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ASYMPTOTIC BEHAVIOR OF NONOSCILLATORY SOLUTIONS OF SECOND ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

MUSTAFA HASANBULLI AND YURI V. ROGOVCHENKO

(*communicated by R. P. Agarwal*)

Abstract. We study asymptotic behavior of solutions of second order nonlinear neutral differential equations of the form

$$(x(t)+p(t)x(t-\tau))''+f\left(t,x(t),x(\rho(t)),x'(t),x'(\sigma(t))\right)=0.$$

First we prove that solutions can be indefinitely continued to the right. Then, using the celebrated Bihari integral inequality, we obtain conditions for all nonoscillatory solutions to behave at infinity like nontrivial linear functions. Our theorems complement and extend recent results reported in the literature.

1. Introduction

In this paper, we are concerned with asymptotic behavior of nonoscillatory solutions of the second order nonlinear neutral differential equation

$$(x(t)+p(t)x(t-\tau))''+f\left(t,x(t),x(\rho(t)),x'(t),x'(\sigma(t))\right)=0. \tag{1}$$

By a solution of Eq. (1) we mean a continuous function $x(t)$ which satisfies the differential equation on $[t_x,+\infty)$ for some $t_x \geqslant t_0$, that is, the function $x(t)$, defined on $[t_x,+\infty)$, such that $x(t)+p(t)x(t-\tau)$ is twice continuously differentiable and $x(t)$ satisfies (1) for $t \geqslant t_x$. A nontrivial solution of Eq. (1) is called oscillatory if it does not have the largest zero and nonoscillatory otherwise. In what follows, we assume that Eq. (1) possesses nontrivial nonoscillatory solutions.

This research is motivated by recent papers regarding existence of solutions to nonlinear second order differential equations

$$u''+f(t,u,u')=0, \qquad t \geqslant t_0 \geqslant 1 \tag{2}$$

and

$$u''+f(t,u)=0, \qquad t \geqslant t_0 \geqslant 1 \tag{3}$$

which behave at infinity like solutions of the simplest second order differential equation, $u''=0$, see, for instance, Constantin [4], Constantin and Villari [5], Lipovan

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[13], Mustafa and Rogovchenko [15, 16, 17, 18], Philos et al. [21], Rogovchenko and Rogovchenko [22, 23], Rogovchenko [24], Rogovchenko and Villari [25], Seifert [26], Serrin and Zou [27], Yin [28], Zhao [29] and the references cited therein. Very recently, interesting applications of the analysis of nonoscillatory solutions to ordinary differential equations for some important classes of equations in mathematical physics have been discussed in Agarwal et al. [1], Agarwal and Mustafa [2], Hesaaraki and Moradifam [10].

Two types of behavior of asymptotically linear solutions of Eqs. (2) and (3) have been studied more extensively. Namely, Constantin [4], Rogovchenko and Rogovchenko [23], Yin [28], and Zhao [29] explored conditions which guarantee asymptotic representation

$$u(t) = At + o(t) \qquad \text{as } t \to +\infty, \tag{4}$$

whereas Lipovan [13], Mustafa [14], and Mustafa and Rogovchenko [15] derived conditions for a more precise asymptotic development

$$u(t) = At + B + o(1) \qquad \text{as } t \to +\infty,$$

where $A, B \in \mathbb{R}$ and $A \neq 0$. Our goal is to establish sufficient conditions for all nonoscillatory solutions of nonlinear neutral differential equation (1) to have asymptotic representation (4) as $t \to +\infty$.

Many authors were concerned with the oscillatory and asymptotic properties of solutions of different classes of neutral differential equations. In particular, Kulcsár [11] obtained sufficient conditions for the convergence to zero of nonoscillatory solutions of the second order linear neutral differential equation

$$(x(t) - px(t-\tau))'' + q(t)x(\sigma(t)) = 0.$$

Graef and Spikes [8] derived two sets of sufficient conditions which guarantee that any bounded nonoscillatory solution of a forced nonlinear neutral differential equation

$$[y(t) + P(t)y(g(t))]'' - Q(t)f(y(t-\sigma)) = R(t) \tag{5}$$

tends to zero as $t \to +\infty$, while Grammatikopoulos et al. [9] established similar conditions for nonoscillatory solutions of Eq. (5) in the case $R(t) \equiv 0$. Further studies in this direction have been undertaken by Graef et al. [7] who derived sufficient conditions for solutions of neutral differential equation (5) with $R(t) \equiv 0$ to have one of the following properties: (a) the nonoscillatory solutions are bounded or tend to zero; (b) the bounded solutions are either oscillatory or tend to zero; (c) the unbounded solutions are either oscillatory or tend to infinity.

