Mathematical
Inequalities
& Applications
Volume 10, Number 3 (2007), 535–546

ON THE CHEBYSHEV FUNCTIONAL

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(*communicated by I. Perić*)

Abstract. In this paper we prove an inequality for certain orthoprojectors. For orthoprojectors of rank one we obtain a Chebyshev type inequality. Grüss-Lupaş type inequalities are also discussed.

1. Introduction and summary

Let V be a real linear space with an inner product $\langle \cdot, \cdot \rangle$. The *Chebyshev functional* is defined by

$$T_v(x, y) = \|v\|^2 \langle x, y \rangle - \langle x, v \rangle \langle y, v \rangle \quad \text{for } x, y \in V, \tag{1}$$

where $v \in V$ is a given nonzero vector and $\|v\|^2 = \langle v, v \rangle$.

The purpose of this note is to prove Chebyshev and Grüss-Lupaş type inequalities in the framework of Eaton systems connected with group-induced cone orderings [1, 3]. That is, we shall give some bounds on the values of Chebyshev functional (1).

In Section 2 we present some notions related to Eaton systems. Also, a general inequality is given for certain orthoprojectors (see Theorem 2.1). Equality case for this inequality is studied in Theorem 2.2.

Section 3 is devoted to Chebyshev type inequalities. They are particular cases of the above-mentioned result applied for special orthoprojectors of rank one. For the E-system induced by the permutation group acting on $\mathbb{R}^n$, one obtains the classical Chebyshev sum inequality.

Grüss-Lupaş type inequalities are investigated in Section 4. Here we develop recent results of Izumino, Pečarić and Tepeš [4]. We base on an identity generated by certain class of linear operators. In particular, this method leads to an estimation of the Chebyshev functional using the second order differences.

2. Projection inequality for Eaton systems

Throughout the paper, V is a finite-dimensional real linear space with a (real) inner product $\langle \cdot, \cdot \rangle$ unless otherwise indicated. Assume G is a closed subgroup of the orthogonal group $O(V)$ acting on V. For given $x, y \in V$, we write $y \preceq_G x$ if y belongs to the convex hull of the orbit Gx. The ordering $\preceq_G$ on V is called G*-majorization*.

Mathematics subject classification (2000): 26D15, 26D20, 15A39, 06F20.
Key words and phrases: orthoprojector, Chebyshev type inequality, Grüss-Lupaş inequality.

If there exists a closed convex cone $D \subset V$ such that

$(A1)$ for each $a \in V$ there exist $g \in G$ and $b \in D$ satisfying $a = gb$,

$(A2)$ $\langle a, gb \rangle \leqslant \langle a, b \rangle$ for all $a, b \in D$ and $g \in G$,

then we say that (V, G, D) is an *Eaton system* (in short, *E-system*). Condition $(A1)$ asserts that vectors of the space can be decomposed with the aid of the operators of the group and the vectors of the cone (see (2)). Condition $(A2)$ generalizes *von Neumann trace inequality* for matrices (see [1, p. 17]).

If axioms $(A1)-(A2)$ are satisfied, the ordering $\preceq_G$ is said to be *group-induced cone ordering* [1, 3]. A related notion is a *normal decompostion (ND) system* introduced by Lewis [5, 6, 7].

It can be shown (see e.g. [1, p. 15)], [11, p. 14]) that under $(A1)$ and $(A2)$ there exists an idempotent operator $(\cdot)_\downarrow : V \to V$ with the range D such that

$$\{a_\downarrow\} = D \cap Ga \text{ for each } a \in V.$$

Then each $a \in V$ has its decomposition

$$a = ga_\downarrow \text{ for some } g \in G. \tag{2}$$

Examples of (2) cover such important results as the Spectral Theorem for Hermitian matrices and the Singular Value Theorem for complex matrices. In these cases the operator $(\cdot)_\downarrow$ is the eigenvalue map and the singular values map, respectively (see [1, p. 17-18]).

The triple $(V, G, (\cdot)_\downarrow)$ is called a *normal decomposition system* and the operator $(\cdot)_\downarrow$ is called a *normal map* [5, 6, 7].

See [1, 2, 3, 5, 6, 7, 9, 11, 12] for examples and applications of group-induced cone orderings, Eaton systems and ND systems (see also Examples 2.4 and 3.3-3.4).

We denote

$$M_G(V) = \{a \in V : ga = a \text{ for all } g \in G\}.$$

It is known that $M_G(V) \subset D$ [12, Theorem 3.1].

