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SEVERAL INEQUALITIES FOR THE LARGEST SINGULAR VALUE AND THE SPECTRAL RADIUS OF MATRICES

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Abstract. For nonnegative matrices $A=(a_{ij})\in\mathbb{R}^{n\times m}$, $B=(b_{ij})\in\mathbb{R}^{m\times n}$ and any $t\in[0,1]$, we present $\sigma(S_t(A,B))\leqslant\sigma(A)^t\sigma(B)^{1-t}$, in which $S_t(A,B)=(a_{ij}^t b_{ji}^{1-t})$ and $\sigma(\cdot)$ denotes the largest singular value. Using the result obtained, the inequality $\sigma(A\circ B)\leqslant\sqrt{\sigma(A\circ A)\sigma(B\circ B)}$ for matrices $A=(a_{ij})$ and $B=(b_{ij})\in\mathbb{C}^{n\times m}$ is established. Here, $A\circ B=(a_{ij}\bar{b}_{ij})$, and $\bar{b}_{ij}$ denotes the complex conjugate of b_{ij}. Finally, some inequalities for the spectral radius are also studied.

1. Introduction and notation

If all entries of matrix A are nonnegative (positive), we say that A is nonnegative (positive). By Perron-Frobenius theory [10], a nonnegative matrix has an eigenvalue equal to its spectral radius, i.e., the Perron root. We denote by $\mathbb{C}^{n\times m}$ and $\mathbb{R}_+^{n\times m}$ the $n\times m$ matrices and the $n\times m$ nonnegative matrices, respectively, and by A^T, A^H, $\mathrm{tr}(A)$, $\sigma(A)$ and $\rho(A)$ the transpose, the conjugate transpose, the trace, the largest singular value and the spectral radius of the matrix A, respectively. For $A=(a_{ij})$ and $B=(b_{ij})\in\mathbb{C}^{n\times m}$, we write $A\geqslant B$ if $A-B$ is nonnegative, and define $|A|:=(|a_{ij}|)$ and $A\circ B:=(a_{ij}\bar{b}_{ij})$, where $\bar{b}_{ij}$ is the complex conjugate of b_{ij}. For $A=(a_{ij})\in\mathbb{R}_+^{n\times m}$ and $B=(b_{ij})\in\mathbb{R}_+^{m\times n}$, we define

$$S_t(A,B):=(a_{ij}^t b_{ji}^{1-t}),\quad t\in[0,1],$$

and, in particular, if $n=m$, we define

$$S_t(A):=S_t(A,A),\quad t\in[0,1].$$

The matrix $C\in\mathbb{C}^{n\times n}$ is said to be reducible if either (a) $n=1$ and C is a zero matrix; or (b) $n\geqslant 2$, there exists an $n\times n$ permutation matrix P such that $P^TCP=\begin{pmatrix} B & E\\ 0 & D\end{pmatrix}$, where $B\in\mathbb{C}^{k\times k}$, $1\leqslant k\leqslant n-1$; Otherwise, we say that C is irreducible; see [2, p.18,

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Definition 1.15]. The irreducible matrix $A \in \mathbb{R}_+^{n\times n}$ is said to be primitive if A has only one eigenvalue with modulus $\rho(A)$; see [2, p.40, Definition 2.10].

The directed graph of the matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, denoted by $D(A)$, is the directed graph on n nodes $v_1, v_2, \cdots, v_n$ such that there is a directed arc in $D(A)$ from v_i to v_j, denoted by $v_i \to v_j$, if and only if $a_{ij} \neq 0$; see [1,2]. Let $W = v_{i_1}v_{i_2}\cdots v_{i_k}$ be a sequence arising in n nodes. Then we define

$$W^{-1} := v_{i_k}v_{i_{k-1}}\cdots v_{i_1},\ w_A(W) := a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{k-1}i_k}.$$

If $v_{i_1} \to v_{i_2}, v_{i_2} \to v_{i_3}, \cdots, v_{i_{k-1}} \to v_{i_k}$ exist, then W is said to be a directed path in $D(A)$. A cycle $W = v_{i_1}v_{i_2}\cdots v_{i_k}v_{i_1}$ is a closed path with its k nodes distinct. $\gamma(A)$ denotes the set of cycles in $D(A)$.