For higher order equations, Y. Naito [20] obtained a necessary and sufficient condition for the neutral differential equation

$$\frac{d^n}{dt^n}\left[x(t) - h(t)x(\tau(t))\right] + f(t, x(g(t))) = 0$$

to have a positive solution satisfying

$$\lim_{t\to+\infty} \frac{x(t) - h(t)x(\tau(t))}{t^k} = c > 0,$$

whereas M. Naito [19] proved that n-th order nonlinear neutral differential equation

$$\frac{d^n}{dt^n}\left[x(t)+\lambda x(t-\tau)\right]+\sigma F(t,x(g(t)))=0$$

has a solution satisfying

$$\lim_{t\to+\infty}\frac{x(t)}{t^k}=c>0$$

if and only if

$$\int_{t_0}^{+\infty} t^{n-k-1}F(t,c\,[g(t)]^k)dt<+\infty \qquad \text{for some } c>0.$$

Recently, Džurina [6] extended results of the second author [24] on asymptotic integration of Eq. (2) to second order nonlinear neutral differential equation

$$\left(x(t)+px(t-\tau)\right)''+f\left(t,x(t)\right)=0$$

establishing conditions under which all nonoscillatory solutions behave like linear functions $at+b$ as $t\to+\infty$ for some $a,b\in\mathbb{R}$ and stated without proof similar theorem for

$$\left(x(t)+px(t-\tau)\right)''+f\left(t,x(t),x'(t)\right)=0.$$

We shall show that, in a similar manner, one can extend more general results on asymptotic behavior of solutions of the nonlinear differential equation (2) due to S. Rogovchenko and the second author [22, 23] to nonlinear neutral differential equation (1).

2. Auxiliary results

Let $\mathbb{R}^+=[0,+\infty)$. In the sequel, we suppose that the function $f\left(t,u_1,u_2,v_1,v_2\right)$ satisfies the following conditions:

$(A1)$ $f\left(t,u_1,u_2,v_1,v_2\right)$ is continuous in

$$D=\left\{(t,u_1,u_2,v_1,v_2): t\geqslant t_0\geqslant 1,\ u_1,u_2,v_1,v_2\in\mathbb{R}\right\};$$

$(A2)$ there exist functions $h_1,\ \ldots,\ h_5,\ g_1,\ldots,\ g_4\in C\left[\mathbb{R}^+,\mathbb{R}^+\right]$ such that either

$$\left|f\left(t,u_1,u_2,v_1,v_2\right)\right|\leqslant h_1(t)\,g_1\left(\frac{|u_1|}{t}\right)+h_2(t)\,g_2\left(\frac{|u_2|}{\rho(t)}\right)+h_3(t), \tag{6}$$

or

$$\left|f\left(t,u_1,u_2,v_1,v_2\right)\right|\leqslant h_4(t)\,g_3\left(\frac{|u_1|}{t}\right)g_4\left(\frac{|u_2|}{\rho(t)}\right)+h_5(t), \tag{7}$$

where, for $s>0$, the functions $g_i(s)$, $i=1,\ldots,4$, are positive, non-decreasing, and

$$\int_{t_0}^{+\infty} h_i(s)\,ds=H_i<+\infty, \qquad i=1,\ldots,5.$$

$(A3)$ $\rho,\sigma\in C\left[\mathbb{R}^+,\mathbb{R}^+\right]$, $\rho(t)\leqslant t$, $\sigma(t)\leqslant t$, $\lim_{t\to+\infty}\rho(t)=+\infty$, and $\lim_{t\to+\infty}\sigma(t)=+\infty$;

$(A4)$ $p \in C[\mathbb{R}^+, \mathbb{R}^+]$, $0 \leqslant p(t) \leqslant p_* < 1$, and $\lim_{t\to+\infty} p(t) = p_0$.

For $t \geqslant t_0$, introduce the functions G_1 and G_2 by

$$G_1(t) \stackrel{\text{def}}{=} \int_{t_0}^{t} \frac{ds}{g_1(s) + g_2(s)}, \qquad G_2(t) \stackrel{\text{def}}{=} \int_{t_0}^{t} \frac{ds}{g_3(s)\, g_4(s)}.$$

We need the following result which, although independent of Eq. (1), helps to study behavior of nonoscillatory solutions of this equation, cf. Džurina [6, Lemma 1], Györi and Ladas [12, Lemma 6.1.1].