THEOREM 2.1. *Let (V, G, D) be an Eaton system and let I and P stand, respectively, for the identity operator on V and for the orthoprojector from V onto $M_G(V)$. Then the following inequality holds:*

$$\langle x, (I-P)y \rangle \geqslant 0 \text{ for } x, y \in D. \tag{3}$$

Proof. Since G is a closed subgroup of the compact group $O(V)$, G is compact. Let μ denotes the Haar probability measure on G [10]. Denote

$$P_0 a = \int_G ga \, d\mu(g) \text{ for } a \in V.$$

Using condition $(A2)$ we obtain

$$\langle x, y - P_0 y \rangle = \langle x, y - \int_G gy \, d\mu(g) \rangle = \int_G \langle x, y - gy \rangle \, d\mu(g) \geqslant 0 \text{ for } x, y \in D. \tag{4}$$

It is now sufficient to prove $P = P_0$. To this end we shall show that the operator P_0 is symmetric and idempotent, and that the range of P_0 is $M_G(V)$.

Since G is unimodular [10, p. 81], for any $a,b \in V$ we get

$$\begin{aligned}\langle a, P_0 b\rangle &= \langle a, \int_G gb\, d\mu(g)\rangle = \int_G \langle a, gb\rangle\, d\mu(g) \\ &= \int_G \langle g^{-1}a, b\rangle\, d\mu(g) = \int_G \langle ga, b\rangle\, d\mu(g) = \langle P_0 a, b\rangle,\end{aligned}$$

which gives the symmetry of P_0.

To see that P_0 is idempotent, we employ the G-invariance of the Haar integral as follows. For $a \in V$ we have

$$\begin{aligned}{P_0}^2 a &= \int_G g\left(\int_G \widetilde{g}a\, d\mu(\widetilde{g})\right) d\mu(g) = \int_G \left(\int_G g\widetilde{g}a\, d\mu(\widetilde{g})\right) d\mu(g) \\ &= \int_G \left(\int_G \widetilde{g}a\, d\mu(\widetilde{g})\right) d\mu(g) = \int_G \widetilde{g}a\, d\mu(\widetilde{g}) \cdot \mu(G) = P_0 a.\end{aligned}$$

It remains to show $P_0 V = M_G(V)$. For $a \in M_G(V)$, we have

$$P_0 a = \int_G ga\, d\mu(g) = \int_G a\, d\mu(g) = a.$$

This implies $M_G(V) \subset P_0 V$.

The opposite inclusion can be obtained as follows. If $a \in P_0 V$ then $a = P_0 b$ for some $b \in V$. So, for each $g \in G$, by the G-invariance, we get

$$ga = gP_0 b = g\int_G \widetilde{g}b\, d\mu(\widetilde{g}) = \int_G g\widetilde{g}b\, d\mu(\widetilde{g}) = \int_G \widetilde{g}b\, d\mu(\widetilde{g}) = P_0 b = a.$$

This gives $a \in M_G(V)$, as wanted.

In summary, we have proved that

$$Pa = \int_G ga\, d\mu(g) \quad \text{for } a \in V.$$

This together with (4) completes the proof of Theorem 2.1. □

We now study the case of equality in (3). We say that a linear subspace $W \subset V$ is *G-invariant* if $ga \in W$ for all $g \in G$ and $a \in W$. If W is G-invariant, the group G is said to be *irreducible* on W (and W is said to be *G-irreducible*), if the only G-invariant subspaces of W are W and $\{0\}$.

In what follows, we assume that $M_G(V) \neq V$. In general, one has the orthogonal decomposition

$$V = V_0 + V_1 + \ldots + V_m \tag{5}$$

for some positive integer m, where $V_0 = M_G(V)$, and V_i, $i = 0, 1, \ldots, m$, are mutually orthogonal G-invariant subspaces in V, and, additionally, V_i, $i = 1, \ldots, m$, are nonzero and G-irreducible.

Denote $W = V_1 + \ldots + V_m$. Obviuosly, W is the orthogonal complement of $V_0 = M_G(V)$. The linear operator $Q = I - P$ is the orthoprojector from V onto W (see Theorem 2.1).

THEOREM 2.2. *Under the assumptions of Theorem* 2.1*, assume that* (5) *is satisfied.*

If W is G-irreducible, that is, if $m = 1$ and $W = V_1$, then equality holds in (3) *if and only if $x \in M_G(V)$ and/or $y \in M_G(V)$.*

Proof. Let $x \in M_G(V)$ and/or $y \in M_G(V)$. Then $Qx = 0$ and/or $Qy = 0$, and therefore

$$0 = \langle Qx, Qy \rangle = \langle x, Q^2 y \rangle = \langle x, Qy \rangle = \langle x, (I-P)y \rangle ,$$

as required.

Conversely, assume $\langle x, (I-P)y \rangle = 0$ for some $x, y \in D$, that is $\langle Qx, Qy \rangle = 0$. Suppose that W is G-irreducible. It is evident that W is G-invariant. It is known that $(W, G|_W, D \cap W)$ is an Eaton system (see [12, p. 111]). Applying [12, Theorem 3.2] we obtain $\langle a, b \rangle > 0$ for all nonzero $a, b \in D \cap W$.

