Mathematical
Inequalities
& Applications
Volume 10, Number 3 (2007), 597–605

SOME STABILITY RESULTS FOR SET INTEGRO–DIFFERENTIAL EQUATIONS

BASHIR AHMAD AND S. SIVASUNDARAM

(*communicated by V. Lakshmikantham*)

Abstract. We study the stability criteria for set integro-differential equations in terms of Lyapunov-like functions. Sufficient conditions for the stability of the null solution of set integro-differential equations are presented.

1. Introduction

The study of set differential equations has been initiated as an independent subject and several results of interest can be found in [2-3, 6-9]. The interesting feature of the set differential equations is that the results obtained in this new framework become the corresponding results of ordinary differential equations as the Hukuhara derivative and the integral used in formulating the set differential equations reduce to the ordinary vector derivative and integral when the set under consideration is a single valued mapping. Moreover, in the present setup, we have only semilinear complete metric space to work with, instead of complete normed linear space required in the study of the ordinary differential systems. Furthermore, set differential equations, that are generated by multivalued differential inclusions when the multivalued functions involved do not possess convex values, can be used as a tool for studying multivalued differential inclusions [11]. Set differential equations can also be utilized to investigate fuzzy differential equations [7].

In this paper, we discuss stability criteria for set integro-differential equations. In section 3, we present some basic results for the set integro-differential equations while the stability criteria is developed in section 4. It has been found that the formulation of the set integro-differential equations has an intrinsic disadvantage that the diameter of the solution is nondecreasing as time increases and consequently the behavior of the solutions, in some cases, does not match with the solutions of ordinary integro-differential equations from which the set integro-differential equations are generated. A criterion is also devised to overcome this inconsistency problem.

Mathematics subject classification (2000): 34K20, 34D20, 45J05.
Key words and phrases: set integro-differential equations, stability.

Paper MIA-10-56

2. Terminology and preliminaries

Let $K(R^n)$ denote the collection of nonempty, compact and convex subsets of R^n. We define the Hausdorff metric as

$$D[X,Y] = \max[\sup_{y \in Y} d(y,X), \sup_{x \in X} d(x,Y)], \tag{1}$$

where $d(y,X) = \inf[d(y,x) : x \in X]$ and X,Y are bounded subsets of R^n. Notice that $K(R^n)$ with the metric is a complete metric space. Moreover, $K(R^n)$ equipped with the natural algebraic operations of addition and nonnegative scalar multiplication becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space [1, 11]. The Hausdorff metric (1) satisfies the following properties: $\forall\, X,Y,Z \in K(R^n)$ and $\mu \in R_+$, we have

$$D[X+Z, Y+Z] = D[X,Y] \quad \text{and} \quad D[X,Y] = D[Y,X], \tag{2}$$

$$D[\mu X, \mu Y] = \mu D[X,Y], \tag{3}$$

$$D[X,Y] \leqslant D[X,Z] + D[Z,Y]. \tag{4}$$

DEFINITION 1. The set $Z \in K(R^n)$ satisfying $X = Y+Z$ is known as the Hukuhara difference of the sets X and Y in $K(R^n)$ and is denoted as $X - Y$.

DEFINITION 2. For any interval $I \in R$, the mapping $F : I \to K(R^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$ if there exists an element $D_H F(t_0) \in K(R^n)$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t_0+h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t_0) - F(t_0-h)}{h}, \tag{5}$$

exist in the topology of $K(R^n)$ and each one is equal to $D_H F(t_0)$.

If $F : I \to K(R^n)$ is Hukuhara differentiable, then the real valued function $t \to \text{diam}[F(t)]$, $t \in I$ is nondecreasing on I. Moreover, Hukuhara differentiability of F on I and $\text{diam}[F(t)] > 0, t \in I$ does not necessarily imply that $F(t)$ is monotone relative to the set inclusion [9].