LEMMA 1. *Let* $x(t) > 0$ *(or* $x(t) < 0$*) eventually,* $\tau > 0$, *and let* $p(t)$ *satisfy* $(A4)$. *Define*

$$w(t) = x(t) + p(t)\frac{t-\tau}{t} x(t-\tau). \tag{8}$$

If there exists a finite limit $\lim_{t\to+\infty} w(t) = c$, *then*

$$\lim_{t\to+\infty} x(t) = \frac{c}{1+p_0}. \tag{9}$$

Proof. Suppose that $x(t) > 0$. It is clear from (8) that $c \geqslant 0$ and (9) yields

$$\liminf_{t\to+\infty} x(t) \leqslant \frac{c}{1+p_0} \leqslant \limsup_{t\to+\infty} x(t).$$

Assume that there exist $\alpha_1, \alpha_2 \geqslant 0$ and sequences μ_n, ν_n diverging to $+\infty$ such that

$$\limsup_{t\to+\infty} x(t) = \lim_{t\to+\infty} x(\mu_n) = \frac{c+\alpha_1}{1+p_0},$$
$$\liminf_{t\to+\infty} x(t) = \lim_{t\to+\infty} x(\nu_n) = \frac{c-\alpha_2}{1+p_0}.$$

We have to prove that $\alpha_1 = \alpha_2 = 0$. Consider the following two cases.

Case 1. Assume that $\alpha_1 > 0$ and $\alpha_1 \geqslant \alpha_2 \geqslant 0$. It follows from (8) that, for any $\varepsilon > 0$,

$$w(t) \geqslant x(t) + p(t)\frac{t-\tau}{t}\frac{c-\alpha_2-\varepsilon}{1+p_0}. \tag{10}$$

Letting in (10) $t = \mu_n$ and passing to the limit as $n \to +\infty$, we obtain

$$c \geqslant \frac{c+\alpha_1}{1+p_0} + p_0\frac{c-\alpha_2-\varepsilon}{1+p_0},$$

or, equivalently,

$$\alpha_1 \leqslant p_0(\alpha_2 + \varepsilon). \tag{11}$$

Chose now $\varepsilon = (2p_0)^{-1}(1-p_0)\alpha_2$. Since $p_0 < 1$, (11) yields

$$\alpha_1 \leqslant \frac{1}{2}\alpha_2(p_0+1) < \alpha_2,$$

which contradicts our initial assumption that $\alpha_1 \geqslant \alpha_2$.

Case 2. Assume now that $\alpha_2 > 0$ and $\alpha_2 \geqslant \alpha_1 \geqslant 0$. Similarly to Case 1, (8) implies that, for any $\varepsilon > 0$,

$$w(t) \leqslant x(t) + p(t) \frac{t-\tau}{t} \frac{c + \alpha_1 + \varepsilon}{1 + p_0}. \tag{12}$$

Let in (12) $t = v_n$ and pass to the limit as $n \to +\infty$ to obtain

$$c \leqslant \frac{c - \alpha_2}{1 + p_0} + p_0 \frac{c + \alpha_1 + \varepsilon}{1 + p_0},$$

which is equivalent to

$$\alpha_2 \leqslant p_0 (\alpha_1 + \varepsilon). \tag{13}$$

Choose $\varepsilon = (2p_0)^{-1} (1 - p_0) \alpha_1$. Using (13) and the fact that $p_0 < 1$, we conclude that

$$\alpha_2 \leqslant \frac{1}{2} \alpha_1 (p_0 + 1) < \alpha_1,$$

which contradicts our assumption that $\alpha_2 \geqslant \alpha_1$.
The proof is complete. □

REMARK 2. In the case $p(t) = p$, Lemma 1 reduces to Džurina's result [6, Lemma 1].

Let $z(t_0) = c_1$ and $z'(t_0) = c_2$. In what follows, we shall use the notation

$$c_* \overset{\text{def}}{=} |c_1| + |c_2|.$$

Further, define $z(t)$ by

$$z(t) = x(t) + p(t) x(t - \tau). \tag{14}$$

The next result provides useful estimate for solutions of Eq. (1).