On the other hand, $Qx, Qy \in D \cap W$. Therefore the assumption that both Qx and Qy are nonzero vectors leads to $\langle Qx, Qy \rangle > 0$, a contradiction. Thus we get $Qx = 0$ and/or $Qy = 0$.

Hence $x \in M_G(V)$ and/or $y \in M_G(V)$, completing the proof. □

We say that an Eaton system (V, G, D) is *effective* if $M_G(V) = \{0\}$.

REMARK 2.3. For an effective Eaton system (V, G, D) we have $P \equiv 0$. Therefore in this case (3) reduces to

$$\langle x, y \rangle \geqslant 0 \quad \text{for } x, y \in D \tag{6}$$

(cf. [12, Theorem 3.2]).

EXAMPLE 2.4. Let

$$V = \{a = (a_1, \ldots, a_n)^T \in \mathbb{R}^n : \sum_{i=1}^n a_i = 0\}$$

with the standard inner product $\langle a, b \rangle = \sum_{i=1}^n a_i b_i$.

Set

$$G = \mathbb{P}_n = \text{the group of all } n\text{-by-}n \text{ permutation matrices.}$$

Recall that a *permutation matrix* is a matrix with entries all 0 or 1 such that every row and every column contains exactly one entry equal to 1. Take

$$D = \{a \in \mathbb{R}^n : a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n, \sum_{i=1}^n a_i = 0\}.$$

Then (V, G, D) is an Eaton system (cf. [1, p. 16]). In addition, V is effective and G-irreducible, that is $V_0 = \{0\}$ and $W = V_1$.

In this situation, (6) takes the form

$$\sum_{i=1}^n x_i y_i \geqslant 0 \quad \text{for } x, y \in D. \tag{7}$$

This is a particular case of the *Chebyshev sum inequality* (see (14)). Equality holds in (7) if and only if $x = 0$ and/or $y = 0$ (see Theorem 2.2).

For an Eaton system (V,G,D), one obtains

$$V = \bigcup_{g\in G} gD \tag{8}$$

by $(A1)$.

We say that two vectors $x,y \in V$ are *synchronous* (with respect to (V,G,D)) if there exists $g \in G$ such that $x,y \in gD$.

REMARK 2.5. It is not hard to check that (V,G,D) is an Eaton system iff (V,G,gD) is an Eaton system for each $g \in G$.

In consequence, under the assumptions of Theorem 2.1, inequality (3) extends to

$$\langle x,(I-P)y\rangle \geqslant 0 \text{ for synchronous vectors } x,y \in V. \tag{9}$$

In particular, for each $x \in V$ inequality (9) holds for $y = x$, because x and $y = x$ are synchronous vectors by (8). In this case, (9) asserts that the operator $Q = I - P$ is positive semidefinite. In fact, Q is the orthoprojector from V onto the subspace W orthogonal to $M_G(V)$.

3. Chebyshev type inequality

In this section we investigate the particular situation when the subspace $M_G(V)$ is one-dimensional.

THEOREM 3.1. *Let (V,G,D) be an Eaton system and let $M_G(V) = \operatorname{span} v$ for some nonzero $v \in V$.*

Then the following Chebyshev type inequality holds:

$$\|v\|^2\langle x,y\rangle - \langle x,v\rangle\langle y,v\rangle \geqslant 0 \text{ for } x,y \in D. \tag{10}$$

If, in addition, the subspace $W = (\operatorname{span} v)^{\perp}$ is G-irreducible, then equality holds in (10) if and only if x and/or y is a scalar multiple of v.

Proof. It is sufficient to prove that

$$\langle x,y\rangle - \langle x,\frac{v}{\|v\|}\rangle\langle y,\frac{v}{\|v\|}\rangle \geqslant 0 \text{ for } x,y \in D. \tag{11}$$

Observe that the left-hand side of (11) is equal to $\langle x, y - \langle y,\frac{v}{\|v\|}\rangle\frac{v}{\|v\|}\rangle$.

The operator $\widetilde{P}: V \to V$ defined by

$$\widetilde{P}a = \langle a,\frac{v}{\|v\|}\rangle\frac{v}{\|v\|}, \quad a \in V,$$

is the orthoprojector from V onto $\operatorname{span} v$. Since $M_G(V) = \operatorname{span} v$, $\widetilde{P}$ is equal to the orthoprojector P from V onto $M_G(V)$.

Therefore (11) follows from (3). In consequence, (10) is valid.

It is obvious that if x and/or y is a scalar multiple of v, then equality holds in (10). Conversely, suppose that equality holds in (10) and that W is G-irreducible. By Theorem 2.2, $x \in M_G(V) = \operatorname{span} v$ and/or $y \in M_G(V) = \operatorname{span} v$. This completes the proof of Theorem 3.1. $\square$

REMARK 3.2. (a) Inequality (10) can be extended as follows:

$$\|v\|^2\langle x,y\rangle - \langle x,v\rangle\langle y,v\rangle \geqslant 0 \text{ for synchronous vectors } x,y\in V. \tag{12}$$

To see this, apply (10) together with the fact that $v\in M_G(V)$ and $gv=v$ for all $g\in G$.

