p_1-1)}G^{-2\beta n(2p_1-1)-2\alpha n(p_2+p_1)}, \end{aligned} \tag{3.14}$$

in which the left side is equal to

$$\begin{aligned}
&\mathcal{H}_f^{\alpha(p_2+p_1-2)}(p_2,1-p_2)\mathcal{H}_f^{-\alpha(p_2+p_1)}(p_1,1-p_1)\left(\frac{f(1-p_1)}{f(p_1)}\frac{f(p_2)}{f(1-p_2)}\right)^{\beta}\\
&\quad=\mathcal{H}_f^{\alpha(p_2+p_1-2)+\beta(2p_2-1)}(p_2,1-p_2)\mathcal{H}_f^{-\alpha(p_2+p_1)+\beta(1-2p_1)}(p_1,1-p_1)\\
&\quad=\mathcal{H}_f^{p_2+p_1-1}(p_2,1-p_2)\mathcal{H}_f^{-(p_2+p_1-1)}(p_1,1-p_1),
\end{aligned}$$

the right side is equal to 1, because

$$\begin{aligned}
&2n(p_2+p_1-1)-2\beta n(2p_1-1)-2\alpha n(p_2+p_1)\\
&\quad=2n(p_2+p_1-1)-\frac{2n(2p_1-1)+2n(p_2+p_1)(p_2-p_1)}{p_2-p_1+1}\\
&\quad=2n[(p_2+p_1-1)-\frac{(2p_1-1)+(p_2+p_1)(p_2-p_1)}{p_2-p_1+1}\\
&\quad=2n\left((p_2+p_1-1)-\frac{p_2^2-(p_1^2-2p_1+1)}{p_2-p_1+1}\right)\\
&\quad=0.
\end{aligned}$$

Consequently, there is

$$\mathcal{H}_f^{p_2+p_1-1}(p_2,1-p_2)\mathcal{H}_f^{-(p_2+p_1-1)}(p_1,1-p_1)>1$$

from (3.14), which is equivalent to $\mathcal{H}_f(p_2,1-p_2)>\mathcal{H}_f(p_1,1-p_1)$ for $p_2+p_1-1>0$, i.e. $\mathcal{H}_f(p,1-p)$ is strictly increasing in p on $(1,+\infty)$ if $f(x,y)$ is symmetric with respect to x and y.

If $p\in(-\infty,0)$. Assume $p_1,p_2\in(-\infty,0)$ with $p_1<p_2$, then $1-p_2,1-p_1\in(1,+\infty)$ with $1-p_2<1-p_1$, so the inequality (3.7) is valid, i.e. $\mathcal{H}_f(p_1,1-p_1)>\mathcal{H}_f(p_2,1-p_2)$, which shows that $\mathcal{H}_f(p,1-p)$ is strictly decreasing in p on $(-\infty,0)$.

Combining 1) with 2), the proof is completed. □

Proof of Corollary 1.2. It proves only in the case of $J=(x-y)(xI)_x<0$.

1) By Lemma 4, $\ln G_{f,t}$ is strictly convexin t on $(0,+\infty)$, and strictly concave on $(-\infty,0)$. So when $p,q\in(0,+\infty)$, by using the well-known Hermite-Hadamard inequality, we have

$$\ln G_{f,\frac{p+q}{2}}<\frac{1}{p-q}\int_q^p \ln G_{f,t}\mathrm{d}t<\frac{\ln G_{f,p}+\ln G_{f,q}}{2}, \tag{3.15}$$

i.e. inequality (1.17) holds. When $p,q\in(-\infty,0)$, (3.15) is reversed, and inequality (1.17) is also reverse with it.

2) If $f(x,y)$ is defined on $\mathbb{R}_+\times\mathbb{R}_+$ and symmetric with respect to x and y further, there is (2.12), meanwhile $T'(t)$ is strictly log-convex in t on $(0,+\infty)$ and log-concave on $(-\infty,0)$. Using Lemma 6, we can understand the second part of Corollary (2) is valid. □

4. Some conclusions concerning L, A, E and D

By Theorem 1, the log-convexity of $\mathcal{H}_f$ depends on the sign of $J = (x-y)(xI)_x$, which implies that the judgement for log-convexity comes down to a computation of J finally. In this section we will combine Theorem 1 with Corollary 1 and 2 to present some conclusions about log-convexity of $\mathcal{H}_f$ by some straightforward computations, where $f(x,y) = L(x,y)$, $A(x,y)$, $E(x,y)$ and $D(x,y)$.

