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INEQUALITIES FOR THE INCENTER SIMPLICES

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Abstract. Let $I_i (i = 0, 1, 2, ..., n)$ denote the incenter of facet F_i of an n-dimensional simplex Ω_A and we call $\Omega_I = \text{conv}\{I_0, I_1, ..., I_n\}$ the incenter simplex of Ω_A. In [3], L. H. Tang and G. S. Leng conjectured

$$V(\Omega_I) \leqslant \frac{1}{n^n} V(\Omega_A),$$

with equality if and only if Ω_A is a regular simplex. In this paper, we give a positive answer of the conjecture. Further, we improve the condition of the equality holds.

1. Introduction

The setting for this paper is n-dimensional Euclidean space $\mathbb{E}^n$. Let $\Omega_A = \text{conv}\{A_0, A_1, ..., A_n\}$ denote an n-dimensional simplex in $E^n (n \geqslant 3)$ with vertices $A_0, A_1, ..., A_n$, and $V(\Omega_A)$ the n-dimensional volume of Ω_A. Let F_i denote the $(n-1)$-dimensional facet spanned by the vertex set $\{A_0, A_1, ..., A_{i-1}, A_{i+1}, ..., A_n\}$ and I_i the incenter of the $(n-1)$-dimensional facet $F_i (i = 0, 1, 2, ..., n)$. Let ρ_{ij} be the length of edge A_iA_j. It is easy to get $\rho_{ij} = \rho_{ji}, \rho_{ii} = 0$, and the matrix (ρ_{ij}) is a $n \times n$ positive definite matrix.

Let P be an interior point of Ω_A and H_i the foot of the perpendicular drawn from P to the facet $F_i (i = 0, 1, 2, ..., n)$. We call $\Omega_H = \text{conv}\{H_0, H_1, ..., H_n\}$ the orthocentric simplex for P and Ω_A. In [2], H. M. Su conjectured that

$$V(\Omega_H) \leqslant \frac{1}{n^n} V(\Omega_A),$$

with equality if and only if P is the circumcenter of simplex Ω_A.

In [7], Y. Zhang gave a positive answer of the conjecture and improved the condition of the equality holds. He proved the following theorem.

THEOREM 1. *Let P be an interior point of an n-dimensionalsimplex Ω_A and H_i the foot of the perpendicular drawn from P to the facet $F_i (i = 0, 1, 2, ..., n)$. Then*

$$V(\Omega_H) \leqslant \frac{1}{n^n} V(\Omega_A). \tag{1.1}$$

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Let $(\lambda_0, \lambda_1, ..., \lambda_n)$ *be the gauge barycenter coordinate of* P *about* Ω_A *and* θ_{ij} *the internal dihedral angle between facets* F_i *and* F_j *of* Ω_A. *The equality in* (1.1) *holds if and only if*

$$\lambda_k = \frac{\cos\theta_{ij}}{n(\cos\theta_{ij} + \cos\theta_{ik}\cos\theta_{kj})},$$

where $i,j,k = 0,1,\cdots,n$ *and* $i \neq j, i \neq k, j \neq k,$.

Specially, when P is the incenter of simplex Ω_A in Theorem 1, then $H_i(i = 1,2...,n)$ become the tangent points of Ω_A with its inscribe ball. We call simplex Ω_H the tangent points simplex for P and Ω_A, and obtain the following result.

COROLLARY 1. ([1]) *Let* $T_i(i = 1,2...,n)$ *be the tangent points of an n-dimensional simplex* Ω_A *with its inscribe ball and* $\Omega_T = \mathrm{conv}\{T_0, T_1, ..., T_n\}$. *Then*

$$V(\Omega_T) \leqslant \frac{1}{n^n} V(\Omega_A). \tag{1.2}$$

with equality if and only if Ω_A *is a regular simplex.*

When P is the circumcenter of simplex Ω_A in Theorem 1, then $H_i(i = 1,2...,n)$ become the circumcenter of $F_i(i = 1,2,...,n)$. We call Ω_H the circumcenter simplex of Ω_A, and get the following result.

COROLLARY 2. *Let* $O_i(i = 1,2...,n)$ *the circumcenter of facet* F_i *of an n-dimensional simplex* Ω_A *and* $\Omega_O = \mathrm{conv}\{O_0, O_1, ..., O_n\}$. *Then*

$$V(\Omega_O) \leqslant \frac{1}{n^n} V(\Omega_A), \tag{1.3}$$

with equality if and only if Ω_A *is a regular simplex.*

Let I_i be the incenter of $F_i(i = 0,1,...,n)$. We call $\Omega_I = \mathrm{conv}\{I_0, I_1, ..., I_n\}$ the incenter simplex of Ω_A. It is easy to see that it is not the special case of Theorem 1. A natural question is that for the incenter simplex, whether exists inequality analogous to (1.1).