(b) It is easily seen that the case $y=x$ of (12) leads to Cauchy-Schwarz inequality

$$\|v\|^2\|x\|^2 \geqslant \langle v,x\rangle^2. \tag{13}$$

EXAMPLE 3.3. Take V to be the Euclidean space $\mathbb{R}^n$ with the standard inner product. Consider $G=\mathbb{P}_n$ = the group of all n-by-n permutation matrices, and

$$D=\{a=(a_1,\ldots,a_n)^T\in\mathbb{R}^n : a_1\geqslant a_2\geqslant\ldots\geqslant a_n\}.$$

Then (V,G,D) is an Eaton system [1, p. 16].

Moreover, $a_{\downarrow}=(a_{[1]},\ldots,a_{[n]})$, where $a_{[1]}\geqslant\ldots\geqslant a_{[n]}$ are the entries of a in nonincreasing order. In addition, $M_G(V)=\operatorname{span} v$ for $v=(1,\ldots,1)^T$.

In this case, we have the following interpretation of (12): for any synchronous n-tuples $x=(x_1,\ldots,x_n)^T$ and $y=(y_1,\ldots,y_n)^T$

$$n\sum_{i=1}^n x_iy_i \geqslant \sum_{i=1}^n x_i\sum_{i=1}^n y_i \tag{14}$$

with equality iff x and/or y is a scalar multiple of $(1,\ldots,1)^T$ (cf. [13, Section 4]). This result is *the classical Chebyshev sum inequality*.

Here the synchronicity of x and y means

$$(x_i-x_j)(y_i-y_j)\geqslant 0 \text{ for all } 1\leqslant i,j\leqslant n.$$

EXAMPLE 3.4. Let

$$V=\mathbb{H}_n = \text{the real space of all } n\text{-by-}n \text{ Hermitian matrices},$$
$$G=\{U(\cdot)U^* : U\in\mathbb{U}_n\} = \text{the group of all unitary similarities},$$

and

$$D=\{a\in\mathbb{D}_n : a_{11}\geqslant a_{22}\geqslant\ldots\geqslant a_{nn}\}$$
$$= \text{the convex cone of all real } n\text{-by-}n \text{ diagonal matrices}$$
$$\text{with decreasingly ordered diagonal entries.}$$

The inner product on V is given by $\langle a,b\rangle=\operatorname{tr}ab$, the trace of the matrix ab.

By virtue of the Spectral Theorem, condition $(A1)$ holds. On the other hand, the von Neumann trace inequality implies $(A2)$ (see [1, p. 17] for details). Therefore (V,G,D) is an Eaton system. Here $M_G(V)=\operatorname{span} I_n$, where I_n denotes the n-by-n identity matrix. Additionally, $a_{\downarrow}=\operatorname{diag}\lambda(a)$, where $\lambda(a)=(\lambda_1(a),\ldots,\lambda_n(a))^T$ is the vector of the eigenvalues of a Hermitian matrix a with $\lambda_1(a)\geqslant\ldots\geqslant\lambda_n(a)$.

It now follows from Theorem 3.1 that the following Chebyshev type inequality is valid:

$$n\operatorname{tr} xy \geqslant \operatorname{tr} x \operatorname{tr} y \quad \text{for } x, y \in D.$$

In other words, by (2),

$$n\operatorname{tr} x_{\downarrow} y_{\downarrow} \geqslant \operatorname{tr} x \operatorname{tr} y \quad \text{for } x, y \in V,$$

or, equivalently,

$$n\sum_{i=1}^{n} \lambda_i(x)\lambda_i(y) \geqslant \operatorname{tr} x \operatorname{tr} y \quad \text{for } x, y \in V. \tag{15}$$

This is a matrix version of (14).

Equality holds in (15) iff x and/or y is a scalar multiple of I_n.

4. Grüss-Lupaş type inequality

Assume $\|v\| = 1$. Remind that

$$T_v(x,y) = \langle x, y\rangle - \langle x, v\rangle \langle y, v\rangle = \langle x - \langle x, v\rangle v, y\rangle \quad \text{for } x, y \in V. \tag{16}$$

In this section our main goal is to provide an estimate for the modulus of $T_v(x,y)$. Here we develop a method used in [4].

In the sequel V is a finite-dimensional real inner product space and $\{e_i : i = 1, 2, \dots, n\}$ is a basis in V. Denote by $\{d_i : i = 1, 2, \dots, n\}$ the dual basis of $\{e_i\}$, i.e., $\langle e_i, d_j\rangle = \delta_{ij}$, the Kronecker symbol, $i, j = 1, \dots, n$.