Case 1. For $f(x,y) = L(x,y) = \dfrac{x-y}{\ln x - \ln y}(x,y>0, x\neq y)$, then

$$\mathcal{H}_L(a,b;p,q) = \begin{cases} \left(\dfrac{q}{p}\dfrac{a^p-b^p}{a^q-b^q}\right)^{\frac{1}{p-q}}, & p\neq q, pq\neq 0, \\ G_{L,p}(a,b), & p=q\neq 0, \\ L^{\frac{1}{p}}(a^p,b^p), & p\neq 0, q=0, \\ L^{\frac{1}{q}}(a^q,b^q), & p=0, q\neq 0, \\ G(a,b), & p=q=0, \end{cases} \tag{4.1}$$

where $G_{L,p}(a,b) = E^{\frac{1}{p}}(a^p,b^p) := E_p(a,b)$, $E(a,b) = e^{-1}\left(\dfrac{a^a}{b^b}\right)^{\frac{1}{a-b}}$, $G(a,b)=\sqrt{ab}$.

$$\begin{aligned} I &= (\ln f)_{xy} = \frac{1}{(x-y)^2} - \frac{1}{xy(\ln x - \ln y)^2}, \\ J &= (x-y)(xI)_x = (x-y)\left(-\frac{x+y}{(x-y)^3} + \frac{2}{xy(\ln x - \ln y)^3}\right) \\ &= \frac{2}{xy(x-y)^2}\left(L^3(x,y) - \frac{x+y}{2}\left(\sqrt{xy}\right)^2\right). \end{aligned}$$

By the well-known inequality $L(x,y) > \left(\dfrac{x+y}{2}\right)^{\frac{1}{3}}\left(\sqrt{xy}\right)^{\frac{2}{3}}$ (see[7]), we get $J > 0$.

REMARK 3. That $E(a,b) = e^{-1}\left(\dfrac{a^a}{b^b}\right)^{\frac{1}{a-b}}$ $(a,b>0$ with $a\neq b)$ is called exponential mean of unequal positive numbers a and b (see[17]), and is also called identic mean and denoted by $I(a,b)$. For avoiding confusion, we adopt our terms and notations in what follows.

Case 2. For $f(x,y) = A(x,y) = \dfrac{x+y}{2}(x,y>0)$, then

$$\mathcal{H}_A(a,b;p,q) = \begin{cases} \left(\dfrac{a^p+b^p}{a^q+b^q}\right)^{\frac{1}{p-q}}, & p\neq q, \\ G_{A,p}(a,b), & p=q, \end{cases} \tag{4.2}$$

where $G_{A,p}(a,b) = Z^{\frac{1}{p}}(a^p, b^p) := Z_p(a,b), Z(a,b) = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}$.

$$I = (\ln f)_{xy} = -\frac{1}{(x+y)^2},$$

$$J = (x-y)(xI)_x = \frac{(x-y)^2}{(x+y)^3} > 0.$$

REMARK 4. That $Z(a,b) = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}$ is a mean value of positive numbers a and b, and is called power-exponential mean temporarily.

Case 3. For $f(x,y) = E(x,y) = e^{-1}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}$ $(x,y > 0, x \neq y)$, then

$$\mathcal{H}_E(a,b;p,q) = \begin{cases} \left(\frac{E(a^p,b^p)}{E(a^q,b^q)}\right)^{\frac{1}{p-q}}, & p \neq q, pq \neq 0, \\ G_{E,p}(a,b), & p = q \neq 0, \\ E^{\frac{1}{p}}(a^p,b^p), & p \neq 0, q = 0, \\ E^{\frac{1}{q}}(a^q,b^q), & p = 0, q \neq 0, \\ G(a,b), & p = q = 0, \end{cases} \tag{4.3}$$

where $G_{E,p}(a,b) = Y^{\frac{1}{p}}(a^p,b^p) := Y_p(a,b), Y(a,b) = Ee^{1-\frac{G^2}{L^2}}$.

$$I = (\ln f)_{xy} = \frac{1}{(x-y)^3}\left(2(x-y) - (x+y)(\ln x - \ln y)\right),$$

$$J = (x-y)(xI)_x = \frac{-3(x^2-y^2) + (x^2+4xy+y^2)(\ln x - \ln y)}{(x-y)^3}$$

$$= -\frac{6(\ln x - \ln y)}{(x-y)^3}\left(\frac{x^2-y^2}{\ln x^2 - \ln y^2} - \frac{\frac{x^2+y^2}{2} + 2xy}{3}\right).$$

By the well-known inequality $L(x,y) < \dfrac{\frac{x+y}{2} + 2\sqrt{xy}}{3}$ (see[1]), we get $J > 0$.

REMARK 5. That $Y(a,b) = Ee^{1-\frac{G^2}{L^2}}$ is also a mean value of positive numbers a and b, and is called exponent-geometric mean temporarily.