The research on matrix singular values and spectral radius plays an important role in matrix theory and numerical algebra; see, e.g., [1, 3, 4, 6, 7, 9]. We shall present some new and interesting inequalities for matrix singular values in Section 2. In Section 3, several inequalities for the spectral radius will be also studied.

2. Inequalities for the largest singular value

We begin with two lemmas.

LEMMA 2.1. [5, Theorem 2.8.3]. *Let* a_{ij} $(i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ *be nonnegative numbers, and* $\alpha_1, \alpha_2, \cdots, \alpha_m$ *be positive numbers with* $\sum_{k=1}^{m} \frac{1}{\alpha_k} \geqslant 1$. *Then*

$$\sum_{i=1}^{n} a_{i1}a_{i2}\cdots a_{im} \leqslant \Big(\sum_{i=1}^{n} a_{i1}^{\alpha_1}\Big)^{\frac{1}{\alpha_1}}\Big(\sum_{i=1}^{n} a_{i2}^{\alpha_2}\Big)^{\frac{1}{\alpha_2}}\cdots\Big(\sum_{i=1}^{n} a_{im}^{\alpha_m}\Big)^{\frac{1}{\alpha_m}}.$$

LEMMA 2.2. [10, p.28, Theorem 1.11]. *Let* $A \in \mathbb{R}_+^{n\times n}$, *and let* x *be a positive vector. If there exists a real number* α *such that* $\alpha x \geqslant Ax$, *then* $\rho(A) \leqslant \alpha$.

Based on the two lemmas above, we give our main result as follows.

THEOREM 2.1. *Let* $A = (a_{ij}) \in \mathbb{R}_+^{n\times m}$ *and* $B = (b_{ij}) \in \mathbb{R}_+^{m\times n}$. *Then, for any* $t \in [0, 1]$,

$$\sigma(S_t(A, B)) \leqslant \sigma(A)^t\sigma(B)^{1-t}.$$

Proof. Trivial for $t = 0, 1$.

Assume that $t \in (0, 1)$. Then we first show that the inequality holds for positive matrices A and B. By the Perron-Frobenius theory, there exist positive eigenvectors $x = (x_1, x_2, \cdots, x_n)^T$ and $y = (y_1, y_2, \cdots, y_n)^T$ such that

$$AA^Tx = \rho(AA^T)x,\ B^TBy = \rho(B^TB)y.$$

Taking $z := (x_1^t y_1^{1-t}, x_2^t y_2^{1-t}, \cdots, x_n^t y_n^{1-t})^T$ and $C := S_t(A,B)$, by Lemma 2.1 we have, for any $1 \leqslant i \leqslant n$,

$$\begin{aligned}
\sum_{j=1}^{n} (CC^T)_{ij} z_j &= \sum_{j=1}^{n} \left(\sum_{k=1}^{m} C_{ik} C_{jk} \right) z_j \\
&= \sum_{j=1}^{n} \left(\sum_{k=1}^{m} a_{ik}^t b_{ki}^{1-t} a_{jk}^t b_{kj}^{1-t} \right) z_j \\
&= \sum_{j=1}^{n} \left(\sum_{k=1}^{m} (a_{ik} a_{jk})^t (b_{ki} b_{kj})^{1-t} \right) z_j \\
&\leqslant \sum_{j=1}^{n} \left(\sum_{k=1}^{m} a_{ik} a_{jk} \right)^t \left(\sum_{k=1}^{m} b_{ki} b_{kj} \right)^{1-t} z_j \\
&= \sum_{j=1}^{n} \left((AA^T)_{ij} x_j \right)^t \left((B^T B)_{ij} y_j \right)^{1-t} \\
&\leqslant \left(\sum_{j=1}^{n} (AA^T)_{ij} x_j \right)^t \left(\sum_{j=1}^{n} (B^T B)_{ij} y_j \right)^{1-t} \\
&= \left(\rho(AA^T) x_i \right)^t \left(\rho(B^T B) y_i \right)^{1-t} \\
&= \rho(AA^T)^t \rho(B^T B)^{1-t} z_i,
\end{aligned}$$

