Mathematical
Inequalities
& Applications
Volume 10, Number 3 (2007), 619–625

A NOTE ON ABSOLUTE RIESZ SUMMABILITY FACTORS

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(*communicated by L. Leindler*)

Abstract. In the present paper a theorem on $|\bar{N},p_n|_k$ summability factors of infinite series has been proved under more weaker conditions. Also we have obtained a new result concerning the $|C,1|_k$ summability factors.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α and t_n^α the n-th Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, i.e.,

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{2}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad and \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{3}$$

A series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geqslant 1$, if (see [6], [8])

$$\sum_{n=1}^{\infty} \frac{|t_n^\alpha|^k}{n} < \infty. \tag{4}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geqslant 1). \tag{5}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{6}$$

Mathematics subject classification (2000): 40D15, 40F05, 40G99.

Key words and phrases: absolute summability, almost increasing sequence.

Paper MIA-10-58

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [7]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geqslant 1$, if (see [2], [3])

$$\sum_{n=1}^{\infty}(P_n/p_n)^{k-1}|\Delta\sigma_{n-1}|^k < \infty, \tag{7}$$

where

$$\Delta\sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v, \quad n \geqslant 1. \tag{8}$$

In the special case $p_n = 1$ for all values of n $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ summability.

2. Known results.

Bor [4] has proved the following theorem for $|\bar{N}, p_n|_k$ summability.

THEOREM A. *Let $k \geqslant 1$ and (X_n) be a positive non-decreasing sequence and there be sequences (β_n) and (λ_n) such that*

$$|\Delta\lambda_n| \leqslant \beta_n, \tag{9}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{10}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{11}$$

$$|\lambda_n|X_n = O(1) \tag{12}$$

and

$$\sum_{v=1}^{n}\frac{|s_v|^k}{v} = O(X_n) \quad as \quad n \to \infty. \tag{13}$$

Suppose further, the sequence (p_n) is such that

$$P_n = O(np_n), \tag{14}$$

$$P_n\Delta p_n = O(p_n p_{n+1}). \tag{15}$$

Then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$.

It may be noticed if we take $k = 1$ in Theorem A, then we get a result due to Mishra and Srivastava (see [10]).

Later on Bor [5] has proved Theorem A under weaker conditions in the following form.

THEOREM B. *Let $k \geqslant 1$ and (X_n) be a positive non-decreasing sequence and the sequences (β_n) and (λ_n) are such that conditions $(9) - (15)$ of Theorem A are satisfied with the condition (13) replaced by:*

$$\sum_{v=1}^{n}\frac{|t_v|^k}{v} = O(X_n) \quad as \quad n \to \infty. \tag{16}$$

where (t_n) *is the n-th* $(C,1)$ *mean of the sequence* (na_n), *then the series* $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ *is summable* $|\bar{N}, p_n|_k$.

It may be noted that condition (16) is weaker than condition (13).

3. Main result

The aim of this paper is to prove Theorem B under weaker conditions. Therefore we need the concept of almost increasing sequence. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leqslant b_n \leqslant Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = ne^{(-1)^n}$.

Now we shall prove the following theorem.

THEOREM *Let* (X_n) *be an almost increasing sequence. If the conditions* (9)−(12) *and* (14)−(16) *are satisfied, then the series* $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ *is summable* $|\bar{N}, p_n|_k$, $k \geqslant 1$.

REMARK. It should be noted that from the hypotheses of the Theorem, (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [4]).

4. Two lemmas

We require the following lemmas for the proof of the theorem.

LEMMA 1. ([9]) *If* (X_n) *an almost increasing sequence, then under the conditions* (10) − (11) *we have that*

$$nX_n\beta_n = O(1), \tag{17}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{18}$$

LEMMA 2. ([5]) *If the conditions* (14) *and* (15) *are satisfied, then* $\Delta(P_n/p_n n^2) = O(1/n^2)$.

5. Proof of the Theorem

Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{np_n}$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{rp_r} = \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) \frac{a_v P_v \lambda_v}{vp_v}. \tag{19}$$

Then for $n \geqslant 1$

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v \lambda_v}{v p_v} \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v v \lambda_v}{v^2 p_v}. \end{aligned}$$

Using Abel's transformation, we get

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \Delta \left(\frac{P_{v-1} P_v \lambda_v}{v^2 p_v} \right) \sum_{r=1}^{v} r a_r + \frac{\lambda_n}{n^2} \sum_{v=1}^{n} v a_v \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v}{p_v} (v+1) t_v p_v \frac{\lambda_v}{v^2} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v (v+1) \frac{t_v}{v^2 p_v} \\ &\quad - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_{v+1} (v+1) t_v \Delta (P_v / v^2 p_v) + \lambda_n t_n (n+1)/n^2 \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad say \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,r}|^k < \infty, \quad for \quad r = 1,2,3,4. \tag{20}$$