Case 4. For $f(x,y) = D(x,y) = |x-y| (x,y > 0, x \neq y)$, then

$$\mathcal{H}_D(a,b;p,q) = \begin{cases} \left|\frac{a^p - b^p}{a^q - b^q}\right|^{\frac{1}{p-q}}, & p \neq q, pq \neq 0, \\ G_{D,p}(a,b), & p = q \neq 0, \end{cases} \tag{4.4}$$

where $G_{D,p}(a,b) = e^{\frac{1}{p}} E^{\frac{1}{p}}(a^p,b^p) := e^{\frac{1}{p}} E_p(a,b)$.

$$I = (\ln f)_{xy} = \frac{1}{(x-y)^2},$$
$$J = (x-y)(xI)_x = -\frac{x+y}{(x-y)^2} < 0.$$

Applying Theorem 1, Corollary 1 mechanically and 2, meanwhile note that $L(x,y)$, $A(x,y)$, $E(x,y)$ and $D(x,y)$ is symmetric with respect to x and y, and

$$f(1,1) := \lim_{x\to 1} f(x,1) = \begin{cases} 1 & \text{if } f = L, A, E; \\ 0 & \text{if } f = D, \end{cases}$$

and then

$$\mathcal{H}_f(0,1) := \lim_{p\to 0} \mathcal{H}_f(p,1-p) = \begin{cases} f(a,b) & \text{if } f = L, A, E; \\ \text{do not exist} & \text{if } f = D, \end{cases}$$
$$\mathcal{H}_f(1,0) := \lim_{p\to 1} \mathcal{H}_f(p,1-p) = \begin{cases} f(a,b) & \text{if } f = L, A, E; \\ \text{do not exist} & \text{if } f = D, \end{cases}$$

we immediately obtain the following conclusions.

Conclusion 1. For $f(x,y) = L(x,y)$, $A(x,y)$ and $E(x,y)$,

1) $\mathcal{H}_f(p,q)$ are strictly log-concave with respect to either p or q on $(0,+\infty)$, and strictly log-convex on $(-\infty,0)$.

2) $\mathcal{H}_f(p,1-p)$ are strictly increasing in p on $(-\infty,\frac{1}{2})$, and strictly decreasing on $(\frac{1}{2},+\infty)$.

3) If $p+q>0$, then

$$G_{f,\frac{p+q}{2}} > \mathcal{H}_f(p,q) > \sqrt{G_{f,p}G_{f,q}}. \tag{4.5}$$

(4.5) is reversed if $p+q<0$.

Conclusion 2. 1) $\mathcal{H}_D(p,q)$ is strictly log-convex with respect to either p or q on $(0,+\infty)$, and strictly log-concave on $(-\infty,0)$.

2) $\mathcal{H}_D(p,1-p)$ is strictly decreasing in p on $(-\infty,0)$ or $(0,\frac{1}{2})$, and strictly increasing on $(\frac{1}{2},1)$ or $(1,+\infty)$.

3) If $p,q>0$, there is

$$G_{D,\frac{p+q}{2}} < \mathcal{H}_D(p,q) < \sqrt{G_{D,p}G_{D,q}}. \tag{4.6}$$

(4.6) is reversed if $p,q<0$.

5. The refinements of some classical inequalities and new inequalities

By applying the above conclusions we will present a series of new inequalities concerning logarithm mean, exponential mean (identic mean), power-exponential mean and exponential-geometry mean, meanwhile propose estimations of the upper and lower bounds of extended mean.

EXAMPLE 1. *A group of chains of inequalities for homogeneous mean*. By part 2) of Conclusion 1, $\mathcal{H}_f(p, 1-p)$ is strictly monotone decreasing in p on $(\frac{1}{2}, +\infty)$ for $f = L, A$ and E, so there are

$$\begin{aligned}\mathcal{H}_f(2,-1) &< \mathcal{H}_f(\tfrac{3}{2},-\tfrac{1}{2}) < \mathcal{H}_f(\tfrac{4}{3},-\tfrac{1}{3}) < \mathcal{H}_f(1,0) < \mathcal{H}_f(\tfrac{4}{5},\tfrac{1}{5}) \\ &< \mathcal{H}_f(\tfrac{3}{4},\tfrac{1}{4}) < \mathcal{H}_f(\tfrac{2}{3},\tfrac{1}{3}) < \mathcal{H}_f(\tfrac{3}{5},\tfrac{2}{5}) < \mathcal{H}_f(\tfrac{1}{2},\tfrac{1}{2}),\end{aligned} \tag{5.1}$$

i.e.