which, by Lemma 2.2, implies

$$\rho(CC^T) \leqslant \rho(AA^T)^t \rho(B^T B)^{1-t}.$$

Thus,

$$\sigma(S_t(A,B)) = \sqrt{\rho(CC^T)} \leqslant \sqrt{\rho(AA^T)^t \rho(B^T B)^{1-t}} = \sigma(A)^t \sigma(B)^{1-t}.$$

Secondly, we show that the inequality holds for all nonnegative matrices. For each $\varepsilon > 0$, set $A(\varepsilon) := (a_{ij} + \varepsilon)$ and $B(\varepsilon) := (b_{ij} + \varepsilon)$. It follows from the result obtained above that

$$\sigma(S_t(A(\varepsilon), B(\varepsilon))) \leqslant \sigma(A(\varepsilon))^t \sigma(B(\varepsilon))^{1-t}.$$

By the continuity of eigenvalues, we get

$$\begin{aligned}
\sigma(S_t(A,B)) &= \lim_{\varepsilon \to 0} \sigma(S_t(A(\varepsilon), B(\varepsilon))) \\
&\leqslant \lim_{\varepsilon \to 0} \sigma(A(\varepsilon))^t \sigma(B(\varepsilon))^{1-t} = \sigma(A)^t \sigma(B)^{1-t}.
\end{aligned}$$

This completes the proof of Theorem 2.1. □

REMARK 2.1. By Theorem 2.1 and the weighted arithmetic-geometric mean inequality [8, Appendix B], it is easy to get, for any $t \in [0,1]$,

$$\begin{aligned}
\sigma(S_t(A,B)) &\leqslant \sigma(A)^t \sigma(B)^{1-t} \\
&\leqslant t\sigma(A) + (1-t)\sigma(B) \leqslant \max\{\sigma(A), \sigma(B)\}.
\end{aligned}$$

So $\max\{\sigma(A), \sigma(B)\}$ is the upper bound of the function

$$\varphi(t) := \sigma(S_t(A,B)), \ t \in [0,1].$$

Using $\sigma(B) = \sigma(B^T)$ and Theorem 2.1, it is not difficult to get the following results.

COROLLARY 2.1. *Let A and $B \in \mathbb{R}_+^{m\times n}$. Then, for any $t \in [0,1]$,*

$$\sigma(S_t(A,B^T)) \leqslant \sigma(A)^t \sigma(B)^{1-t}.$$

COROLLARY 2.2. *Let A and $B \in \mathbb{R}_+^{n\times n}$. Then*

$$\sigma(S_{\frac{1}{2}}(A,B)) \leqslant \sqrt{\sigma(A)\sigma(B)}, \quad \sigma(S_{\frac{1}{2}}(A,B^T)) \leqslant \sqrt{\sigma(A)\sigma(B)}.$$

THEOREM 2.2. *Let $A \in \mathbb{R}_+^{n\times n}$. Then, for any $t \in [0,1]$,*

$$\sigma(S_{\frac{1}{2}}(A)) \leqslant \sigma(S_t(A)) \leqslant \sigma(A).$$

Proof. It is trivial by the equality $S_{\frac{1}{2}}(A) = S_{\frac{1}{2}}(S_t(A),S_t(A))$ and Theorem 2.1. □

REMARK 2.2. Since $S_{\frac{1}{2}}(A)$ is symmetric, we get $\rho(S_{\frac{1}{2}}(A)) = \sigma(S_{\frac{1}{2}}(A))$. Thus, Theorem 2.2 shows that $\rho(S_{\frac{1}{2}}(A))$ and $\sigma(A)$ are lower and upper bounds for $\sigma(S_t(A)), t \in [0,1]$, respectively.