By embedding $K(R^n)$ as a complete cone in a corresponding Banach space and taking into account the result on differentiation of Bochner integral, it is found that if

$$F(t) = X_0 + \int_0^t \phi(\eta)d\eta, \;\; X_0 \in K(R^n), \tag{6}$$

where $\phi : I \to K(R^n)$ is integrable in the sense of Bochner, then $D_H F(t)$ exists and

$$D_H F(t) = \phi(t) \;\text{ a.e.} \quad \text{on } I. \tag{7}$$

Also, for any compact set $I \subset R_+$, the Hukuhara integral is defined by

$$\int_I F(\eta)d\eta = [\int_I f(\eta)d\eta : \text{f is a continuous selector of F}].$$

Consider the set integro-differential equation

$$D_H U(t) = F(t, U(t)) + \int_{t_0}^{t} K(t, \eta, U(\eta)) d\eta, \quad U(t_0) = U_0 \in K(R^n), \quad t_0 > 0, \tag{8}$$

where $F \in C[R_+ \times K(R^n), K(R^n)], K \in C[R_+ \times R_+ \times K(R^n), K(R^n)]$.

The mapping $U \in C^1[J, K(R^n)]$, $J = [t_0, t_0 + T]$, $T > 0$, is said to be a solution of (8) on J if it satisfies (8) on J. Since $U(t)$ is continuously differentiable, we have

$$U(t) = U_0 + \int_{t_0}^{t} D_H U(\eta) d\eta, \ t \in J, \tag{9}$$

which can be put in the form [10]

$$U(t) = U_0 + \int_{t_0}^{t} [F(\eta, U(\eta)) + \int_{\eta}^{t} K(\sigma, \eta, U(\eta)) d\sigma] d\eta, \ t \in J, \tag{10}$$

where the integral is in the sense of Hukuhara integral [4-5]. Thus, we can say that $U(t)$ is a solution of (8) if and only if it satisfies (10) on J.

3. Some basic results in set integro-differential equations

In order to establish the stability criteria for set integro-differential equations, we present some basic results relative to such equations.

THEOREM 1. (Comparison result) *Assume that* $F \in C[R_+ \times K(R^n), K(R^n)], K \in C[R_+ \times R_+ \times K(R^n), K(R^n)]$ *and for* $t \in R_+, X, Y \in K(R^n)$,

$$D[F(t,X) + \int_{t_0}^{t} K(t, \eta, X) d\eta, F(t,Y) + \int_{t_0}^{t} K(t, \eta, Y) d\eta] \leqslant g(t, D[X,Y]) + \int_{t_0}^{t} G(t, \eta, D[X,Y]) d\eta,$$

where $g \in C[R_+ \times R_+, R_+]$ *and* $G \in C[R_+ \times R_+ \times R_+, R_+]$. *Moreover, we require that there exists the maximal solution* $r(t, t_0, w_0)$ *of the scalar integro-differential equation*

$$w'(t) = g(t, w(t)) + \int_{t_0}^{t} G(t, \eta, w(\eta)) d\eta, \quad w(t_0) = w_0 \geqslant 0, \quad t \geqslant t_0.$$

Then, if $U(t) = U(t, t_0, U_0)$, $V(t) = V(t, t_0, V_0)$ *are any two solutions of* (8) *such that* $U_0, \ V_0 \in K(R^n)$ *exist for* $t \geqslant t_0$ *and* $U(t_0) = U_0, \ V(t_0) = V_0$, *we have*

$$D[U(t), V(t)] \leqslant r(t, t_0, w_0), \quad t \geqslant t_0,$$

provided that $D[U_0, V_0] \leqslant w_0$.

Proof. Since $U(t)$, $V(t)$ are solutions of (8), the differences $U(t+h)-U(t)$, $V(t+h)-V(t)$ exist for small $h>0$. For $t\in R_+$, we set $m(t)=D(U(t),V(t))$. Using the properties (2)-(4) of Hausdorff metric, we have