$$\begin{aligned}\left(\frac{f(a^2,b^2)}{f(a^{-1},b^{-1})}\right)^{\frac{1}{3}} &< \left(\frac{f(a^{\frac{3}{2}},b^{\frac{3}{2}})}{f(a^{-\frac{1}{2}},b^{-\frac{1}{2}})}\right)^{\frac{1}{2}} < \left(\frac{f(a^{\frac{4}{3}},b^{\frac{4}{3}})}{f(a^{-\frac{1}{3}},b^{-\frac{1}{3}})}\right)^{\frac{3}{5}} < \frac{f(a,b)}{f(1,1)} \\ &< \left(\frac{f(a^{\frac{4}{5}},b^{\frac{4}{5}})}{f(a^{\frac{1}{5}},b^{\frac{1}{5}})}\right)^{\frac{5}{3}} < \left(\frac{f(a^{\frac{3}{4}},b^{\frac{3}{4}})}{f(a^{\frac{1}{4}},b^{\frac{1}{4}})}\right)^{2} < \left(\frac{f(a^{\frac{2}{3}},b^{\frac{2}{3}})}{f(a^{\frac{1}{3}},b^{\frac{1}{3}})}\right)^{3} \\ &< \left(\frac{f(a^{\frac{3}{5}},b^{\frac{3}{5}})}{f(a^{\frac{2}{5}},b^{\frac{2}{5}})}\right)^{5} < a^{\frac{\sqrt{a}f_x(\sqrt{a},\sqrt{b})}{f(\sqrt{a},\sqrt{b})}} b^{\frac{\sqrt{b}f_y(\sqrt{a},\sqrt{b})}{f(\sqrt{a},\sqrt{b})}}\end{aligned} \tag{5.2}$$

1) For $f(x,y) = L(x,y)$, notice $f(1,1) = 1$, we get

$$\begin{aligned}\left(\frac{-(b^2-a^2)}{2(b^{-1}-a^{-1})}\right)^{\frac{1}{3}} &< \left(\frac{-\frac{1}{2}(b^{\frac{3}{2}}-a^{\frac{3}{2}})}{\frac{3}{2}(b^{-\frac{1}{2}}-a^{-\frac{1}{2}})}\right)^{\frac{1}{2}} < \left(\frac{-\frac{1}{3}(b^{\frac{4}{3}}-a^{\frac{4}{3}})}{\frac{4}{3}(b^{-\frac{1}{3}}} - a^{-\frac{1}{3}})\right)^{\frac{3}{5}} \\ &< L(a,b) < \left(\frac{\frac{1}{5}(b^{\frac{4}{5}}-a^{\frac{4}{5}})}{\frac{4}{5}(b^{\frac{1}{5}}-a^{\frac{1}{5}})}\right)^{\frac{5}{3}} < \left(\frac{\frac{1}{4}(b^{\frac{3}{4}}-a^{\frac{3}{4}})}{\frac{3}{4}(b^{\frac{1}{4}}-a^{\frac{1}{4}})}\right)^{2} \\ &< \left(\frac{\frac{1}{3}(b^{\frac{2}{3}}-a^{\frac{2}{3}})}{\frac{2}{3}(b^{\frac{1}{3}}-a^{\frac{1}{3}})}\right)^{3} < \left(\frac{\frac{2}{5}(b^{\frac{3}{5}}-a^{\frac{3}{5}})}{\frac{3}{5}(b^{\frac{2}{5}}-a^{\frac{2}{5}})}\right)^{5} \\ &< E^2(\sqrt{a},\sqrt{b}),\end{aligned} \tag{5.3}$$

i.e

$$\begin{aligned}\left(\frac{ab(b+a)}{2}\right)^{\frac{1}{3}} &< \left(\sqrt{ab}\frac{(b+\sqrt{ba}+a)}{3}\right)^{\frac{1}{2}} < \left((ab)^{\frac{1}{3}}\frac{(b^{\frac{1}{3}}+a^{\frac{1}{3}})(b^{\frac{2}{3}}+a^{\frac{2}{3}})}{4}\right)^{\frac{3}{5}} \\ &< L(a,b) < \left(\frac{(b^{\frac{1}{5}}+a^{\frac{1}{5}})(b^{\frac{2}{5}}+a^{\frac{2}{5}})}{4}\right)^{\frac{5}{3}} < \left(\frac{b^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+a^{\frac{1}{2}})}{3}\right)^{2} \\ &< \left(\frac{b^{\frac{1}{3}}+a^{\frac{1}{3}}}{2}\right)^{3} < \left(\frac{2(b^{\frac{2}{5}}+b^{\frac{1}{5}}a^{\frac{1}{5}}+a^{\frac{2}{5}})}{3(b^{\frac{1}{5}}+a^{\frac{1}{5}})}\right)^{5} \\ &< E^2(\sqrt{a},\sqrt{b}).\end{aligned} \tag{5.4}$$

Inequality's chain (5.4) may be concisely denoted by

$$\begin{aligned}G^{\frac{2}{3}}A^{\frac{1}{3}} &< \sqrt{GH} < G^{\frac{2}{5}}A_{\frac{1}{3}}^{\frac{1}{5}}A_{\frac{2}{3}}^{\frac{2}{5}} \\ &< L < A_{\frac{1}{5}}^{\frac{1}{3}}A_{\frac{2}{5}}^{\frac{2}{3}} < H_{\frac{1}{2}} \\ &< A_{\frac{1}{3}} < H_{\frac{2}{5}}^{2}A_{\frac{1}{5}}^{-1} < E_{\frac{1}{2}},\end{aligned} \tag{5.5}$$

where $A_p = \left(\dfrac{a^p + b^p}{2}\right)^{\frac{1}{p}}$, $E_p = E^{\frac{1}{p}}(a^p, b^p)$, $H_p = \left(\dfrac{a^p + (\sqrt{ab})^p + b^p}{3}\right)^{\frac{1}{p}}$, $H = H_1$.