The following is the monotonicity property of the function

$$f(t) := \sigma(S_t(A)), t \in [0,1].$$

THEOREM 2.3. *Let $A \in \mathbb{R}_+^{n\times n}$. Then $f(t)$ is decreasing in $[0,0.5]$ and increasing in $[0.5,1]$.*

Proof. Assume that $0 \leqslant t_1 < t_2 \leqslant 0.5$. Then we set $\alpha = \frac{t_1+t_2-1}{2t_1-1}$, and then have $0 \leqslant \alpha < 1$. A direct computation yields that

$$a_{ij}^{t_2} a_{ji}^{1-t_2} = (a_{ij}^{t_1} a_{ji}^{1-t_1})^{\alpha} (a_{ji}^{t_1} a_{ij}^{1-t_1})^{1-\alpha},$$

which, together with Theorem 2.1, leads to

$$\begin{aligned} f(t_2) &= \sigma(S_{t_2}(A)) = \sigma(S_\alpha(S_{t_1}(A),S_{t_1}(A))) \\ &\leqslant \sigma(S_{t_1}(A)) = f(t_1). \end{aligned}$$

Hence, $f(t)$ is decreasing in $[0,0.5]$. Likewise, assume that $0.5 \leqslant t_1 < t_2 \leqslant 1$ and $\alpha = \frac{t_1+t_2-1}{2t_2-1}$. Then $0 \leqslant \alpha < 1$, and

$$a_{ij}^{t_1} a_{ji}^{1-t_1} = (a_{ij}^{t_2} a_{ji}^{1-t_2})^{\alpha} (a_{ji}^{t_2} a_{ij}^{1-t_2})^{1-\alpha},$$

which, by Theorem 2.1, implies

$$\begin{aligned} f(t_1) &= \sigma(S_{t_1}(A)) = \sigma(S_\alpha(S_{t_2}(A),S_{t_2}(A))) \\ &\leqslant \sigma(S_{t_2}(A)) = f(t_2). \end{aligned}$$

So, $f(t)$ is increasing in [0.5,1]. □

As an application of Theorem 2.1, some inequalities for the largest singular values of complex matrices are derived as follows.

THEOREM 2.4. *Let A and $B \in \mathbb{C}^{n\times m}$. Then*

$$\sigma(A\circ B) \leqslant \sqrt{\sigma(A\circ A)\sigma(B\circ B)}.$$

Proof. Clearly,

$$|(A\circ B)(A\circ B)^H| \leqslant |(A\circ B)||(A\circ B)^H| = (|A|\circ|B|)(|A|\circ|B|)^T.$$

By the equality $|A|\circ|B| = S_{\frac{1}{2}}(|A|\circ|A|,(|B|\circ|B|)^T)$, [9, p.38, Corollary 2.1] and Corollary 2.2, we get

$$\begin{aligned}\sigma(A\circ B) &\leqslant \sigma(|A|\circ|B|)\\ &\leqslant \sqrt{\sigma(|A|\circ|A|)\sigma(|B|\circ|B|)} = \sqrt{\sigma(A\circ A)\sigma(B\circ B)}.\end{aligned}$$

The proof is completed. □

From $\sigma(B) = \sigma(B^H)$ and Theorem 2.4, the following results are immediate.

COROLLARY 2.3. *Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$. Then*

$$\sigma(A\circ B^H) \leqslant \sqrt{\sigma(A\circ A)\sigma(B\circ B)}.$$

COROLLARY 2.4. *Let $A \in \mathbb{C}^{n\times n}$. Then $\sigma(A\circ A^H) \leqslant \sigma(A\circ A)$.*

COROLLARY 2.5. *Let $A \in \mathbb{C}^{n\times n}$, and let P be an $n\times n$ permutation matrix. Then*

$$\sigma(A\circ(P^TAP)) \leqslant \sigma(A\circ A), \quad \sigma(A\circ(P^TA^HP)) \leqslant \sigma(A\circ A).$$

Proof. By Theorem 2.4, it is easy to get

$$\begin{aligned}\sigma(A\circ(P^TAP)) &\leqslant \sqrt{\sigma(A\circ A)\sigma((P^TAP)\circ(P^TAP))}\\ &= \sqrt{\sigma(A\circ A)\sigma(P^T(A\circ A)P)} = \sigma(A\circ A).\end{aligned}$$

Analogously, $\sigma(A\circ(P^TA^HP)) \leqslant \sigma(A\circ A)$ can be obtained. □

Using Lemma 2.1 and the proof of Theorem 2.1, we present the following theorem generalizing Theorem 2.1.