Then the following formula holds:

$$\langle a, b\rangle = \sum_{i=1}^{n} \langle a, e_i\rangle \langle b, d_i\rangle \quad \text{for } a, b \in V. \tag{17}$$

We assume that for each $i = 1, \dots, n$ the space V possesses the orthogonal decomposition

$$V = V_{1i} + V_{2i} \tag{18}$$

for some (orthogonal) subspaces V_{1i} and V_{2i}.

For $k = 1, 2$, suppose that $A_{ki}, B_{ki}, C_{ki} : V_{ki} \to V_{ki}$ are linear operators (matrices) such that

$$B_{ki}^{*} A_{ki} = I_{ki} \quad \text{and} \quad C_{ki}^{*} C_{ki} = I_{ki}, \tag{19}$$

where I_{ki} is the identity operator on V_{ki}.

Here B_{ki}^{*} is the dual operator of B_{ki} in the sense that $\langle B_{ki}^{*} w, v\rangle = \langle w, B_{ki} v\rangle$ for all $v, w \in V_{ki}$. (For C_{ki} analogously.)

Then the identity

$$\langle a, b\rangle = \langle A_{1i} C_{1i}\, a_{1i}, B_{1i} C_{1i}\, b_{1i}\rangle + \langle A_{2i} C_{2i}\, a_{2i}, B_{2i} C_{2i}\, b_{2i}\rangle \quad \text{for } a, b \in V \tag{20}$$

holds, where $a = a_{1i} + a_{2i}$ and $b = b_{1i} + b_{2i}$ with $a_{ki}, b_{ki} \in V_{ki}$, $k = 1, 2$.

THEOREM 4.1. *Under the above assumptions, the following Grüss-Lupaş type inequality holds:*

$$|T_v(x,y)| \leqslant \sum_{i=1}^{n} \big(|\langle A_{1i}C_{1i}\, w_{1i}, B_{1i}C_{1i}\, v_{1i} \rangle| + |\langle A_{2i}C_{2i}\, w_{2i}, B_{2i}C_{2i}\, v_{2i} \rangle| \big) \cdot |\langle v, e_i \rangle| \cdot |\langle y, d_i \rangle| \tag{21}$$

for $x,y,v \in V$ *with* $\|v\| = 1$*, where* $w^{(i)} = x - \frac{\langle x,e_i \rangle}{\langle v,e_i \rangle} v$ *with* $\langle v, e_i \rangle \neq 0$*, and* w_{ki} *and* v_{ki} *are the projections of* $w^{(i)}$ *and* v*, respectively, into* V_{ki}*,* $k = 1,2$*.*

Proof. Because of (16) and (17) we can write

$$T_v(x,y) = \sum_{i=1}^{n} \langle x - \langle x, v \rangle v, e_i \rangle \langle y, d_i \rangle .$$

Hence

$$|T_v(x,y)| \leqslant \sum_{i=1}^{n} |\langle x - \langle x, v \rangle v, e_i \rangle| \cdot |\langle y, d_i \rangle| . \tag{22}$$

Denoting $Px = \langle x, v \rangle v$ and $\eta_i = \frac{\langle x,e_i \rangle}{\langle v,e_i \rangle}$ we have

$$\begin{aligned} \langle Px - x, e_i \rangle &= \langle Px, e_i \rangle - \langle x, e_i \rangle = \langle Px, e_i \rangle - \eta_i \langle v, e_i \rangle \\ &= \langle P(x - \eta_i v), e_i \rangle = \langle Pw^{(i)}, e_i \rangle \\ &= \langle \langle w^{(i)}, v \rangle v, e_i \rangle = \langle w^{(i)}, v \rangle \langle v, e_i \rangle . \end{aligned}$$

In consequence, we derive

$$|\langle x - \langle x, v \rangle v, e_i \rangle| = |\langle w^{(i)}, v \rangle| |\langle v, e_i \rangle| . \tag{23}$$

Combining (22) and (23), we get

$$|T_v(x,y)| \leqslant \sum_{i=1}^{n} |\langle w^{(i)}, v \rangle| \cdot |\langle v, e_i \rangle| \cdot |\langle y, d_i \rangle| .$$

Finally, employing (20) for $a = w^{(i)}$ and $b = v$ yields (21). This completes the proof of Theorem 4.1. □

Remark that for any $\mu \in \mathbb{R}$

$$T_v(x,y) = \langle x - \langle x, v \rangle v, y \rangle = \langle x - \langle x, v \rangle v, y - \mu v \rangle = T_v(x, y_\mu) \ \text{ for } x, y \in V$$

where $\|v\| = 1$ and $y_\mu = y - \mu v$.

Using a similar argument as in the above proof, for $\mu = \langle y, v \rangle$ we get

$$|\langle y - \langle y, v \rangle v, d_i \rangle| = |\langle \widetilde{w}^{(i)}, v \rangle| |\langle v, d_i \rangle| ,$$

where $\widetilde{w}^{(i)} = y - \frac{\langle y,d_i \rangle}{\langle v,d_i \rangle} v$ with $\langle v, d_i \rangle \neq 0$.