That $L < A_{\frac{1}{3}}$ is well-known Tong-Po Ling's inequality (see [8]), and $L < H_{\frac{1}{2}}$ was found by Gao Jia and Jinde Cao in [5]. Now the chain of inequalities (5.5) not only presents a new proof of Ling's and Gao's inequalities, but also strengthen them. In addition, that $L > G^{\frac{2}{3}}A^{\frac{1}{3}}$ established by E. B. Leach and M. C. Sholander (see[7]) is strengthened to

$$L > G^{\frac{2}{5}}A^{\frac{1}{5}}_{\frac{1}{3}}A^{\frac{2}{5}}_{\frac{2}{3}} > \sqrt{GH}.$$

2) For $f(x,y) = A(x,y)$, notice $f(1,1) = 1$, likewise we get

$$\begin{aligned} G^{\frac{2}{3}}A_2^{\frac{2}{3}}A^{-\frac{1}{3}} &< G^{\frac{1}{2}}A^{\frac{3}{4}}_{\frac{3}{2}}A^{-\frac{1}{4}}_{\frac{1}{2}} < G^{\frac{2}{5}}A^{\frac{4}{5}}_{\frac{4}{3}}A^{-\frac{1}{5}}_{\frac{1}{3}} \\ &< A < A^{\frac{4}{3}}_{\frac{4}{5}}A^{-\frac{1}{3}}_{\frac{1}{5}} < A^{\frac{3}{2}}_{\frac{3}{4}}A^{-\frac{1}{2}}_{\frac{1}{4}} \\ &< A^{2}_{\frac{2}{3}}A^{-1}_{\frac{1}{3}} < A^{3}_{\frac{3}{5}}A^{-2}_{\frac{2}{5}} < Z_{\frac{1}{2}}, \end{aligned} \tag{5.6}$$

where $A_p = (\dfrac{a^p + b^p}{2})^{\frac{1}{p}}$, $Z_p = Z^{\frac{1}{p}}(a^p, b^p)$.

3) For $f(x,y) = E(x,y)$, notice $f(1,1) = 1$, $\dfrac{E(a^2,b^2)}{E(a,b)} = Z(a,b)$, likewise we get

$$\begin{aligned} G^{\frac{2}{3}}Z_1^{\frac{1}{3}} &< G^{\frac{1}{2}}E^{\frac{3}{4}}_{\frac{3}{2}}E^{-\frac{1}{4}}_{\frac{1}{2}} < G^{\frac{2}{5}}Z^{\frac{1}{5}}_{\frac{1}{3}}Z^{\frac{2}{5}}_{\frac{2}{3}} \\ &< E < Z^{\frac{1}{3}}_{\frac{1}{5}}Z^{\frac{2}{3}}_{\frac{2}{5}} < E^{\frac{3}{2}}_{\frac{3}{4}}E^{-\frac{1}{2}}_{\frac{1}{4}} \\ &< Z_{\frac{1}{3}} < E^{3}_{\frac{3}{5}}E^{-2}_{\frac{2}{5}} < Y_{\frac{1}{2}}, \end{aligned} \tag{5.7}$$

where $Z_p = Z^{\frac{1}{p}}(a^p, b^p)$, $E_p = E^{\frac{1}{p}}(a^p, b^p)$, $Y_p = Y^{\frac{1}{p}}(a^p, b^p)$.

REMARK 6. Being similar to $\dfrac{L(a^2,b^2)}{L(a,b)} = A(a,b)$, that $\dfrac{E(a^2,b^2)}{E(a,b)} = Z(a,b)$ is a new identical equation for mean. In fact,

$$\begin{aligned} E(a,b)Z(a,b) &= e^{-1}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} b^{\frac{b}{b+a}}a^{\frac{a}{b+a}} \\ &= e^{-1}b^{\frac{b}{b+a}+\frac{b}{b-a}}a^{\frac{a}{b+a}-\frac{a}{b-a}} \\ &= e^{-1}b^{\frac{2b^2}{b^2-a^2}}a^{\frac{-2a^2}{b^2-a^2}} = e^{-1}\left(\frac{(b^2)^{b^2}}{(a^2)^{a^2}}\right)^{\frac{1}{b^2-a^2}} \\ &= E(a^2,b^2). \end{aligned}$$