THEOREM 2.5. *Let $A_1 = (a_{ij}^1)$, $A_2 = (a_{ij}^2), \cdots, A_l = (a_{ij}^l) \in \mathbb{R}_+^{n\times m}$, $\sum_{k=1}^l \alpha_k = 1$ with $\alpha_k \geqslant 0$ $(k = 1,2,\cdots,l)$ and*

$$S_{\alpha_1\cdots\alpha_l}(A_1,\cdots,A_l) = ((a_{ij}^1)^{\alpha_1}(a_{ij}^2)^{\alpha_2}\cdots(a_{ij}^l)^{\alpha_l}).$$

Then

$$\sigma(S_{\alpha_1\cdots\alpha_l}(A_1,\cdots,A_l)) \leqslant \sigma(A_1)^{\alpha_1}\sigma(A_2)^{\alpha_2}\cdots\sigma(A_l)^{\alpha_l}.$$

3. Inequalities for the spectral radius

Let $A_1 = (a_{ij}^1), A_2 = (a_{ij}^2), \cdots, A_m = (a_{ij}^m) \in \mathbb{R}_+^{n\times n}$ and $\sum_{k=1}^m \alpha_k \geqslant 1$, where $\alpha_k > 0, k = 1,2,\cdots,m$. Then define $C := (c_{ij})$ with

$$c_{ij} = (a_{ij}^1)^{\alpha_1}(a_{ij}^2)^{\alpha_2}\cdots(a_{ij}^m)^{\alpha_m}, i,j = 1,2,\cdots,n.$$

Elsner, Hershkowitz and Pinkus [6] proved

$$\rho(C) \leqslant \rho(A_1)^{\alpha_1}\rho(A_2)^{\alpha_2}\cdots\rho(A_m)^{\alpha_m}.$$

Especially, setting $A_1 = A, A_2 = B^T, \alpha_1 = t, \alpha_2 = 1-t,\ t \in (0,1)$, we derive $C = S_t(A,B) = (a_{ij}^t b_{ji}^{1-t})$, and then

$$\rho(S_t(A,B)) \leqslant \rho(A)^t\rho(B)^{1-t}.$$

In this section, we shall study further this inequality. New proof method, sufficient condition for equality and some interesting results are obtained. We begin with the following lemmas.

LEMMA 3.1. [2, p. 46, Theorem 2.18]. *Let $A \in \mathbb{R}_+^{n\times n}$. Then A is primitive if and only if there exists some positive integer m such that A^m is positive.*

LEMMA 3.2. [2, p. 49, Exercise 6]. *Let $A \in \mathbb{R}_+^{n\times n}$ be primitive. Then*

$$\lim_{m\to\infty}(\mathrm{tr}(A^m))^{\frac{1}{m}} = \rho(A).$$

LEMMA 3.3. *Let A and $B \in \mathbb{R}_+^{n\times n}$, and let $S_t(A,B)$ be primitive for some $t \in (0,1)$. Then A and B are both primitive.*