Therefore, applying a representation of type (20), one can obtain a version of (21) with an estimate for $\langle \widetilde{w}^{(i)}, v\rangle$ analogously as for $\langle w^{(i)}, v\rangle$. Namely,

$$|T_v(x,y)| \leqslant \sum_{i=1}^{n} (|\langle A_{1i}C_{1i}\, w_{1i}, B_{1i}C_{1i}\, v_{1i}\rangle| + |\langle A_{2i}C_{2i}\, w_{2i}, B_{2i}C_{2i}\, v_{2i}\rangle|) \cdot |\langle v, e_i\rangle| \times$$
$$\times (|\langle \widetilde{A}_{1i}\widetilde{C}_{1i}\, \widetilde{w}_{1i}, \widetilde{B}_{1i}\widetilde{C}_{1i}\, v_{1i}\rangle| + |\langle \widetilde{A}_{2i}\widetilde{C}_{2i}\, \widetilde{w}_{2i}, \widetilde{B}_{2i}\widetilde{C}_{2i}\, v_{2i}\rangle|) \cdot |\langle v, d_i\rangle|$$

for $x,y,v \in V$ with $\|v\| = 1$, and w_{ki} and v_{ki} are the projections of $w^{(i)}$ and v, respectively, into V_{ki}, $k = 1,2$.

For $k = 1,2$, $\widetilde{A}_{ki}$, $\widetilde{B}_{ki}$ and $\widetilde{C}_{ki}$ are operators (matrices) such that $\widetilde{B}^*_{ki}\widetilde{A}_{ki} = I_{ki}$ and $\widetilde{C}^*_{ki}\widetilde{C}_{ki} = I_{ki}$.

Additionally, $\widetilde{w}_{ki}$ are the projections of $\widetilde{w}^{(i)}$ into $\widetilde{V}_{ki}$, $k = 1,2$, satisfying $V = \widetilde{V}_{1i} + \widetilde{V}_{2i}$ with orthogonal $\widetilde{V}_{1i}$ and $\widetilde{V}_{2i}$.

We now show how our theory works. In Example 4.2 we specialize Theorem 4.1 to obtain a recent result of Izumino, Pečarić and Tepeš [4, Theorem 2.3].

EXAMPLE 4.2. Let V be $\mathbb{R}^n$ and let $p = (p_1,\dots,p_n)^T$ be a probabilistic vector, i.e., $\sum_{i=1}^n p_i = 1$ and $p_i > 0$. Consider the inner product defined by

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i p_i \tag{24}$$

for $x = (x_1,\dots,x_n)^T$ and $y = (y_1,\dots,y_n)^T$.

That is, $\langle x,y\rangle = \langle x, Ly\rangle_s$, where $\langle \cdot,\cdot\rangle_s$ is the standard inner product and $L = \operatorname{diag}(p_1,\dots,p_n)$ is the diagonal matrix with the p_i's on the main diagonal.

Let $e_i = (1/\sqrt{p_i})(0,\dots,0,1,0,\dots,0)^T$ with 1 at the ith position, $i = 1,\dots,n$. Then $\langle e_i, e_j\rangle = \delta_{ij}$, the Kronecker delta. In other words, e_i's constitute an orthonornal basis in $\mathbb{R}^n$ with respect to $\langle \cdot,\cdot\rangle$. So, $d_i = e_i$, $i = 1,\dots,n$, is the dual basis of e_i's.

Putting

$$V_{1i} = \{(z_1,\dots,z_{i-1},0,\dots,0)^T : z_1,\dots,z_{i-1} \in \mathbb{R}\}$$

and

$$V_{2i} = \{(0,\dots,0,z_i,\dots,z_n)^T : z_i,\dots,z_n \in \mathbb{R}\}$$

we obtain (18).

For simplicity we identify the vectors $(z_1,\dots,z_{i-1},0,\dots,0)^T$ and $(z_1,\dots,z_{i-1})^T$. Thus V_{1i} is identified with $\mathbb{R}^{i-1}$. Similarly for V_{2i}.

For $1 \leqslant m \leqslant n$ we denote by A_m and B_m the m-by-m matrices

$$A_m = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & & 0 & 0 \\ \vdots & & \ddots & \ddots & & \\ 0 & 0 & 0 & & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad \text{and} \quad B_m = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}. \tag{25}$$

Furthermore, we set

$$A_{1i} = A_{i-1}, \quad B_{1i} = B_{i-1}, \quad A_{2i} = A_{n-i+1}, \quad B_{2i} = B_{n-i+1},$$

and

$$C_{1i} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_{2i} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & & 1 & 0 \\ \vdots & & & & \vdots \\ 0 & 1 & & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{26}$$

Here C_{1i} and C_{2i} are matrices of sizes $(i-1)$-by-$(i-1)$ and $(n-i+1)$-by-$(n-i+1)$, respectively.