$$\begin{aligned}
m(t+h)-m(t) &= D[U(t+h),V(t+h)]-D[U(t),V(t)]\\
&\leqslant D[U(t+h),U(t)+h\{F(t,U(t))+\int_{t_0}^{t}K(t,\eta,U(\eta))d\eta\}]+D[U(t)\\
&\quad +h\{F(t,U(t))+\int_{t_0}^{t}K(t,\eta,U(\eta))d\eta\},V(t)+h\{F(t,V(t))+\int_{t_0}^{t}K(t,\eta,V(\eta))d\eta\}]\\
&\quad +D[V(t)+h\{F(t,V(t))+\int_{t_0}^{t}K(t,\eta,V(\eta))d\eta\},V(t+h)]-D[U(t),V(t)]\\
&\leqslant D[U(t+h),U(t)+h\{F(t,U(t))+\int_{t_0}^{t}K(t,\eta,U(\eta))d\eta\}]\\
&\quad +D[V(t)+h\{F(t,V(t))+\int_{t_0}^{t}K(t,\eta,V(\eta))d\eta\},V(t+h)]\\
&\quad +hD[F(t,U(t))+\int_{t_0}^{t}K(t,\eta,U(\eta))d\eta,F(t,V(t))+\int_{t_0}^{t}K(t,\eta,V(\eta))d\eta],
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{m(t+h)-m(t)}{h} &\leqslant D[\frac{U(t+h)-U(t)}{h},F(t,U(t))+\int_{t_0}^{t}K(t,\eta,U(\eta))d\eta]\\
&\quad +D[F(t,V(t))+\int_{t_0}^{t}K(t,\eta,V(\eta))d\eta,\frac{V(t+h)-V(t)}{h}]\\
&\quad +D[F(t,U(t))+\int_{t_0}^{t}K(t,\eta,U(\eta))d\eta,F(t,V(t))+\int_{t_0}^{t}K(t,\eta,V(\eta))d\eta],
\end{aligned}$$

Taking $\limsup$ as $h\to 0^+$ yields

$$\begin{aligned}
D^+m(t) &= \limsup_{h\to 0^+}\frac{1}{h}[m(t+h)-m(t)]\\
&\leqslant D[F(t,U(t))+\int_{t_0}^{t}K(t,\eta,U(\eta))d\eta,F(t,V(t))+\int_{t_0}^{t}K(t,\eta,V(\eta))d\eta]\\
&\leqslant g(t,D[U,V])+\int_{t_0}^{t}G(t,\eta,D[U,V])d\eta\\
&\leqslant g(t,D[U_0,V_0])+\int_{t_0}^{t}G(t,\eta,D[U_0,V_0])d\eta.
\end{aligned}$$

Which together with the fact that $D[U_0,V_0]\leqslant w_0$ and by the comparison theorem for ordinary integro-differential equations [10] gives

$$D[U(t),V(t)]\leqslant r(t,t_0,w_0),\quad t\geqslant t_0.$$

This completes the proof of the theorem.

The following existence and uniqueness theorem is more general than Lipschitz condition whose proof is based on the comparison result.

THEOREM 2. *Assume that*

(A_1) $F \in C[J \times B(U_0,b), K(R^n)]$, $K \in C[J \times J \times B(U_0,b), K(R^n)]$, *where* $B(U_0,b) = [U \in K(R^n) : D[U,U_0] \leqslant b]$ *and* $D[F(t,U),\theta] \leqslant M_0$ *on* $J \times B(U_0,b)$, $\int_\eta^t D[K(\sigma,\eta,U(\eta)),\theta]d\sigma \leqslant N_0$ *on* $J \times J \times B(U_0,b)$, *where* θ *is the zero element of* R^n *regarded as a one point set.*

(A_2) $D[F(t,U),F(t,V)] \leqslant g(t,D[U,V])$ *on* $J \times B(U_0,b)$, $D[K(t,\eta,U),K(t,\eta,V)] \leqslant G(t,\eta,D[U,V])$ *on* $J \times J \times B(U_0,b)$, *where* $g \in C[J \times [0,2b],R_+]$, $G \in C[J \times J \times [0,2b],R_+]$, $g(t,w) \leqslant M_1$ *on* $J \times [0,2b]$, $G(t,\eta,w) \leqslant N_1$ *on* $J \times J \times [0,2b]$, $g(t,0) = 0, G(t,\eta,0) = 0$, $g(t,w)$ *and* $G(t,\eta,w)$ *are nondecreasing in* w *for each* $t \in J$, $(t,\eta) \in J \times J$.