Now, applying Hölder's inequality, we have that

$$\begin{aligned} \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,1}|^k &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v |t_v| |\lambda_v| \frac{1}{v} \right\}^k \\ &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k p_v |t_v|^k |\lambda_v|^k \frac{1}{v^k} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k p_v |t_v|^k |\lambda_v|^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k |\lambda_v|^{k-1} |\lambda_v| p_v |t_v|^k \frac{1}{P_v} \frac{1}{v^k} \\ &= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^{k-1} |\lambda_v| |t_v|^k \frac{1}{v^k} \\ &= O(1) \sum_{v=1}^{m} v^{k-1} \frac{1}{v^k} |\lambda_v| |t_v|^k \\ &= O(1) \sum_{v=1}^{m} |\lambda_v| \frac{|t_v|^k}{v} \end{aligned}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\frac{|t_r|^k}{r} + O(1)|\lambda_m|\sum_{v=1}^{m}\frac{|t_v|^k}{v}$$
$$= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v + O(1)|\lambda_m|X_m$$
$$= O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)|\lambda_m|X_m = O(1)$$

as $m \to \infty$, by virtue of the hypotheses of the Theorem and Lemma 1.

Now using the fact that $(P_v/v) = O(p_v)$ by (14), we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k = O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}|\Delta\lambda_v|p_v|t_v|\right\}^k$$
$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k|\Delta\lambda_v|^k|t_v|^kp_v\right\}\times$$
$$\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k|\Delta\lambda_v|^k|t_v|^kp_v\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}|\Delta\lambda_v|^k|t_v|^k$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}|\Delta\lambda_v|^{k-1}|\Delta\lambda_v||t_v|^k$$
$$= O(1)\sum_{v=1}^{m}v^{k-1}\frac{1}{v^{k-1}}|\Delta\lambda_v||t_v|^k$$
$$= O(1)\sum_{v=1}^{m}\beta_v|t_v|^k = O(1)\sum_{v=1}^{m}v\beta_v\frac{|t_v|^k}{v}$$
$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\frac{|t_r|^k}{r} + O(1)m\beta_m\sum_{v=1}^{m}\frac{|t_v|^k}{v}$$
$$= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_vX_v + O(1)m\beta_mX_m$$
$$= O(1) \quad as \quad m \to \infty,$$

in view of the hypotheses of the Theorem and Lemma 1.

Now, since $\Delta(\frac{P_v}{p_v v^2}) = O(\frac{1}{v^2})$ by Lemma 2, we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}|^k = O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_n P_{n-1}^k}\left\{\sum_{v=1}^{n-1}P_v|\lambda_{v+1}||t_v|\frac{1}{v}\frac{v+1}{v}\right\}^k$$
$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_n P_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}p_v|\lambda_{v+1}|\frac{1}{v}|t_v|\right\}^k$$
$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_n P_{n-1}}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k p_v\frac{1}{v^k}|\lambda_{v+1}|^k|t_v|^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k p_v\frac{1}{v^k}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}||t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_n P_{n-1}}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}\frac{1}{v^k}|\lambda_{v+1}||t_v|^k$$
$$= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{|t_v|^k}{v}$$
$$= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_{v+1}|\sum_{r=1}^{v}\frac{|t_r|^k}{r} + O(1)|\lambda_{m+1}|\sum_{v=1}^{m}\frac{|t_v|^k}{v}$$
$$= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|X_v + O(1)|\lambda_{m+1}|X_m$$
$$= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|X_{v+1} + O(1)|\lambda_{m+1}|X_{m+1}$$
$$= O(1)\sum_{v=2}^{m}|\Delta\lambda_v|X_v + O(1)|\lambda_{m+1}|X_{m+1}$$
$$= O(1)\sum_{v=1}^{m}\beta_v X_v + O(1)|\lambda_{m+1}|X_{m+1} = O(1)$$

as $m \to \infty$, by virtue of the hypotheses of the Theorem and Lemma 1.

Finally, as in $T_{n,3}$, we have that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,4}|^k = O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\frac{n+1}{n}\right)^k\frac{1}{n^k}|\lambda_n|^k|t_n|^k$$
$$= O(1)\sum_{n=1}^{m}n^{k-1}\frac{1}{n^k}|\lambda_n|^{k-1}|\lambda_n||t_n|^k$$
$$= O(1)\sum_{n=1}^{m}|\lambda_n|\frac{|t_n|^k}{n} = O(1) \quad as \quad m \to \infty.$$

Therefore, we get that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,r}|^k = O(1) \quad as \quad m\to\infty, \quad for \quad r=1,2,3,4.$$

This completes the proof of the Theorem.

If we take $p_n = 1$ for all values of n, then we get a new result concerning the $|C,1|_k$ summability factors.

Acknowledgement. The author expresses his sincerest thanks to Professor László Leindler for his invaluable suggestions for the improvement of this paper.

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(Received May 10, 2006)

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