REMARK 7. It is easy to verify that

$$X_p^2(a,b) = X_{\frac{p}{2}}(a^2,b^2) \tag{5.8}$$

are valid for $X = A.H, E, Z$. Put

$$X_1 = X, \tag{5.9}$$

then (5.5)-(5.7) may be rewritten into

$$E > H_{\frac{4}{5}}^2 A_{\frac{2}{5}}^{-1} > A_{\frac{2}{3}} > H > A_{\frac{2}{5}}^{\frac{1}{3}} A_{\frac{4}{5}}^{\frac{2}{3}} > L_2 = \sqrt{LA} \tag{5.10}$$

$$Z > A_{\frac{6}{5}}^3 A_{\frac{4}{5}}^{-2} > A_{\frac{4}{3}}^2 A_{\frac{2}{3}}^{-1} > A_{\frac{3}{2}}^{\frac{3}{2}} A_{\frac{1}{2}}^{-\frac{1}{2}} > A_{\frac{8}{5}}^{\frac{4}{3}} A_{\frac{2}{5}}^{-\frac{1}{3}} > A_2 \tag{5.11}$$

$$Y > E_{\frac{6}{5}}^3 E_{\frac{4}{5}}^{-2} > Z_{\frac{2}{3}} > E_{\frac{3}{2}}^{\frac{3}{2}} E_{\frac{1}{2}}^{-\frac{1}{2}} > Z_{\frac{2}{5}}^{\frac{1}{3}} Z_{\frac{4}{5}}^{\frac{2}{3}} > E_2 = \sqrt{EZ} \tag{5.12}$$

That $E > A_{\frac{2}{3}}$ is well-known Stolarsky's inequality (see [15]). (5.10) indicates that $H_{\frac{4}{5}}^2 A_{\frac{2}{5}}^{-1}$ can be inserted in between E and $A_{\frac{2}{3}}$, so (5.10) strengthens Stolarsky's inequality. It follows that Tong-Po Ling's and Stolarsky's inequality are unified into the same chain of inequalities and refined by (5.5) or (5.10). At the same time they are generalized to the case of arithmetic mean and exponential mean (identic mean) by (5.6) or (5.11) and (5.7) or (5.12) in parallel.

REMARK 8. There include some concise or brand-new inequalities in (5.5)-(5.12), such as

$$E > A_{\frac{2}{3}} > H > L_2 \tag{5.13}$$

from (5.10), as well as $Z > A_2$ from (5.11), i.e.

$$Z > \sqrt{\frac{a^2+b^2}{2}}. \tag{5.14}$$

While $Z > A_{\frac{3}{2}}^{\frac{3}{2}} A_{\frac{1}{2}}^{-\frac{1}{2}}$ may be rewritten into $Z > \dfrac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}} = a+b-\sqrt{ab}$, i.e.

$$\frac{Z+G}{2} > A. \tag{5.15}$$

By (5.7) we get

$$E < Z_{\frac{1}{3}} < Y_{\frac{1}{2}}. \tag{5.16}$$

Combining (5.13) with (5.16), we get a new chain of inequalities concerning L, H, A, E, Z and Y :

$$L_2 < H < A_{\frac{2}{3}} < E < Z_{\frac{1}{3}} < Y_{\frac{1}{2}}. \tag{5.17}$$

EXAMPLE 2. *A reversed chain of inequalities for exponential mean (identic mean).* By part 2) of Conclusion 2, noticed $D(1,1)$ does not exist, we have

$$\mathcal{H}_D(\frac{1}{2},\frac{1}{2}) < \mathcal{H}_D(\frac{3}{5},\frac{2}{5}) < \mathcal{H}_D(\frac{2}{3},\frac{1}{3}) < \mathcal{H}_D(\frac{3}{4},\frac{1}{4}) < \mathcal{H}_D(\frac{4}{5},\frac{1}{5}), \tag{5.18}$$

i.e.

$$e^2E^2(\sqrt{a},\sqrt{b}) < \left(\frac{b^{\frac{3}{5}}-a^{\frac{3}{5}}}{b^{\frac{2}{5}}-a^{\frac{2}{5}}}\right)^5 < \left(\frac{b^{\frac{2}{3}}-a^{\frac{2}{3}}}{b^{\frac{1}{3}}-a^{\frac{1}{3}}}\right)^3 < \left(\frac{b^{\frac{3}{4}}-a^{\frac{3}{4}}}{b^{\frac{1}{4}}-a^{\frac{1}{4}}}\right)^2 < \left(\frac{b^{\frac{4}{5}}-a^{\frac{4}{5}}}{b^{\frac{1}{5}}-a^{\frac{1}{5}}}\right)^{\frac{5}{3}}, \tag{5.16}$$

i.e.