Proof. Since $S_t(A,B)$ is primitive, by Lemma 3.1, there exists a positive integer m such that $S_t(A,B)^m$ is positive. For any $1 \leqslant i,j \leqslant n$, by Lemma 2.1 we obtain

$$\begin{aligned}
0 &< (S_t(A,B)^m)_{ij} \\
&= \sum_{i_1=1}^n\sum_{i_2=1}^n\cdots\sum_{i_{m-1}=1}^n a_{ii_1}^t b_{i_1 i}^{1-t} a_{i_1 i_2}^t b_{i_2 i_1}^{1-t}\cdots a_{i_{m-1}j}^t b_{ji_{m-1}}^{1-t} \\
&= \sum_{i_1=1}^n\sum_{i_2=1}^n\cdots\sum_{i_{m-1}=1}^n (a_{ii_1}a_{i_1i_2}\cdots a_{i_{m-1}j})^t(b_{i_1i}b_{i_2i_1}\cdots b_{ji_{m-1}})^{1-t} \\
&\leqslant (\sum_{i_1=1}^n\sum_{i_2=1}^n\cdots\sum_{i_{m-1}=1}^n a_{ii_1}a_{i_1i_2}\cdots a_{i_{m-1}j})^t(\sum_{i_1=1}^n\sum_{i_2=1}^n\cdots\sum_{i_{m-1}=1}^n b_{i_1i}b_{i_2i_1}\cdots b_{ji_{m-1}})^{1-t} \\
&= ((A^m)_{ij})^t((B^m)_{ji})^{1-t}.
\end{aligned}$$

Thus, A^m and B^m are both positive, and then A and B are both primitive. □

THEOREM 3.1. *Let* $A=(a_{ij})$ *and* $B=(b_{ij})\in\mathbb{R}_+^{n\times n}$. *Then, for any* $t\in[0,1]$,

$$\rho(S_t(A,B))\leqslant\rho(A)^t\rho(B)^{1-t}. \tag{1}$$

If $S_t(A,B)$ *is primitive for some* $t\in(0,1)$, $D(A)=D(B^T)$ *and* $w_A(W)=w_B(W^{-1})$ *for any* $W\in\gamma(A)$, *then equality in* (1) *holds.*

Proof. Assume that $t=0$ or $t=1$. Then (1) holds clearly. Assume that $t\in(0,1)$. Then we first show that (1) holds for primitive matrix $S_t(A,B)$. It follows from Lemma 3.3 that A and B are both primitive. For any positive integer k, by Lemma 2.1 we have

$$\begin{aligned}
\operatorname{tr}(S_t(A,B)^k) &= \sum_{i=1}^n\sum_{i_1=1}^n\cdots\sum_{i_{k-1}=1}^n a_{ii_1}^t b_{i_1i}^{1-t}a_{i_1i_2}^t b_{i_2i_1}^{1-t}\cdots a_{i_{k-1}i}^t b_{ii_{k-1}}^{1-t}\\
&= \sum_{i=1}^n\sum_{i_1=1}^n\cdots\sum_{i_{k-1}=1}^n (a_{ii_1}a_{i_1i_2}\cdots a_{i_{k-1}i})^t(b_{i_1i}b_{i_2i_1}\cdots b_{ii_{k-1}})^{1-t}\\
&\leqslant \Big(\sum_{i=1}^n\sum_{i_1=1}^n\cdots\sum_{i_{k-1}=1}^n a_{ii_1}a_{i_1i_2}\cdots a_{i_{k-1}i}\Big)^t\\
&\quad\times\Big(\sum_{i=1}^n\sum_{i_1=1}^n\cdots\sum_{i_{k-1}=1}^n b_{i_1i}b_{i_2i_1}\cdots b_{ii_{k-1}}\Big)^{1-t}\\
&= (\operatorname{tr}(A^k))^t(\operatorname{tr}(B^k))^{1-t},
\end{aligned}$$

and then, from Lemma 3.2,

$$\begin{aligned}
\rho(S_t(A,B)) &= \lim_{k\to\infty}(\operatorname{tr}(S_t(A,B)^k))^{\frac{1}{k}}\\
&\leqslant \lim_{k\to\infty}(\operatorname{tr}(A^k))^{\frac{t}{k}}(\operatorname{tr}(B^k))^{\frac{1-t}{k}}\\
&= \rho(A)^t\rho(B)^{1-t}.
\end{aligned}$$