LEMMA 3. (*i*) *Assume that* $f(t, u_1, u_2, v_1, v_2)$ *satisfies* (6). *Then, for all* $t \geqslant t_0$, *one has*

$$\max \left[\frac{|z(t)|}{t}, \frac{|z(\rho(t))|}{\rho(t)} \right] \leqslant \Phi_1(t), \tag{15}$$

where

$$\Phi_1(t) \overset{def}{=} c_* + \int_{t_0}^{t} h_1(s) g_1 \left(\frac{|z(s)|}{s} \right) ds + \int_{t_0}^{t} h_2(s) g_2 \left(\frac{|z(\rho(s))|}{\rho(s)} \right) ds + \int_{t_0}^{t} h_3(s) ds.$$

(*ii*) *Assume that* $f(t, u_1, u_2, v_1, v_2)$ *satisfies* (7). *Then, for all* $t \geqslant t_0$, *one has*

$$\max \left[\frac{|z(t)|}{t}, \frac{|z(\rho(t))|}{\rho(t)} \right] \leqslant \Phi_2(t), \tag{16}$$

where

$$\Phi_2(t) \overset{def}{=} c_* + \int_{t_0}^{t} h_4(s) g_3 \left(\frac{|z(s)|}{s} \right) g_4 \left(\frac{|z(\rho(s))|}{\rho(s)} \right) ds + \int_{t_0}^{t} h_5(s) ds.$$

Proof. Part (i). Let $x(t)$ be a nonoscillatory solution of Eq. (1). Clearly,

$$|z(t)| \geqslant |x(t)|, \tag{17}$$

and it follows from Eq. (1) that

$$z''(t) = -f\left(t, x(t), x(\rho(t)), x'(t), x'(\sigma(t))\right), \tag{18}$$

where $z(t)$ is defined in (14). Integrating (18) twice from t_0 to t, we obtain

$$z'(t) = c_2 - \int_{t_0}^{t} f\left(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s))\right) ds, \tag{19}$$

$$z(t) = c_2(t-t_0) + c_1 - \int_{t_0}^{t} (t-s) f\left(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s))\right) ds, \tag{20}$$

and it follows from (20) that, for $t \geqslant t_0$,

$$|z(t)| \leqslant t\left(c_* + \int_{t_0}^{t} \left|f\left(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s))\right)\right| ds\right).$$

Using (6), (17), and monotonicity of the functions g_1 and g_2, we have

$$\begin{aligned}
&\left|f\left(t, x(t), x(\rho(t)), x'(t), x'(\sigma(t))\right)\right| \\
&\quad \leqslant h_1(t)\, g_1\left(\frac{|x(t)|}{t}\right) + h_2(t)\, g_2\left(\frac{|x(\rho(t))|}{\rho(t)}\right) + h_3(t) \\
&\quad \leqslant h_1(t)\, g_1\left(\frac{|z(t)|}{t}\right) + h_2(t)\, g_2\left(\frac{|z(\rho(t))|}{\rho(t)}\right) + h_3(t).
\end{aligned}$$

Hence, for all $t \geqslant t_0$,

$$\begin{aligned}
\frac{|z(t)|}{t} \leqslant c_* &+ \int_{t_0}^{t} h_1(s)\, g_1\left(\frac{|z(s)|}{s}\right) ds \\
&+ \int_{t_0}^{t} h_2(s)\, g_2\left(\frac{|z(\rho(s))|}{\rho(s)}\right) ds + \int_{t_0}^{t} h_3(s)\, ds = \Phi_1(t).
\end{aligned} \tag{21}$$

Clearly, $\Phi_1(t)$ is increasing because, for all $t \geqslant t_0$, $\Phi_1'(t) > 0$. Then, by the assumption $(A3)$, one has

$$|z(\rho(t))| \leqslant \rho(t)\Phi_1(\rho(t)) \leqslant \rho(t)\Phi_1(t),$$

or

$$\frac{|z(\rho(t))|}{\rho(t)} \leqslant \Phi_1(t). \tag{22}$$

Now (15) follows from (21) and (22).