(A_3) $w(t) = 0$ *is the only solution of*

$$w'(t) = g(t,w(t)) + \int_{t_0}^t G(t,\eta,w(\eta))d\eta, \quad w(t_0) = w_0.$$

Then the successive approximations defined by

$$U_{n+1}(t) = U_0 + \int_{t_0}^t [F(\eta,U_n(\eta)) + \int_\eta^t K(\sigma,\eta,U_n(\eta))d\sigma]d\eta, \; n = 0,1,2,...,$$

exist on $J_0 = [t_0, t+\alpha]$, *where* $\alpha = \min(a, b/(M+N)]$, $M = \max(M_0,M_1)$, $N = \max(N_0,N_1)$, *as continuous functions and converge uniformly to the unique solution* $U(t,t_0,U_0)$ *of* (8) *on* J_0.

Now, we present a global existence result dealing with continuous dependence of solution of (8) with respect to initial value (t_0,U_0). As the reasoning and working of the proof of this theorem is similar to the one employed in proving comparison theorem, so we omit its proof.

THEOREM 3. *Assume that*

(B_1) $F \in C[R_+ \times K(R^n), K(R^n)]$ *and for* $(t,X) \in R_+ \times K(R^n)$,

$$D[F(t,X),\theta] \leqslant q(t,D[X,\theta]),$$

where $q \in C[R_+ \times R_+, R_+]$ *and* $q(t,w)$ *is nondecreasing in* w *for each* $t \in R_+$, *and* θ *is the zero element of* $K(R^n)$ *regarded as a one point set.*

(B_2) $K \in C[R_+ \times R_+ \times K(R^n), K(R^n)]$ *and let*

$$D[K(t,\eta,X),\theta] \leqslant Q(t,\eta,D[X,\theta]),$$

where $Q(t,\eta,w)$ *is nondecreasing in* w *for each* $(t,\eta) \in R_+ \times R_+$.

(B_3) *The maximal solution* $r(t,t_0,w_0)$ *of*

$$w'(t) = q(t,w(t)) + \int_{t_0}^t Q(t,\eta,w(\eta))d\eta, \quad w(t_0) = w_0,$$

exists for $t \geqslant t_0$ *and for every* $w_0 \geqslant 0$.

(B_4) *There exists a local solution* $U(t) = U(t, t_0, U_0)$ *of* (8) *for every* $(t_0, U_0) \in R_+ \times K(R^n)$.

Then, for every $U_0 \in K(R^n)$ *such that* $D[U_0, \theta] \leqslant w_0$, *the initial value problem* (8) *possesses a solution* $U(t) = U(t, t_0, U_0)$ *defined for* $t \geqslant t_0$ *satisfying* $D[U(t), \theta] \leqslant r(t, t_0, w_0)$, $t \geqslant t_0$.

4. Stability criteria

The following comparison theorem provides a basis to investigate the stability criteria of set integro-differential equation (8) in term of Lyapunov-like functions.

THEOREM 4. *Assume that*

(C_1) $V \in C[R_+ \times K(R^n), K(R^n)]$ *and* $|V(t,X) - V(t,Y)| \leqslant LD[X,Y]$, *where* L *is the local Lipschitz constant,* $X, Y \in K(R^n)$, $t \in R_+$.

(C_2) $g \in C[R_+ \times R_+, R_+]$, $G \in C[R_+ \times R_+ \times R_+, R_+]$ *and for* $X \in K(R^n)$, $t \in R_+$,

$$D^+V(t,X) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, X + h\{F(t,X) + \int_{t_0}^{t} K(t,\eta,X)d\eta\}) - V(t,X))]$$
$$\leqslant g(t, V(t,X)) + \int_{t_0}^{t} G(t,\eta,V(\eta,X))d\eta.$$

Then, if $U(t) = U(t, t_0, U_0)$ *is any solution of* (8) *existing on* $[t_0, \infty)$ *such that* $V(t_0, U_0) \leqslant w_0$, *we have*

$$V(t, U(t)) \leqslant r(t, t_0, w_0), t \in [t_0, \infty),$$

where $r(t, t_0, w_0)$ *is the maximal solution of*

$$w'(t) = g(t, w(t)) + \int_{t_0}^{t} G(t, \eta, w(\eta))d\eta, \quad w(t_0) = w_0 \geqslant 0, \tag{11}$$

existing on $[t_0, \infty)$.

