AN APPLICATION OF λ–METHOD ON INEQUALITIES OF SHAFER–FINK'S TYPE

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Abstract. In this article λ-method of Mitrinović-Vasić [1] is applied to improve the upper bound for the arc sin function of L. Zhu [4].

1. Inequalities of Shafer-Fink's type

D. S. Mitrinović in [1] considered the lower bound of the arc sin function, which belongs to R. E. Shafer. Namely, the following statement is true.

THEOREM 1.1. *For $0 \leqslant x \leqslant 1$ the following inequalities are true*:

$$\frac{3x}{2 + \sqrt{1 - x^2}} \leqslant \frac{6(\sqrt{1+x} - \sqrt{1-x})}{4 + \sqrt{1+x} + \sqrt{1-x}} \leqslant \arcsin x. \tag{1}$$

A. M. Fink proved the following statement in [2].

THEOREM 1.2. *For $0 \leqslant x \leqslant 1$ the following inequalities are true*:

$$\frac{3x}{2 + \sqrt{1 - x^2}} \leqslant \arcsin x \leqslant \frac{\pi x}{2 + \sqrt{1 - x^2}}. \tag{2}$$

B. J. Malešević proved the following statement in [3].

THEOREM 1.3. *For $0 \leqslant x \leqslant 1$ the following inequalities are true*:

$$\frac{3x}{2 + \sqrt{1 - x^2}} \leqslant \arcsin x \leqslant \frac{\frac{\pi}{\pi-2}x}{\frac{2}{\pi-2} + \sqrt{1 - x^2}} \leqslant \frac{\pi x}{2 + \sqrt{1 - x^2}}. \tag{3}$$

The main result of the article [3] can be formulated with the next statement.

PROPOSITION 1.4. *In the family of the functions*:

$$f_b(x) = \frac{(b + 1)x}{b + \sqrt{1 - x^2}} \quad (0 \leqslant x \leqslant 1), \tag{4}$$

according to the parameter $b > 0$, *the function* $f_2(x)$ *is the greatest lower bound of the* arc sin x *function and the function* $f_{2/(\pi-2)}(x)$ *is the least upper bound of the* arc sin x *function.*

L. Zhu proved the following statement in [4].

THEOREM 1.5. *For* $x \in [0, 1]$ *the following inequalities are true*:

$$\frac{3x}{2 + \sqrt{1 - x^2}} \leqslant \frac{6(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} \leqslant \text{arc sin } x$$

$$\leqslant \frac{\pi(\sqrt{2} + \frac{1}{2})(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} \leqslant \frac{\pi x}{2 + \sqrt{1 - x^2}}. \tag{5}$$

In this article we further improve the upper bound of the arc sin function. Namely, in the next section we will give proof of the following theorem:

THEOREM 1.6. *For* $x \in [0, 1]$ *the following inequalities are true*:

$$\frac{3x}{2 + \sqrt{1 - x^2}} \leqslant \frac{6(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} \leqslant \text{arc sin } x$$

$$\leqslant \frac{\frac{\pi(2-\sqrt{2})}{\pi-2\sqrt{2}}(\sqrt{1 + x} - \sqrt{1 - x})}{\frac{\sqrt{2}(4-\pi)}{\pi-2\sqrt{2}} + \sqrt{1 + x} + \sqrt{1 - x}} \tag{6}$$

$$\leqslant \frac{\pi(\sqrt{2} + \frac{1}{2})(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} \leqslant \frac{\pi x}{2 + \sqrt{1 - x^2}}.$$

REMARK 1.7. Using numerical method from [5] we have the following conclusions:

1°. For values $x \in (0, 0.387\,266\,274\ldots)$ the following inequality is true:

$$\text{arc sin } x < \frac{\frac{\pi}{\pi-2}x}{\frac{2}{\pi-2} + \sqrt{1 - x^2}} < \frac{\pi(\sqrt{2} + \frac{1}{2})(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}}, \tag{7}$$

and for values $x \in (0.387\,266\,274\ldots, 1)$ the following inequality is true:

$$\text{arc sin } x < \frac{\pi(\sqrt{2} + \frac{1}{2})(\sqrt{1 + x} - \sqrt{1 - x})}{4 + \sqrt{1 + x} + \sqrt{1 - x}} < \frac{\frac{\pi}{\pi-2}x}{\frac{2}{\pi-2} + \sqrt{1 - x^2}}. \tag{8}$$

Numerically determined constant $c = 0.387\,266\,274\ldots$ is the unique number where the previous bounds have the same values over $(0, 1)$.