$$e^2E^2(\sqrt{a},\sqrt{b}) < \left(\frac{b^{\frac{2}{5}}+b^{\frac{1}{5}}a^{\frac{1}{5}}+a^{\frac{2}{5}}}{b^{\frac{1}{5}}+a^{\frac{1}{5}}}\right)^5 < (b^{\frac{1}{3}}+a^{\frac{1}{3}})^3 < \left(b^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+a^{\frac{1}{2}})\right)^2 < \left((b^{\frac{1}{5}}+a^{\frac{1}{5}})(b^{\frac{2}{5}}+a^{\frac{2}{5}})\right)^{\frac{5}{3}} \tag{5.17}$$

If replace a,b with a^2,b^2 , divide the terms by e^2 and take the square roots of them in (5.17), then it may be denoted concisely by

$$E < \frac{\sqrt{486}}{8e}H^2_{\frac{4}{5}}A^{-1}_{\frac{2}{5}} < \frac{\sqrt{8}}{e}A_{\frac{2}{3}} < \frac{3}{e}H < \frac{\sqrt[3]{32}}{e}A^{\frac{1}{3}}_{\frac{2}{5}}A^{\frac{2}{3}}_{\frac{4}{5}}, \tag{5.21}$$

which is a reversed chain of inequalities of five items in left side of (5.10).

REMARK 9. By (5.10) and (5.21), we get

$$A^{\frac{1}{3}}_{\frac{2}{5}}A^{\frac{2}{3}}_{\frac{4}{5}} < H < A_{\frac{2}{3}} < H^2_{\frac{4}{5}}A^{-1}_{\frac{2}{5}} < E < \frac{\sqrt{486}}{8e}H^2_{\frac{4}{5}}A^{-1}_{\frac{2}{5}} < \frac{\sqrt{8}}{e}A_{\frac{2}{3}} < \frac{3}{e}H < \frac{\sqrt[3]{32}}{e}A^{\frac{1}{3}}_{\frac{2}{5}}A^{\frac{2}{3}}_{\frac{4}{5}}. \tag{5.19}$$

It follows that

$$1 < E/A^{\frac{1}{3}}_{\frac{2}{5}}A^{\frac{2}{3}}_{\frac{4}{5}} < \sqrt[3]{32}/e \approx 1.16794, \tag{5.23}$$

$$1 < E/H < 3/e \approx 1.10364, \tag{5.24}$$

$$1 < E/A_{\frac{2}{3}} < \sqrt{8}/e \approx 1.04052, \tag{5.25}$$

$$1 < E/H^2_{\frac{4}{5}}A^{-1}_{\frac{2}{5}} < \sqrt{486}/8e \approx 1.01376. \tag{5.26}$$

Inequalities (5.23)-(5.26) indicate that regardless the size of positive numbers a and b, the relative error estimating exponential mean E by $A^{\frac{1}{3}}_{\frac{2}{5}}A^{\frac{2}{3}}_{\frac{4}{5}}$, H, $A_{\frac{2}{3}}$ and $H^2_{\frac{4}{5}}A^{-1}_{\frac{2}{5}}$ are approximate to 17%, 10%, 4% and 1% respectively.

EXAMPLE 3. *Estimations of the lower and upper bounds of the extended mean.* From part 3) of Conclusion 2, and notice

$$G_{D,p} = e^{\frac{1}{p}}E^{\frac{1}{p}}(x^p,y^p) = e^{\frac{1}{p}}E_p, \tag{5.27}$$

we have

$$e^{\frac{2}{p+q}}E_{\frac{p+q}{2}} < \mathcal{H}_D(p,q) < \sqrt{e^{\frac{1}{p}}E_p e^{\frac{1}{q}}E_q}, \text{if } p,q>0 \text{ and } p\neq q. \tag{5.28}$$

If notice further

$$\begin{aligned}\mathcal{H}_D(p,q) &= \left|\frac{b^p-a^p}{b^q-a^q}\right|^{\frac{1}{p-q}} = \left(\frac{p}{q}\right)^{\frac{1}{p-q}}\left(\frac{q}{p}\frac{b^p-a^p}{b^q-a^q}\right)^{\frac{1}{p-q}}\\ &= e^{\frac{1}{L(p,q)}}\mathcal{H}_L(p,q),\end{aligned} \tag{5.29}$$

then (5.28) can be rewritten into

$$\begin{aligned}e^{\frac{1}{A(p,q)}-\frac{1}{L(p,q)}}E_{\frac{p+q}{2}} &< \mathcal{H}_L(p,q)\\ &< e^{\frac{1}{A_{-1}(p,q))}-\frac{1}{L(p,q)}}\sqrt{E_pE_q},\end{aligned} \tag{5.30}$$

where $A(p,q)=\dfrac{p+q}{2}$, $A_{-1}\,(p,q)=\dfrac{2pq}{p+q}$, $L(p,q)=\dfrac{p-q}{\ln(p/q)}$, $p,q>0$ with $p\neq q$.