Proof. Define $m(t) = V(t, U(t))$ so that $m(t_0) = V(t_0, U_0) \leqslant w_0$. For small $h > 0$, we consider

$$m(t+h)-m(t) = V(t+h, U(t+h)) - V(t, U(t))$$
$$= V(t+h, U(t+h)) - V(t+h, U(t) + h\{F(t, U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta\})$$
$$+ V(t+h, U(t) + h\{F(t, U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta\}) - V(t, U(t))$$
$$\leqslant LD[U(t+h), U(t) + h\{F(t, U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta\}]$$
$$+ V(t+h, U(t) + h\{F(t, U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta\}) - V(t, U(t)),$$

where we have used the Lipschitz condition described in (C_1). Thus,

$$D^+m(t) = \limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)]$$
$$\leqslant D^+V(t,U(t)) + L\limsup_{h\to 0^+} \frac{1}{h}\left[D[U(t+h), U(t)\right.$$
$$\left. + h\{F(t,U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta\}]\right].$$

Let $U(t+h) = U(t) + Z(t)$, where $Z(t)$ is the Hukuhara difference of $U(t+h)$ and $U(t)$ for small $h > 0$ and is assumed to exist. Hence, employing the properties of $D[.,.]$, it follows that

$$D[U(t+h), U(t) + h\{F(t,U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta\}]$$
$$= D[U(t) + Z(t), U(t) + h\{F(t,U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta\}]$$
$$= D[Z(t), h\{F(t,U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta\}]$$
$$= D[U(t+h) - U(t), h\{F(t,U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta\}].$$

Consequently, we find that

$$\frac{1}{h}[D[U(t+h), U(t) + h\{F(t,U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta\}]$$
$$= D[\frac{U(t+h) - U(t)}{h}, F(t,U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta],$$

which, in view of the fact that $U(t)$ is a solution of (8), yields

$$\limsup_{h\to 0^+} \frac{1}{h}[D[U(t+h), U(t) + h\{F(t,U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta\}]$$
$$= \limsup_{h\to 0^+} D[\frac{U(t+h) - U(t)}{h}, F(t,U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta]$$
$$= D[U'_H(t), F(t,U(t)) + \int_{t_0}^{t} K(t,\eta,U(\eta))d\eta] = 0.$$

Hence, we have the scalar integro-differential inequality

$$D^+m(t) \leqslant g(t,m(t)) + \int_{t_0}^{t} G(t,\eta,m(\eta))d\eta, \quad m(t_0) \leqslant w_0,$$

which, by following the method of proof for Theorem 1.4.1 (page 13 [10]), provides the desired estimate

$$m(t) \leqslant r(t,t_0,w_0), \quad t \in [t_0,\infty).$$

This proves the assertion of the theorem.

REMARK The set integro-differential equation (SIDE) (8) reduces to an ordinary integro-differential equation (OIDE) when $U(t)$ is single valued and SIDE (8) can be generated from OIDE by setting $F(t,X) = \overline{co}f(t,X)$, $G(t,\eta,X) = \overline{co}g(t,\eta,X)$, $X \in K(R^n)$ where $\overline{co}$ denotes closed convex hull and $f : R_+ \times R^n \to R^n$, $g : R_+ \times R_+ \times R^n \to R^n$ arise from OIDE

$$u'(t) = f(t,u(t)) + \int_{t_0}^{t} g(t,\eta,u(\eta))d\eta, \quad u(t_0) = u_0 \in R^n.$$

Consequently, the solution of OIDE is imbedded in the solutions of SIDE.