Notice that (19) is satisfied for $B^*_{ki} = L^{-1}_{ki} B^T_{ki} L_{ki}$, where $L_{1i} = \operatorname{diag}(p_1, \dots, p_{i-1})$ and $L_{2i} = \operatorname{diag}(p_i, \dots, p_n)$. For this reason, (20) takes the form

$$\langle a, b\rangle = \langle A_{1i}C_{1i}a_{1i}, B_{1i}C_{1i}L_{1i}b_{1i}\rangle_s + \langle A_{2i}C_{2i}a_{2i}, B_{2i}C_{2i}L_{2i}b_{2i}\rangle_s \text{ for } a, b \in \mathbb{R}^n, \tag{27}$$

where

$$a_{1i} = (a_1, \dots, a_{i-1})^T, \quad a_{2i} = (a_i, \dots, a_n)^T,$$
$$b_{1i} = (b_1, \dots, b_{i-1})^T, \quad b_{2i} = (b_i, \dots, b_n)^T.$$

Take

$$x = (x_1, \dots, x_n)^T, \quad y = (y_1, \dots, y_n)^T \quad \text{and} \quad v = (v_1, \dots, v_n)^T$$

with $\|v\|^2 = \sum v_i^2 p_i = 1$.

Let $w^{(i)} = (w_1^{(i)}, \dots, w_n^{(i)})^T$ be defined as in Theorem 4.1, that is $w^{(i)} = x - \eta_i v$ for $\eta_i = x_i / v_i$.

Clearly, the i-th entry of $w^{(i)}$ is zero. Denote

$$\Delta x_j = x_{j+1} - x_j, \quad \Delta v_j = v_{j+1} - v_j, \quad \Delta w_j^{(i)} = w_{j+1}^{(i)} - w_j^{(i)}, \quad \text{and}$$
$$S_j = v_1 p_1 + \dots + v_j p_j, \quad \overline{S}_{j+1} = v_{j+1} p_{j+1} + \dots + v_n p_n.$$

Now, applying (25)-(27) leads to the conclusion that

$$\langle w^{(i)}, v\rangle = \sum_{j=1}^{i-1} (-\Delta x_j + \eta_i \Delta v_j) S_j + \sum_{j=i}^{n-1} (\Delta x_j - \eta_i \Delta v_j) \overline{S}_{j+1}, \tag{28}$$

because $\Delta w_j^{(i)} = \Delta x_j - \eta_i \Delta v_j$.

Consequently, (21) becomes

$$\begin{aligned} &\Big|\sum_{i=1}^n x_i y_i p_i - \sum_{i=1}^n x_i v_i p_i \sum_{i=1}^n y_i v_i p_i\Big| \\ &\qquad \leqslant \sum_{i=1}^n \left(\sum_{j=0}^{i-1} |S_j||\Delta w_j^{(i)}| + \sum_{j=i}^{n} |\overline{S}_{j+1}||\Delta w_j^{(i)}| \right) |v_i||y_i| p_i \end{aligned} \tag{29}$$

with $S_0 = \overline{S}_{n+1} = 0$.

Taking $v = (1,\ldots,1)^T$, we have $\|v\|^2 = \langle v,v\rangle = 1$. Also, $\Delta v_j = 0$ and therefore $\Delta w_j^{(i)} = \Delta x_j$ for all j.

Finally, (29) reduces to the mentioned result of Izumino, Pečarić and Tepeš (see [4, Theorem 2.3]):

$$\left|\sum_{i=1}^n x_i y_i p_i - \sum_{i=1}^n x_i p_i \sum_{i=1}^n y_i p_i\right| \leqslant \sum_{i=1}^n \left(\sum_{j=0}^{i-1} P_j|\Delta x_j| + \sum_{j=i}^{n} \overline{P}_{j+1}|\Delta x_j|\right)|y_i|p_i, \tag{30}$$

where

$$P_j = p_1 + \ldots + p_j, \quad \overline{P}_{j+1} = p_{j+1} + \ldots + p_n, \quad P_0 = \overline{P}_{n+1} = 0.$$

In Example 4.3 we employ Theorem 4.1 to derive an analog of (30) for the second order differences (cf. [4, Theorem 2.3])).

EXAMPLE 4.3. Let V, V_{1i}, V_{2i}, $e_i = d_i$ and $\langle\cdot,\cdot\rangle$ be defined as in Example 4.2. We introduce the following n-by-n matrices

$$A = \begin{pmatrix} 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 2 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ n-1 & n-2 & \ldots & 1 & 0 \\ n & n-1 & \ldots & 2 & 1 \end{pmatrix}.$$

Take A_{1i} and A_{2i} to be, respectively, the $(i-1)$-by-$(i-1)$ upper left corner and the $(n-i+1)$-by-$(n-i+1)$ lower right corner of A. Likewise, let B_{1i} and B_{2i} be, respectively, the $(i-1)$-by-$(i-1)$ upper left corner and the $(n-i+1)$-by-$(n-i+1)$ lower right corner of B. Since A and B are triangular, and $B^T = A^{-1}$, one has the needed property $B_{ki}^T A_{ki} = I_{ki}$.