2°. For values $x \in (0, 1)$ the following inequality is true:

$$\text{arc sin } x < \frac{\frac{\pi(2-\sqrt{2})}{\pi-2\sqrt{2}}(\sqrt{1 + x} - \sqrt{1 - x})}{\frac{\sqrt{2}(4-\pi)}{\pi-2\sqrt{2}} + \sqrt{1 + x} + \sqrt{1 - x}} < \frac{\frac{\pi}{\pi-2}x}{\frac{2}{\pi-2} + \sqrt{1 - x^2}}. \tag{9}$$

2.   The main results

In this article, using λ -method of Mitrinović-Vasić we give an analogous statement to Proposition 1.4. Let us notice that from inequality given by L. Zhu [4]:

$$\frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} \leqslant \arcsin x \leqslant \frac{\pi(\sqrt{2}+\frac{1}{2})(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}}, \qquad (10)$$

for $x \in [0,1]$, we can conclude that the function $\varphi(x) = \arcsin x$ has a lower bound and upper bound in the family of the functions:

$$\Phi_{\alpha,\beta}(x) = \frac{\alpha(\sqrt{1+x}-\sqrt{1-x})}{\beta+\sqrt{1+x}+\sqrt{1-x}} \quad (0 \leqslant x \leqslant 1), \qquad (11)$$

for some values of parameters $\alpha, \beta > 0$. Next for $x = 0$ it is true that $\Phi_{\alpha,\beta}(0) = 0$, for $\alpha, \beta > 0$. On the other hand, for values $x \in (0,1]$ it is true:

$$\Phi_{\alpha_1,\beta_1}(x) > \Phi_{\alpha_2,\beta_2}(x) \iff \alpha_1\beta_2 - \alpha_2\beta_1 > (\alpha_2 - \alpha_1)(\sqrt{1+x}+\sqrt{1-x}), \qquad (12)$$

for $\alpha_{1,2}, \beta_{1,2} > 0$. Let us apply λ -method of Mitrinović-Vasić on the considered two-parameters family $\Phi_{\alpha,\beta}(x)$ in order to determine the bounds of the function $\varphi(x)$ under the following conditions:

$$\Phi_{\alpha,\beta}(0) = \varphi(0) \quad \text{and} \quad \frac{d}{dx}\Phi_{\alpha,\beta}(0) = \frac{d}{dx}\varphi(0). \qquad (13)$$

It follows that $\alpha = \beta + 2$. In that way we get one-parameter subfamily:

$$f_\beta(x) = \Phi_{\beta+2,\beta}(x) = \frac{(\beta+2)(\sqrt{1+x}-\sqrt{1-x})}{\beta+\sqrt{1+x}+\sqrt{1-x}} \quad (0 \leqslant x \leqslant 1), \qquad (14)$$

according to the parameter $\beta > 0$. For that family the condition (13) is true:

$$f_\beta(0) = \varphi(0) \quad \text{and} \quad \frac{d}{dx}f_\beta(0) = \frac{d}{dx}\varphi(0). \qquad (15)$$