Part (ii). Assume that f satisfies (7). Following the same lines as above, we conclude that, for $t \geqslant t_0$,

$$\frac{|z(t)|}{t} \leqslant c_* + \int_{t_0}^{t} h_4(s)\, g_3\left(\frac{|z(s)|}{s}\right) g_4\left(\frac{|z(\rho(s))|}{\rho(s)}\right) ds + \int_{t_0}^{t} h_5(s)\, ds, \tag{23}$$

which, combined with

$$|z(\rho(t))| \leqslant \rho(t)\Phi_2(\rho(t)) \leqslant \rho(t)\Phi_2(t), \tag{24}$$

immediately yields (16). □

The following lemma establishes existence of solutions of Eq. (1) for all $t \geqslant t_0 \geqslant 1$ and resembles the result proved by Mustafa and the second author for Eq. (2) in the case where f satisfies the growth condition

$$|f(t,u,v)| \leqslant h_1(t)\, g_1\left(\frac{|u|}{t}\right) + h_2(t)\, g_2(|v|) + h_3(t),$$

cf. [17, Lemma 3.6, pp. 318-319].

LEMMA 4. *Suppose that there exists a solution $x(t)$ of Eq. (1) defined on $[1,T)$, $1 < T < +\infty$, which cannot be continued to the right of T.*
(i) If $f(t,u_1,u_2,v_1,v_2)$ satisfies (6), then $G_1(+\infty) < +\infty$.
(ii) If $f(t,u_1,u_2,v_1,v_2)$ satisfies (7), then $G_2(+\infty) < +\infty$.

Proof. Part (i). Let $x(t)$ be a solution of Eq. (1) which is defined on $[1,T)$, $1 < T < +\infty$, and cannot be continued to the right of T, and let $z(t)$ be defined by (14). Using estimates (21) and (22), we conclude that, for $t \in [1,T)$,

$$\max\left[\frac{|z(t)|}{T}, \frac{|z(\rho(t))|}{\rho(T)}\right] \leqslant \max\left[\frac{|z(t)|}{t}, \frac{|z(\rho(t))|}{\rho(t)}\right] \leqslant \gamma(t), \tag{25}$$

where $\gamma(t)$ is the maximal solution of the initial value problem

$$\begin{cases} \xi' = (h_1(t) + h_2(t) + h_3(t))\,(g_1(\xi) + g_2(\xi) + 1), \\ \xi(1) = \xi_0 \overset{\text{def}}{=} c_*. \end{cases} \tag{26}$$

Since solution $x(t)$ of Eq. (1) cannot be continued to the right,

$$\lim_{t\to T-} |x(t)| = +\infty,$$

which, in virtue of (17) and (25), implies $\gamma(t) \to +\infty$ as $t \to T-$. Integration of (26) yields, for $t \in [1,T)$,

$$\int_{\xi_0}^{\gamma(t)} \frac{ds}{g_1(s) + g_2(s) + 1} = \int_1^t (h_1(s) + h_2(s) + h_3(s))\, ds. \tag{27}$$

Passing in (27) to the limit as $t \to T-$, we deduce that

$$\int_{\xi_0}^{+\infty} \frac{ds}{g_1(s) + g_2(s) + 1} = \int_1^T (h_1(s) + h_2(s) + h_3(s))\, ds < +\infty. \tag{28}$$

If $G_1(+\infty) = +\infty$, then, according to [15, Lemma 7, p. 346], one has

$$\int_{\xi_0}^{+\infty} \frac{ds}{g_1(s) + g_2(s) + 1} = +\infty,$$

which contradicts (28). Thus, Part (i) is proved.

Part (ii). Let $x(t)$ and $z(t)$ be as in Part (i). Using estimates (23) and (24), we conclude that, for $t \in [1,T)$, inequality (25) holds, where this time $\gamma(t)$ is the maximal solution of the initial value problem

$$\begin{cases} \xi' = (h_4(t)+h_5(t))\,(g_3(\xi)g_4(\xi)+1)\,, \\ \xi(1)=\xi_0, \end{cases} \tag{29}$$

and ξ_0 is defined in (26). Integrating ordinary differential equation in (29) and taking into account that $\gamma(t) \to +\infty$ as $t \to T-$, we obtain, for $t \in [1,T)$,

$$\int_{\xi_0}^{\gamma(t)} \frac{ds}{g_3(s)g_4(s)+1} = \int_1^t (h_4(s)+h_5(s))\,ds. \tag{30}$$

Passing in (30) to the limit as $t \to T-$, we conclude that

$$\int_{\xi_0}^{+\infty} \frac{ds}{g_3(s)g_4(s)+1} = \int_1^T (h_4(s)+h_5(s))\,ds < +\infty. \tag{31}$$

Another application of [15, Lemma 7, p. 346] yields

$$\int_{\xi_0}^{+\infty} \frac{ds}{g_3(s)g_4(s)+1} = +\infty$$

provided that $G_2(+\infty)=+\infty$, which, in virtue of (31), leads to the contradiction.