Combining (4.5) with (5.30), we can get two other estimated expressions of the extended mean $\mathcal{H}_L(p,q)$.

$$e^{\frac{1}{A(p,q)}-\frac{1}{L(p,q)}}E_{\frac{p+q}{2}} < \mathcal{H}_L(p,q) < E_{\frac{p+q}{2}}, \tag{5.31}$$

$$\begin{aligned}\sqrt{E_pE_q} &< \mathcal{H}_L(p,q)\\ &< e^{\frac{1}{A_{-1}(p,q)}-\frac{1}{L(p,q)}}\sqrt{E_pE_q},\end{aligned} \tag{5.32}$$

where $p,q>0$ with $p\neq q$. Inequalities (5.31), (5.32) are reversed if $p,q<0$ with $p\neq q$.

Lastly, we can find out some new inequalities by using the theorem and corollaries in this paper. No longer discuss it here.

REFERENCES

[1] B. C. CARLSON, *The logarithmic mean*, Amer. Math. Monthly, **79**, (1972), 615–618.

[2] P. CZINDER, Z. PÁLES, *An extension of the Hermite-Hadmard inequality and an application for Gini and stolarsky means*, Journal of Inequalities in Pure and Applied Mathematics, **5**, (2) (2004) [ONLINE] Available online at http://jipam.vu.edu.au/images/167_03_JIPAM/167_03.pdf.

[3] C. GINI, *Diuna formula comprensiva delle media*, Metron, **13**, (1938), 3–22.

[4] P. A. HÄSTÖ, *A montonicity property of ratios of symmetric homogeneous means*, Journal of Inequalities in Pure and Applied Mathematics, **3**, (5) (2002) [ONLINE] Available online at http://jipam-old.vu.edu.au/v3n5/013_02.pdf.

[5] GAO JIA, JINDE CAO, *A new upper bound of the logarithmic mean*, Journal of Inequalities in Pure and Applied Mathematics, **4**, (4) (2003) [ONLINE] Available online at http://jipam.vu.edu.au/images/088_03/088_03_www.pdf.

[6] E. B. LEACH, M. C. SHOLANDER, *Extended mean values*, Amer. Math. Monthly, **85**, (1978), 84–90.

[7] E. B. LEACH, M. C. SHOLANDER, *Extended mean values II*, J. Math. Anal. Appl., **92**, (1983), 207–223.

[8] T.-P. LIN, *The power mean and the logarithmic mean*, Amer. Math. Monthly, **81**, (1974), 879–883.

[9] Z. PÁLES, *Inequalities for differences of powers,* J. Math. Anal. Appl. , **131**, (1988), 271–281.

[10] F. QI, *Logarithmically convexities of the Extended Mean values*, Proc. Am. Math. Soc. **130**, (6) (2002), 1787–1796.
[11] F. QI, *Generalized weighted mean values with two parameters*, Proceedings of the Royal Society of London, Series, Mathematical, Physical and Engineering Sciences, **454**, (1978) (1998), 2723–2732.
[12] F. QI, *On a two-parameter family of nonhomogeneous mean values*, Tamkang Journal of Mathematics, **29**, (2) (1998), 155–163.
[13] F. QI, *Generalized Abstracted Mean Values*, Journal of Inequalities in Pure and Applied Mathematics, **1**, (1) (1999) [ONLINE] Available online at http://jipam-old.vu.edu.au/v1n1/013_99.pdf.
[14] K. B. STOLARSKY, *Generalizations of the Logarithmic Mean,* Math. Mag., **48**, (1975), 87–92.
[15] K. B. STOLARSKY, *The power and generalized Logarithmic Means,* Amer. Math. Monthly., **87**, (1980), 545–548.
[16] MING-BAO SUN, *Inequality for two-parameter mean of convex function*, Mathematics in Practice and Theory, **27**, (3) (1997), 193–197.
[17] ZHEN-HANG YANG, *Exponential mean and logarithmic Mean*, Mathematics in Practice and Theory, **4**, (1987), 76–78.
[18] ZHEN-HANG YANG, *Inequalities for power mean of convex function*, Mathematics in Practice and Theory, **20**, (1) (1990), 93–96.
[19] ZHEN-HANG YANG, *Inequalities for general mean of convex function,* Mathematics in Practice and Theory, **33**, (8) (2003), 136–141.
[20] ZHEN-HANG YANG, *Simple discriminance for convexity of homogeneous functions and applications*, Study in College Mathematics, **7**, (4) (2004), 14–19.

(Received May 18, 2005)

Zhejiang Electric Power Vocational Technical College
Hangzhou
Zhejiang, 311600
China
e-mail: yzhkm@163.com