Additionally, we have:

$$\frac{d^2}{dx^2}f_\beta(0) = \frac{d^2}{dx^2}\varphi(0) \quad \text{and} \quad \frac{d^3}{dx^3}f_\beta(0) = \frac{d^3}{dx^3}\varphi(0) + \frac{4-\beta}{4(2+\beta)} \qquad (16)$$

and

$$\frac{d^4}{dx^4}f_\beta(0) = \frac{d^4}{dx^4}\varphi(0) \quad \text{and} \quad \frac{d^5}{dx^5}f_\beta(0) = \frac{d^5}{dx^5}\varphi(0) + \frac{3(128+18\beta-13\beta^2)}{16(2+\beta)^2}. \qquad (17)$$

Let us notice that for the family of the functions $f_\beta(x)$, on the basis of (12), for values $x \in (0,1]$ the following equivalence is true:

$$f_{\beta_1}(x) > f_{\beta_2}(x) \iff \beta_1 < \beta_2, \qquad (18)$$

for $\beta_{1,2} > 0$. Let us emphasize that there is a better upper bound $f_{b_1}(x)$ than upper bound $\Phi_{\pi(\sqrt{2}+1/2),4}(x)$ of the function $\varphi(x)$ over $(0,1]$. It is true that the parameter $\beta = b_1$ fulfils:

$$f_{b_1}(1) = \varphi(1) = \frac{\pi}{2}, \tag{19}$$

hence:

$$b_1 = \frac{\sqrt{2}(4-\pi)}{\pi - 2\sqrt{2}} = 3.876\,452\,527\ldots < 4. \tag{20}$$

Let us prove that the function $f_{b_1}(x)$ is the upper bound of the function $\varphi(x)$ over $[0,1]$. Let us define the function:

$$h(x) = f_{b_1}(x) - \varphi(x) \tag{21}$$

for $0 \leqslant x \leqslant 1$. For the function $h(x)$ we introduce two substitutions $x = \cos t$ $\left(t \in [0, \frac{\pi}{2}]\right)$ and $t = 4\arctan u$ $\left(u \in [0, \tan\frac{\pi}{8}]\right)$ respectively, and we get a new function:

$$\omega(u) = h\big(\cos(4\arctan u)\big) = \frac{\sqrt{2}(b_1+2)(u^2+2u-1)}{(\sqrt{2}-b_1)u^2 - 2\sqrt{2}u - b_1 - \sqrt{2}} - \frac{\pi}{2} + 4\arctan u \tag{22}$$

for $0 \leqslant u \leqslant \tan\frac{\pi}{8} = \sqrt{2} - 1$. Then:

$$\begin{aligned}
\frac{d}{du}\omega(u) = &\Big[\big(4b_1^2 + 2\sqrt{2}b_1^2 - 8b_1 - 4\sqrt{2}b_1 - 8\big)u^4 + \big(-4\sqrt{2}b_1^2 + 8\sqrt{2}b_1 - 32\big)u^3 \\
&+ \big(8b_1^2 - 16b_1 - 16\big)u^2 + \big(-4\sqrt{2}b_1^2 + 8\sqrt{2}b_1 + 32\big)u \\
&+ \big(4b_1^2 - 2\sqrt{2}b_1^2 - 8b_1 + 4\sqrt{2}b_1 - 8\big)\Big]\Big/ \\
&\Big[\big(u^2+1\big)\big(b_1u^2 - \sqrt{2}u^2 + 2\sqrt{2}u + b_1 + \sqrt{2}\big)^2\Big].
\end{aligned} \tag{23}$$

All solutions of the equation $\frac{d}{du}\omega(u) = 0$ are determined by terms:

$$\begin{aligned}
u_{1,4} &= \frac{2\sqrt{2} \mp \sqrt{-b_1^4 + 4b_1^3 + 4b_1^2 - 16b_1}}{b_1^2 - 2b_1 + 2\sqrt{2} - 4}, \\
u_{2,3} &= \sqrt{2} - 1;
\end{aligned} \tag{24}$$

or by numerical values: $u_1 = 0.0869\ldots$, $u_{2,3} = 0.4142\ldots$, $u_4 = 0.8400\ldots$. The function $\omega(u)$ has local maximum at the point u_1 and $\omega(0) = \omega(\sqrt{2}-1) = 0$. Hence $\omega(u) \geqslant 0$ for $u \in [0, \sqrt{2}-1]$. Therefore the function:

$$f_{b_1}(x) = \frac{\frac{\pi(2-\sqrt{2})}{\pi - 2\sqrt{2}}(\sqrt{1+x} - \sqrt{1-x})}{\frac{\sqrt{2}(4-\pi)}{\pi - 2\sqrt{2}} + \sqrt{1+x} + \sqrt{1-x}} \tag{25}$$

is the upper bound of $\varphi(x)$ over $[0,1]$. Let us notice that, for values $x \in (0,1]$, on the basis (12), the following inequalities are true:

$$\varphi(x) < f_{b_1}(x) = \Phi_{b_1+2,b_1}(x) < \Phi_{\pi(\sqrt{2}+1/2),4}(x). \tag{26}$$

Let us prove that the function $f_{b_1}(x)$ is the least upper bound of the function $\varphi(x)$ from the family (14). The following implication is true:

$$b_1 < b \Longrightarrow f_b(1) < f_{b_1}(1) = \varphi(1) = \frac{\pi}{2}. \tag{27}$$

Hence for $b > b_1$ the function $f_b(x)$ is not the upper bound for the function $\varphi(x)$ over $[0, 1]$. According to the previous consideration we can conclude that the function $f_{b_1}(x)$ is the least upper bound of the function $\varphi(x)$ over $[0, 1]$.

The lower bound of the function $f_4(x)$ of the function $\varphi(x)$ over $[0, 1]$, which belongs to R. E. Shafer, according to formulas (15) - (17), has at $x = 0$ the root of the fifth order. Let us prove that the function $f_4(x)$ is the greatest lower bound of the function $\varphi(x)$ from the family (14). For fixed $b \in (b_1, 4)$ let us define the function:

$$g(x) = \begin{cases} \alpha & : x = 0, \\ \dfrac{f_b(x) - \varphi(x)}{x^3} & : x \in (0, 1]; \end{cases} \tag{28}$$

with the constant:

$$\alpha = \frac{\frac{d^3}{dx^3}f_b(0) - \frac{d^3}{dx^3}\varphi(0)}{6} = \frac{4 - b}{24(2 + b)} > 0. \tag{29}$$

The function $g(x)$ is continuous over $[0, 1]$ and the following is true:

$$g(0) > 0 \quad \text{and} \quad g(1) < 0. \tag{30}$$

Therefore we can conclude that there is $c_b \in (0, 1)$ such that $g(c_b) = 0$. Let us notice that $g(0) > 0$ and $g(c_b) = 0$. Then, there is some point $\xi_b \in (0, c_b)$ such that $g(\xi_b) > 0$ $\left(g \in C[0, c_b]\right)$. This is sufficient for conclusion that, for each $b \in (b_1, 4)$, the function $f_b(x)$ is not the lower bound of the function $\varphi(x)$ over $[0, 1]$. According to the previous consideration we can conclude that the function $f_4(x)$ is the greatest lower bound of the function $\varphi(x)$ over $[0, 1]$.

On the basis of the previous consideration the following statement is true.

PROPOSITION 2.1. *In the family of the functions:*

$$f_b(x) = \Phi_{b+2,b}(x) = \frac{(b + 2)(\sqrt{1 + x} - \sqrt{1 - x})}{b + \sqrt{1 + x} + \sqrt{1 - x}} \qquad (0 \leqslant x \leqslant 1), \tag{31}$$

according to the parameter $b > 0$, the function $f_4(x)$ is the greatest lower bound of the $\arcsin x$ function and the function $f_{\sqrt{2}(4-\pi)/(\pi-2\sqrt{2})}(x)$ is the least upper bound of the $\arcsin x$ function.

REMARK 2.2. Let us emphasize that Theorem 1.6 has been recently considered in [6] and [7]. In the article [7] a simple proof of Theorem 1.6 based on "L'Hospital rule for monotonicity" is obtained.

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