This completes the proof of lemma. □

As an immediate consequence of Lemma 4, we obtain the following important continuation result.

COROLLARY 5. *Assume that the nonlinearity f satisfies* (6) (*respectively*, (7)) *and* $G_1(+\infty)=+\infty$ (*respectively*, $G_2(+\infty)=+\infty$). *Then all solutions of Eq.* (1) *can be indefinitely continued to the right.*

3. Existence of asymptotically linear solutions

THEOREM 6. *Suppose that* (6) *holds and* $G_1(+\infty)=+\infty$. *Then any nonoscillatory solution of Eq.* (1) *has the asymptotic representation* (4), *and there exist solutions for which* $A \neq 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1) and $z(t)$ be defined by (14). Then, by virtue of Lemma 3, (15) holds. Since $g_1(s)$ and $g_2(s)$ are non-decreasing for $s>0$, one has

$$g_1\left(\frac{|z(t)|}{t}\right) \leqslant g_1(\Phi_1(t)) \qquad \text{and} \qquad g_2\left(\frac{|z(\rho(t))|}{\rho(t)}\right) \leqslant g_2(\Phi_1(t))\,. \tag{32}$$

Taking into account (32) and the definition of $\Phi_1(t)$, we conclude that

$$\Phi_1(t) \leqslant M + \int_{t_0}^t h_1(s)\,g_1(\Phi_1(s))\,ds + \int_{t_0}^t h_2(s)\,g_2(\Phi_1(s))\,ds,$$

where $M \overset{\text{def}}{=} c_* + H_3$. Observing further that

$$h_1(s)\, g_2(\Phi_1(s)) + h_2(s)\, g_2(\Phi_1(s)) \leqslant (h_1(s) + h_2(s))\,(g_1(\Phi_1(s)) + g_2(\Phi_1(s))),$$

we obtain

$$\Phi_1(t) \leqslant M + \int_{t_0}^{t} (h_1(s) + h_2(s))\,(g_1(\Phi_1(s)) + g_2(\Phi_1(s)))\, ds. \tag{33}$$

Application of the Bihari inequality [3] to (33) yields

$$\Phi_1(t) \leqslant G_1^{-1}\left(G_1(M) + \int_{t_0}^{t} (h_1(s) + h_2(s))\, ds\right),$$

where G_1^{-1} is the inverse of G_1 defined for $x \in (G_1(0+), +\infty)$. Let

$$K_1 \overset{\text{def}}{=} G_1(M) + H_1 + H_2 < +\infty.$$

Since G_1^{-1} is increasing, we conclude that

$$\Phi_1(t) \leqslant G_1^{-1}(K_1) \overset{\text{def}}{=} K_2 < +\infty.$$

Thus,

$$\frac{|z(t)|}{t} \leqslant K_2 \qquad \text{and} \qquad \frac{|z(\rho(t))|}{\rho(t)} \leqslant K_2,$$

where the second inequality follows from (22). On the other hand, for $t \geqslant t_0$,

$$\int_{t_0}^{t} \left|f\left(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s))\right)\right| ds \leqslant g_1(K_2)\, H_1 + g_2(K_2)\, H_2 + H_3 \overset{\text{def}}{=} K_3 < +\infty.$$

Therefore,

$$\lim_{t\to+\infty} \int_{t_0}^{t} \left|f\left(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s))\right)\right| ds$$

exists, and it follows from (19) that there exists a number $\mu \in \mathbb{R}$ such that

$$\lim_{t\to+\infty} z'(t) = \mu.$$

Choosing t_0 appropriately, one can always ensure that $\mu \neq 0$. Furthermore, application of the l'Hospital rule implies that

$$\lim_{t\to+\infty} \frac{z(t)}{t} = \lim_{t\to+\infty} z'(t) = \mu.$$

Set $w(t) = z(t)/t$ and $u(t) = x(t)/t$. Then (14) yields

$$w(t) = u(t) + p(t)\,\frac{t-\tau}{t}\,u(t-\tau).$$

Taking into account that

$$\lim_{t\to+\infty} w(t) = \lim_{t\to+\infty} \frac{z(t)}{t} = \mu \neq 0$$

and using Lemma 1, we conclude that

$$\lim_{t\to+\infty} u(t) = \lim_{t\to+\infty} \frac{x(t)}{t} = \frac{\mu}{1+p_0} \stackrel{\text{def}}{=} A.$$