In [3], L. H. Tang and G. S. Leng conjectured that

$$V(\Omega_I) \leqslant \frac{1}{n^n} V(\Omega_A),$$

holds, with equality if and only if Ω_A is a regular simplex. They proved the conjecture is positive in E^3.

In this paper, we give a positive answer of the conjecture in E^n. Further, we improve the condition of the equality holds. Our result is the following theorem.

THEOREM 2. *Let* Ω_I *be the incenter simplex of* Ω_A. *Then*

$$V(\Omega_I) \leqslant \frac{1}{n^n} V(\Omega_A), \tag{1.4}$$

with equality if and only if there exist $\mu_k \geqslant 0 (k = 0,1,2,...,n)$ *such that* $\rho_{ij} = \mu_i \mu_j (0 \leqslant i < j \leqslant n)$.

2. Some lemmas

To prove Theorem 2, we need some lemmas.

LEMMA 1. ([2]) *Let Ω_A be the coordinate simplex and $\Omega_B = \mathrm{conv}\{B_0,\ B_1, ..., B_n\}$ be an arbitrary simplex in E^n. Let $(\lambda_{i0}, \lambda_{i1}, ..., \lambda_{in})$ be the gauge barycenter coordinate of $B_i (i = 0, 1, ..., n)$ about Ω_A. Then*

$$\frac{V(\Omega_B)}{V(\Omega_A)} = |\det(\lambda_{ij})|. \tag{2.1}$$

LEMMA 2. *Let $F_i \cap F_j (i < j, i, j = 0, 1, ..., n)$ be the $(n-2)$-dimensional subsimplex of simplex Ω_A and V_{ij} be the $(n-2)$-dimensional volume of $F_i \cap F_j$. Then the barycenter coordinate of I_i is*

$$(V_{i0} : V_{i1} : ... : V_{i,i-1} : 0 : V_{i,i+1} : ... : V_{ij} : ... : V_{in}).$$

Proof. By the definition of barycenter coordinate, we can get the barycenter coordinate of I_0 is

$$(0 : V_{A_0I_0A_2...A_n} : V_{A_0A_1I_0A_3...A_n} : ... : V_{A_0A_1...A_{n-1}I_0}),$$

where $V_{A_0A_1...A_{i-1}I_0A_{i+1}...A_n}$ denotes the volume of n-dimensional simplex $\mathrm{conv}\{A_0, A_1, ..., A_{i-1}, I_0, A_{i+1}, ..., A_n\}$.

Since I_0 is the interior point of $F_0 = \mathrm{conv}\{A_1, A_2, ..., A_n\}$, we know that the distances of A_0 to $\mathrm{conv}\{I_0, A_2, ..., A_n\}$ and $\mathrm{conv}\{A_1, I_0, A_3, ..., A_n\}$ are equal, let h_0 denote the distance.

Let r_0 be the distance of I_0 to the boundary of F_0 ($(n-2)$-dimensional subsimplex). Then

$$V^{(n-1)}_{I_0A_2...A_n} = \frac{1}{n-1} V^{(n-2)}_{A_2A_3...A_n} \cdot r_0,$$
$$V^{(n-1)}_{A_1I_0A_3...A_n} = \frac{1}{n-1} V^{(n-2)}_{A_2A_3...A_n} \cdot r_0,$$

where $V^{(k)}$ denotes the volume of k-dimensional simplex.

So

$$\begin{aligned} V_{A_0I_0A_2...A_n} : V_{A_0A_1I_0A_3...A_n} &= \frac{1}{n} V^{(n-1)}_{I_0A_2A_3...A_n} h_0 : \frac{1}{n} V^{(n-1)}_{A_1I_0A_3...A_n} h_0 \\ &= V^{(n-1)}_{I_0A_2A_3...A_n} : V^{(n-1)}_{A_1I_0A_3...A_n} \\ &= \frac{1}{n-1} V^{(n-2)}_{A_2A_3...A_n} r_0 : \frac{1}{n-1} V^{(n-2)}_{A_1A_3...A_n} r_0 \\ &= V^{(n-2)}_{A_2A_3...A_n} r_0 : V^{(n-2)}_{A_1A_3...A_n} r_0 \\ &= V_{01} : V_{02}. \end{aligned}$$