Secondly, we show that (1) holds for the nonnegative matrix $S_t(A,B)$. Define

$$A(\varepsilon):=(a_{ij}+\varepsilon),\quad B(\varepsilon):=(b_{ij}+\varepsilon)\text{ for any }\varepsilon>0.$$

It is easy to find that $A(\varepsilon),B(\varepsilon)$ and $S_t(A(\varepsilon),B(\varepsilon))$ are all primitive. By the result derived above, we have

$$\rho(S_t(A(\varepsilon),B(\varepsilon)))\leqslant\rho(A(\varepsilon))^t\rho(B(\varepsilon))^{1-t}.$$

By the continuity of eigenvalues, it deduces that

$$\begin{aligned}
\rho(S_t(A,B)) &= \lim_{\varepsilon\to0}\rho(S_t(A(\varepsilon),B(\varepsilon)))\\
&\leqslant \lim_{\varepsilon\to0}\rho(A(\varepsilon))^t\rho(B(\varepsilon))^{1-t}=\rho(A)^t\rho(B)^{1-t}.
\end{aligned}$$

Thus, the inequality (1) holds for any $t\in[0,1]$.

Let $S_t(A,B)$ be primitive for some $t\in(0,1)$, $D(A)=D(B^T)$ and $w_A(W)=w_B(W^{-1})$ for any $W\in\gamma(A)$. We first show that

$$w_A(W')=w_B(W'^{-1})\text{ for any path }W'=v_{i_0}v_{i_1}\cdots v_{i_k}\ (v_{i_0}=v_{i_k}). \tag{2}$$

Consider the following cases:

Case (i): $W' \in \gamma(A)$. Clearly, $w_A(W') = w_B(W'^{-1})$.

Case (ii): $W' \notin \gamma(A)$, there exist indices i_{j-1}, i_j such that $a_{i_{j-1}i_j} = 0$. By $D(A) = D(B^T)$ it is obvious that $b_{i_j i_{j-1}} = 0$, and then $0 = w_A(W') = w_B(W'^{-1})$.

Case (iii): $W' \notin \gamma(A)$, there don't exist indices i_{j-1}, i_j such that $a_{i_{j-1}i_j} = 0$. In this case, W' must have repeated nodes. If $v_{i_l} = v_{i_m}$, $l \neq m$, then W' can be split into $W'_1 = v_{i_0} v_{i_1} \cdots v_{i_l} v_{i_{m+1}} \cdots v_{i_k}$ and $W'_2 = v_{i_l} v_{i_{l+1}} \cdots v_{i_m}$. If W'_1 and W'_2 both belong to $\gamma(A)$, then, by Case (i),

$$w_A(W') = w_A(W'_1) w_A(W'_2) = w_B({W'}_1^{-1}) w_B({W'}_2^{-1}) = w_B(W'^{-1}).$$

Otherwise, W'_1 or W'_2 can be split inductively. Finally, we can get that $W'_1, W'_2, \cdots, W'_h$ belong to $\gamma(A)$, and thus

$$w_A(W') = w_A(W'_1) \cdots w_A(W'_h) = w_B({W'}_1^{-1}) \cdots w_B({W'}_h^{-1}) = w_B(W'^{-1}).$$

Secondly, from (2) it follows that

$$\begin{aligned}
\operatorname{tr}(S_t(A,B)^k) &= \sum_{i=1}^{n} \sum_{i_1=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} w_A(v_i v_{i_1} \cdots v_{i_{k-1}} v_i)^t w_B(v_i v_{i_{k-1}} \cdots v_{i_1} v_i)^{1-t} \\
&= \Big(\sum_{i=1}^{n} \sum_{i_1=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} w_A(v_i v_{i_1} \cdots v_{i_{k-1}} v_i)\Big)^t \\
&\quad \times \Big(\sum_{i=1}^{n} \sum_{i_1=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} w_B(v_i v_{i_{k-1}} \cdots v_{i_1} v_i)\Big)^{1-t} \\
&= (\operatorname{tr}(A^k))^t (\operatorname{tr}(B^k))^{1-t},
\end{aligned}$$

which, by Lemmas 3.2 - 3.3, implies

$$\begin{aligned}
\rho(S_t(A,B)) &= \lim_{k\to\infty} (\operatorname{tr}(S_t(A,B)^k))^{\frac{1}{k}} \\
&= \lim_{k\to\infty} (\operatorname{tr}(A^k))^{\frac{t}{k}} (\operatorname{tr}(B^k))^{\frac{1-t}{k}} \\
&= \rho(A)^t \rho(B)^{1-t}.
\end{aligned}$$