Moreover, C_{1i} and C_{2i} are defined as in Example 4.2. We also define

$$\begin{aligned} \Delta^2 x_j &= \Delta x_{j+1} - \Delta x_j = x_j - 2x_{j+1} + x_{j+2}, \\ S_j^2 &= S_1 + \ldots + S_j = jv_1p_1 + (j-1)v_2p_2 + \ldots + 2v_{j-1}p_{j-1} + v_jp_j, \\ \overline{\overline{S}}{}_{j+1}^2 &= \overline{S}_{j+1} + \ldots + \overline{S}_n = v_{j+1}p_{j+1} + 2v_{j+2}p_{j+2} + \ldots + (n-j)v_np_n. \end{aligned}$$

Applying (21), we get the conclusion that

$$\begin{aligned} &\left|\sum_{i=1}^n x_i y_i p_i - \sum_{i=1}^n x_i p_i \sum_{i=1}^n y_i p_i\right| \\ &\quad \leqslant \sum_{i=1}^n \left(\sum_{j=1}^{i-1} |S_j^2||\Delta^2 w_j^{(i)}| + |\Delta w_i^{(i)}||\overline{\overline{S}}{}_{i+1}^2 - S_{i-1}^2| + \sum_{j=i}^{n-2} |\overline{\overline{S}}{}_{j+2}^2||\Delta^2 w_j^{(i)}|\right)|y_i|p_i. \end{aligned}$$

In particular, if $v = (1,\dots,1)^T$ then

$$|\sum_{i=1}^{n} x_i y_i p_i - \sum_{i=1}^{n} x_i p_i \sum_{i=1}^{n} y_i p_i| \leqslant \sum_{i=1}^{n} \left(\sum_{j=1}^{i-1} P_j^2 |\Delta^2 x_j| + |\Delta x_i| |\overline{\overline{P}}_{i+1}^2 - P_{i-1}^2| + \sum_{j=i}^{n-2} \overline{\overline{P}}_{j+2}^2 |\Delta^2 x_j| \right) |y_i| p_i,$$

where

$$P_j^2 = P_1 + \ldots + P_j = jp_1 + (j-1)p_2 + \ldots + 2p_{j-1} + p_j,$$
$$\overline{\overline{P}}_{j+1}^2 = \overline{P}_{j+1} + \ldots + \overline{P}_n = p_{j+1} + 2p_{j+2} + \ldots + (n-j)p_n.$$

REFERENCES

[1] M. L. EATON, *On group induced orderings, monotone functions, and convolution theorems*, in Inequalities in Statistics and Probability (Tong Y.L., Ed.) IMS Lectures Notes Monogr. Ser., **5**, (1984), 13–25.
[2] M. L. EATON, *Lectures on Topics in Probability Inequalities*, CWI Tract 35, Centrum voor Wiskunde en Informatica, Amsterdam, 1987.
[3] M. L. EATON, *Group induced orderings with some applications in statistics*, CWI Newsletter, **16**, (1987), 3–31.
[4] S. IZUMINO, J. E. PEČARIĆ AND B. TEPEŠ, *A Grüss-type inequality and its applications*, J. Inequal. Appl., **3**, (2005), 277–288.
[5] A. S. LEWIS, *Convex analysis on Hermitian matrices*, SIAM J. Optim., **6**, (1996), 164–177.
[6] A. S. LEWIS, *Group invariance and convex matrix analysis*, SIAM J. Matrix Anal. Appl., **17**, (1996), 927–949.
[7] A. S. LEWIS, *Convex analysis on Cartan subspaces*, Nonlinear Anal., **42**, (2000), 813–820.
[8] A. W. MARSHALL AND I. OLKIN, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.
[9] H. F. MIRANDA AND R. C. THOMPSON, *Group majorization, the convex hull of sets of matrices, and the diagonal elements – singular values inequalities*, Linear Algebra Appl., **199**, (1994), 131–141.
[10] L. NACHBIN, *The Haar Integral*, Van Nostrand, Princeton, 1965.
[11] M. NIEZGODA, *Group majorization and Schur type inequalities*, Linear Algebra Appl., **268**, (1998), 9–30.
[12] M. NIEZGODA, *An analytical characterization of both effective and irreducible groups inducing cone orderings*, Linear Algebra Appl., **269**, (1998), 105–114.
[13] M. NIEZGODA, *Bifractional inequalities and convex cones*, Discrete Math., **306**, (2) (2006), 231–243.

(Received March 3, 2006)

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