Then we get the barycenter coordinate of I_0 is

$$(0 : V_{01} : V_{02} : ... : V_{0n}).$$

Similarly, we have the barycenter coordinate of $I_i (i=0,1,...,n)$ is

$$(V_{i0} : V_{i1} : ... : V_{i,i-1} : 0 : V_{i,i+1} : ... : V_{i,j} : ... : V_{in}). \qquad \square$$

LEMMA 3. ([4]) *Let ρ_{ij} be the length of edge $A_iA_j (i,j=0,1,...,n)$ of simplex Ω_A and $R(\Omega_A)$ the circumradius of Ω_A. Then*

$$R^2(\Omega_A) = \frac{1}{(n!V(\Omega_A))^2} \cdot \left| \det(-\frac{1}{2}\rho_{ij}^2) \right|. \tag{2.2}$$

LEMMA 4. ([4]) *Let ρ_{ij} be the length of edge $A_iA_j (i,j=0,1,...,n)$ of simplex Ω_A. Then*

$$\left| \det(-\frac{1}{2}\rho_{ij}^2) \right| \leqslant \frac{n}{2^{n+1}} \prod_{0 \leqslant i<j \leqslant n} \rho_{ij}^{\frac{4}{n}}, \tag{2.3}$$

with equality if and only if all $\dfrac{\rho_{ij}}{\rho_{0i}\rho_{0j}} (i \neq j, i,j=0,1,...,n)$ are equal.

LEMMA 5. ([5]) *Let ρ_{ij} be the length of edge $A_iA_j (i,j=0,1,...,n)$ of simplex Ω_A and $R(\Omega_A)$ the circumradius of Ω_A. Then*

$$\prod_{0 \leqslant i<j \leqslant n} \rho_{ij}^{\frac{2}{n}} \geqslant (n!) \left(\frac{2^{n+1}}{n} \right)^{\frac{1}{2}} R(\Omega_A) V(\Omega_A), \tag{2.4}$$

with equality if and only if exist $\mu_k (\mu_k \geqslant 0, k=0,1,...,n)$ such that $\rho_{ij} = \mu_i\mu_j (i \neq j, i,j=0,1,...,n)$.

LEMMA 6. ([6]) *Let $\Omega_P = \mathrm{conv}\{P_0, P_1, ..., P_m\}$ be a m-dimensional simplex, ρ_{ij} denote the length of edge $P_iP_j (i,j=0,1,...,m)$. Then*

$$V^2(\Omega_P) = \frac{(-1)^{m+1}}{2^m (m!)^2} D(P_0, P_1, ..., P_m), \tag{2.5}$$

where $D(P_0,P_1,...,P_m)$ denotes the $(m+2)$-rank Cauchy-Mengv determinant, i.e.

$$D(P_0,P_1,...,P_m) = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \rho_{ij}^2 & \\ 1 & & & \end{vmatrix}.$$

3. Proof of the Theorem

We keep the notations of the previous two sections.

Proof of Theorem 2. Let Ω_A be the coordinate simplex in E^n. According to Lemma 2, we can get the gauge barycenter coordinate of $I_i (i=0,1,...,n)$

$$I_i(\lambda_{i0}, \lambda_{i1}, ..., \lambda_{in}),$$

applying Lemma 1, we find

$$\frac{V(\Omega_I)}{V(\Omega_A)} = |\det(\lambda_{ij})| = \frac{1}{\prod_{i=0}^{n}\sum_{\substack{j=0\\ i\neq j}}^{n} V_{ij}} |\det(V_{ij})|. \tag{3.1}$$

By Lemma 3, we know

$$\left|\det\left(\rho_{ij}^2\right)\right| = 2^{n+1}(n!)^2\left(R(\Omega_A)V(\Omega_A)\right)^2, \tag{3.2}$$

then we have

$$|\det(\sqrt{m_i}\sqrt{m_j}\rho_{ij}^2)| = 2^{n+1}(n!)^2 m_0 m_1 ... m_n (R(\Omega_A)V(\Omega_A))^2. \tag{3.3}$$

Let $\sqrt{m_i}\sqrt{m_j} = V_{ij}/\rho_{ij}^2 (i,j = 0,1,...,n)$, then

$$m_0 m_1 ... m_n = \frac{1}{\rho_{ij}^{\frac{4}{n}}} \cdot \prod_{0\leqslant i<j\leqslant n} V_{ij}^{\frac{2}{n}},$$

so

$$|\det(V_{ij})| = 2^{n+1}(n!)^2 \cdot \frac{\prod_{0\leqslant i<j\leqslant n} V_{ij}^{\frac{2}{n}}}{\rho_{ij}^{\frac{4}{n}}} \cdot (R(\Omega_A)V(\Omega_A))^2. \tag{3.4}$$