Thus, the equality in (1) holds. The proof is completed. □

From $\rho(B) = \rho(B^H)$, Theorem 3.1 and the proof of Theorems 2.2 - 2.4, Corollaries 2.1 - 2.5, we can derive the following results analogously.

COROLLARY 3.1. *Let A and $B \in \mathbb{R}_+^{n\times n}$. Then, for any $t \in [0,1]$,*

$$\rho(S_t(A,B^T)) \leqslant \rho(A)^t \rho(B)^{1-t}.$$

COROLLARY 3.2. *Let A and $B \in \mathbb{R}_+^{n\times n}$. Then*

$$\rho(S_{\frac{1}{2}}(A,B)) \leqslant \sqrt{\rho(A)\rho(B)}, \quad \rho(S_{\frac{1}{2}}(A,B^T)) \leqslant \sqrt{\rho(A)\rho(B)}.$$

THEOREM 3.2. *Let* $A \in \mathbb{R}_{+}^{n\times n}$. *Then, for any* $t \in [0,1]$,

$$\rho(S_{\frac{1}{2}}(A)) \leqslant \rho(S_t(A)) \leqslant \rho(A). \tag{3}$$

If there exists a positive diagonal matrix D *such that* DAD^{-1} *has a symmetric irreducible component* [3] *with maximum spectral radius, the equalities in* (3) *hold.*

Proof. The proof of (3) is similar to the proof of Theorem 2.2. If there exists a positive diagonal matrix D such that DAD^{-1} has a symmetric irreducible component with maximum spectral radius, then it follows from [3, Theorem 1] that $\rho(S_{\frac{1}{2}}(A)) = \rho(A)$, and then $\rho(S_{\frac{1}{2}}(A)) = \rho(S_t(A)) = \rho(A)$. □

REMARK 3.1. Similar to Remarks 2.1 – 2.2, Theorem 3.1 shows that $\max\{\rho(A),\rho(B)\}$ is the upper bound for $\psi(t) := \rho(S_t(A,B))$, $t \in [0,1]$. From Theorem 3.2, it follows that $\rho(A)$ and $\rho(S_{\frac{1}{2}}(A))$ are upper and lower bounds for $\rho(S_t(A)), t \in [0,1]$, respectively.

THEOREM 3.3. *Let* $A \in \mathbb{R}_{+}^{n\times n}$. *Then the function* $g(t) := \rho(S_t(A))$ *is decreasing in* $[0, 0.5]$ *and increasing in* $[0.5, 1]$.

As an application of Theorem 3.1, inequalities for the spectral radius of complex matrices are also derived.

THEOREM 3.4. *Let* A *and* $B \in \mathbb{C}^{n\times n}$. *Then*

$$\rho(A\circ B) \leqslant \sqrt{\rho(A\circ A)\rho(B\circ B)}, \quad \rho(A\circ B^H) \leqslant \sqrt{\rho(A\circ A)\rho(B\circ B)}.$$

COROLLARY 3.3. *Let* $A \in \mathbb{C}^{n\times n}$. *Then* $\rho(A\circ A^H) \leqslant \rho(A\circ A)$.

COROLLARY 3.4. *Let* $A \in \mathbb{C}^{n\times n}$, *and let* P *be a permutation matrix. Then*

$$\rho(A\circ(P^TAP)) \leqslant \rho(A\circ A), \quad \rho(A\circ(P^TA^HP)) \leqslant \rho(A\circ A).$$

REMARK 3.2. Similar to Theorem 2.5, Theorem 3.1 can also be generalized. We omit it here.

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