The proof is complete now. □

EXAMPLE 7. For $t \geqslant 2$, consider the nonlinear neutral differential equation

$$(x(t) + p(t)x(t-1))'' + a(t)\tanh\left(x'(\sigma(t))\right) + b(t) = 0, \tag{34}$$

where

$$\alpha(t) = \left[(2t+1)^3 (t-1)^2\right]^{-1}, \qquad a(t) = \frac{12t^3\alpha(t)}{\tanh(1+2/t)},$$

$$b(t) = \alpha(t)t^{-2}\left[(4\ln(t-1)-10)t^4 + (5-8\ln(t-1))t^3 + (4\ln(t-1)-3)t^2 + 4t + 1\right],$$

$$p(t) = \frac{t}{2t+1} \qquad \text{and} \qquad \sigma(t) = t/2.$$

By Theorem 6, for any nonoscillatory solution of Eq. (34), (4) holds. In fact, $x(t) = t + \ln t$ is such a solution. We note that neither results reported by Džurina in [6], nor those in the references [7] - [11], [19], [20] apply to Eq. (34).

THEOREM 8. *Suppose that* (7) *holds and* $G_2(+\infty) = +\infty$. *Then the conclusion of Theorem* 6 *holds.*

Proof. Let $x(t)$ and $z(t)$ be as be as in Theorem 6. By Lemma 3,

$$\frac{|z(t)|}{t} \leqslant \Phi_2(t) \qquad \text{and} \qquad \frac{|z(\rho(t))|}{\rho(t)} \leqslant \Phi_2(t). \tag{35}$$

Using (16), (35), and monotonicity of the functions g_3 and g_4, we obtain

$$\Phi_2(t) \leqslant N + \int_{t_0}^{t} h_4(s)\, g_3(\Phi_2(s))\, g_4(\Phi_2(s))\, ds, \tag{36}$$

where $N \stackrel{\text{def}}{=} c_* + H_5$. Application of the Bihari inequality to (36) yields

$$\Phi_2(t) \leqslant G_2^{-1}\left(G_2(N) + \int_{t_0}^{t} h_4(s)\, ds\right),$$

where G_2^{-1} is the inverse of G_2 defined for $x \in (G_2(0+), +\infty)$. Let

$$K_4 \stackrel{\text{def}}{=} G_2(N) + H_4 < +\infty.$$

Then,

$$\Phi_2(t) \leqslant G_2^{-1}(K_4) \overset{\text{def}}{=} K_5 < +\infty,$$

and the proof is completed in the same manner as in Theorem 6. □

EXAMPLE 9. For $t \geqslant 2$, consider the nonlinear neutral differential equation

$$(x(t) + p(t)x(t-1))'' + a(t)\left[\frac{x^2(t)}{x^2(t)+1}\right]^{3/4}\left[\frac{(x'(t))^2}{(x'(t))^2+1}\right]^{1/4} = b(t), \qquad (37)$$

where

$$a(t) = \frac{28t^3\left(t^4 - t^2 + 1\right)^{3/4}\left(2t^4 + 2t^2 + 1\right)^{1/4}}{\left(t^2-1\right)^{3/2}\left(t^2+1\right)^{1/2}\left(2t^2 - t - 1\right)^3},$$

$$b(t) = \frac{2\left(18t^5 - 6t^4 - 8t^3 - 3t^2 + 3t + 1\right)}{t^3\left(2t^2 - t - 1\right)^3} \quad \text{and} \quad p(t) = \frac{1}{2t+1}.$$

By Theorem 8, for any nonoscillatory solution $x(t)$ of Eq. (37), (4) holds. In fact, $x(t) = t - 1/t$ is such a solution. Remarkably, results due to Džurina [6] and other authors, see [7] - [11], [19], [20], do not apply to Eq. (37).

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Mustafa Hasanbulli and Yuri V. Rogovchenko
Department of Mathematics
Eastern Mediterranean University
Famagusta, TRNC, Mersin 10
Turkey
e-mail: mustafa.hasanbulli@emu.edu.tr
e-mail: yuri.rogovchenko@emu.edu.tr