Substituting (3.4) into (3.1) and applying arithmetic-geometric inequality, we get

$$\begin{aligned}\frac{V(\Omega_I)}{V(\Omega_A)} &= \frac{2^{n+1}(n!)^2}{\prod_{i=0}^{n}\sum_{\substack{j=0\\ i\neq j}}^{n} V_{ij}} \cdot \frac{\prod_{0\leqslant i<j\leqslant n} V_{ij}^{\frac{2}{n}}}{\rho_{ij}^{\frac{4}{n}}} \cdot (R(\Omega_A)V(\Omega_A))^2 \\ &\leqslant \frac{2^{n+1}(n!)^2}{n^{n+1}\prod_{0\leqslant i<j\leqslant n} V_{ij}^{\frac{2}{n}}} \cdot \frac{\prod_{0\leqslant i<j\leqslant n} V_{ij}^{\frac{2}{n}}}{\rho_{ij}^{\frac{4}{n}}} \cdot (R(\Omega_A)V(\Omega_A))^2 \\ &= \frac{1}{n^{n+1}\rho_{ij}^{\frac{4}{n}}} \cdot 2^{n+1}(n!)^2 (R(\Omega_A)V(\Omega_A))^2.\end{aligned} \tag{3.5}$$

Applying Lemma 5 to (3.5) yields (1.4).

By arithmetic-geometric inequality and Lemma 5, we obtain that equality in (1.4) holds if and only if both of the following conditions hold:

(I) All $V_{ij}(0 \leqslant i < j \leqslant n)$ are equal;

(II) There exist $\mu_k \geqslant 0 (k = 0,1,...,n)$, such that $\rho_{ij} = \mu_i\mu_j, 0 \leqslant i < j \leqslant n$.

Now we prove that (I) and (II) are equivalent.

In fact, when all $V_{ij}(0 \leqslant i < j \leqslant n)$ are equal, let

$$V_{ij} = V_{n-2}, \quad \delta_{ij} = \begin{cases} 0 & i = j, \\ 1 & i \neq j. \end{cases}$$

Since

$$|\det(V_{ij})| = V_{n-2}^{n+1}\cdot\det(\delta_{ij}) = V_{n-2}^{n+1}|(-1)^n n| = nV_{n-2}^{n+1}.$$

Using (3.4), we get

$$nV_{n-2}^{n+1} = 2^{n+1}(n!)^2\left(V_{n-2}^{\frac{2}{n}}\right)^{C_{n+1}^2}\cdot\frac{(R(\Omega_A)V(\Omega_A))^2}{\prod\limits_{0\leqslant i<j\leqslant n}\rho_{ij}^{\frac{4}{n}}},$$

then

$$\prod_{0\leqslant i<j\leqslant n}\rho_{ij}^{\frac{2}{n}} = (n!)\left(\frac{2^{n+1}}{n}\right)^{\frac{1}{2}}R(\Omega_A)V(\Omega_A). \tag{3.6}$$

By the equality condition of Lemma 5, we know that there exist $\mu_k \geqslant 0(k = 0,1,2,...,n)$, such that

$$\rho_{ij} = \mu_i\mu_j,\ (0\leqslant i<j\leqslant n). \tag{3.7}$$

On the other hand, if (3.7) holds, (3.6) holds too. Substituting (3.6) into (3.4) yields

$$|\det(V_{ij})| = n\prod_{0\leqslant i<j\leqslant n}V_{ij}^{\frac{2}{n}}. \tag{3.8}$$

From Lemma 4, we obtain that there exist $m_k(k=0,1,...,n)$, such that

$$\left|\det\left(\sqrt{m_i}\sqrt{m_j}\rho_{ij}^2\right)\right| \leqslant n(m_0m_1...m_n)\prod_{0\leqslant i<j\leqslant n}\rho_{ij}^{\frac{4}{n}}.$$

Let $\sqrt{m_i}\sqrt{m_j} = V_{ij}/\rho_{ij}^2(i,j=0,1,...,n)$, then

$$|\det(V_{ij})| = n\prod_{0\leqslant i<j\leqslant n}V_{ij}^{\frac{2}{n}}. \tag{3.9}$$

Combining (3.8) and (3.9), and noticing the equality condition in Lemma 4, we get that $\rho_{ij}/\rho_{0i}\rho_{0j}(i\neq j, i,j=0,1,...,n)$ are all equal.

Specially, let ρ_{ij} be all equal, applying Lemma 6, we know that all V_{ij} are equal. So equality holds if and only if there exist $\mu_k\geqslant 0(k=0,1,2,...,n)$, such that

$$\rho_{ij} = \mu_i\mu_j(0\leqslant i<j\leqslant n).$$

The proof is complete. □

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