For the stability criteria of the null solution of (8), one can employ the measure $D[U(t),\theta] = \|U(t)\| = \text{diam}[U(t)]$, $t \geqslant t_0$ so that $\text{diam}[U(t)]$ is nondecreasing in t. This measure needs to be introduced for the generation of SIDE from OIDE otherwise the undesired elements enter the solution and the measure $\|U(t)\|$ becomes unsuitable to develop the stability theory. It has been noticed that the cause of the problem in SIDE is due to the requirement of Hukuhara difference in its formulation. This problem can be overcome by utilizing the existence of Hukuhara difference in the initial conditions also, which in fact makes it possible to match the behavior of the solution of SIDE with the corresponding solutions of OIDE. In order to do so, we suppose that the Hukuhara difference exists for any given initial values $U_0, V_0 \in K(R^n)$ so that we set $U_0 - V_0 = W_0$ and consider the stability of the solution $U(t,t_0,U_0 - V_0) = U(t,t_0,W_0)$ of (8).

We are now in a position to formulate the stability criteria for the trivial solution of (8) as follows:

THEOREM 5. *Assume that the system* (8) *has the trivial solution, and* $(C_1),(C_2)$ *of Theorem* 4 *hold on* $R_+ \times \Omega(\rho)$ *instead of* $K(R^n)$, *where* $\Omega(\rho) = [U \in K(R^n) : \|U\| < \rho]$.

Further, suppose that $b(\|U\|) \leqslant V(t,U) \leqslant a(\|U\|)$ *on* $R_+ \times \Omega(\rho)$, *where* $a,b \in [[0,\rho],R_+]$ *are the usual* $\mathcal{K}$ *class functions.*

Then the stability properties of the trivial solution of (11) *imply the corresponding properties of the trivial solution of* (8) *subject to the condition* $U(t,t_0,U_0 - V_0) = U(t,t_0,W_0)$.

Proof. We only provide the outline of the proof. By Theorem 4 and using the standard method of proof of known results [10], the conclusion of the theorem can be established in a straightforward way.

REFERENCES

[1] A. J. BRANDAO LOPES PINTO, F. S. DE BLASI AND F. IERVOLINO, *Uniqueness and existence theorems for differential equations with compact convex valued solutions*, Boll. Unione. Mat. Italy, **3**, (1970), 47–54.

[2] T. GNANA BHASKAR, V. LAKSHMIKANTHAM, *Set differential equations and flow invariance*, Appl. Anal., **82**, (2003), 357–368.

[3] T. GNANA BHASKAR, V. LAKSHMIKANTHAM, *Lyapunov stability for set differential equations*, Dynam. Systems Appl., **13**, (2004), 1–10.

[4] M. HUKUHARA, *Sur l'aplication semicontinue dont la valeur est un compact convexe*, Funkcial. Ekvak., (1967), 43–68.
[5] M. HUKUHARA, *Integration des applications measurable dont la valeur est un compactconvex*, Funkcial. Ekvak., (1967), 205–229.
[6] V. LAKSHMIKANTHAM, S. LEELA AND A. S. VATSALA, *Setvalued hybrid differential equations and stability in terms of two measures*, J. Hybrid Systems, **2**, (2002), 169–187.
[7] V. LAKSHMIKANTHAM, S. LEELA AND A. S. VATSALA, *Interconnection between set and fuzzy differential equations*, Nonlinear Anal., **54**, (2003), 351–360.
[8] V. LAKSHMIKANTHAM, A. TOLSTONOGOV, *Existence and interrelation between set and fuzzy differential equations*, Nonlinear Anal., **55**, (2003), 255–268.
[9] V. LAKSHMIKANTHAM, S. LEELA AND A. S. VATSALA, *Stability theory for differential equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **11**, (2004), 181–189.
[10] V. LAKSHMIKANTHAM, M. R. M. RAO, *Theory of Integro-differential Equations*, Gordon & Breach, London, 1995.
[11] A. TOLSTONOGOV, *Differential Inclusions In A Banach Space*, Kluwer Academic Publishers, Dordrecht, 2000.

(Received October 14, 2005)

Bashir Ahmad
Department of Mathematics
Faculty of Science
King Abdulaziz University
P.O. Box 80203
Jeddah-21589
Saudi Arabia
e-mail: bashir_qau@yahoo.com

S. Sivasundaram
Department of Mathematics
Embry-Riddle Aeronautical University
Daytona Beach, FL 32114
USA
e-mail: seenithi@gmail.com